CS 1050 - Proofs
Homework 4 and 5
Assigned Tuesday, September 21
Due Thursday, September 30

1. a) Using ideas you've seen in this class, prove that if x, y are any integers and $2^x = 3^y$, then $x = y = 0$.

b) Using part a), show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a, b) = 2^a 3^b$ is one-to-one.

2. For each of the following, show the statement is true by giving a proof, or disprove it by giving a counterexample:

a) The sum of any three even integers is even.

b) The sum of any three odd integers is even.

c) The sum of two primes is never prime.

d) The sum of any four consecutive integers is a multiple of 4.

e) The sum of any five consecutive integers is a multiple of 5.

3. Consider the following sentence:

For every integer x, there is an integer y such that $y^2 = 5 + x^2$.

a) Write this sentence in mathematical notation, using quantifiers.

b) Is this statement true? Either prove it or give a counterexample.

c) Write the negation of this statement in mathematical notation.

4. Consider this sentence:

There is an integer x such that for all integers y, there is an integer z such that

$$(x^2 > 4) \Rightarrow (y + z = x).$$

a) Write this sentence in mathematical notation, using quantifiers.

b) Is this statement true? Either prove it or give a counterexample.

c) Write the negation of this statement in mathematical notation.

5. a) Prove that for $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, $[a = b \pmod{m}]$ implies $[ca = cb \pmod{m}]$.

b) Prove that for $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, $[ca = cb \pmod{m}]$ does not imply that $[a = b \pmod{m}]$ by providing a counterexample.

6. For each of the following, solve and explain how you are using mods to get your answer:

a) What is the units digit of 3^{83}?

b) What is the remainder when 3^{83} is divided by 7?

7. Prove that $9^n + 3$ is always a multiple of 4.

8. Using a truth table, prove the following (where " $\iff$ " means "is equivalent to"):

a) $(p \to q) \iff ((\neg p) \vee q)$.

b) $[p \to (q \to r)] \iff [(p \to q) \to (p \to r)]$.

c) $[(p \leftrightarrow q) \leftrightarrow r)] \iff [p \leftrightarrow (q \leftrightarrow r)]$.

9. a) Disprove: $[(p \to q) \to r)] \iff [p \to (q \to r)]$.

b) Now disprove it without truth tables by giving a counterexample.

10. Twin primes are consecutive odd integers where both are prime. An example is $(3, 5)$.

a) Give 4 more examples of twin prime pairs.

b) Show that there cannot be a set of "triple primes" (i.e., a set of three consecutive odd numbers that are all prime).

11. Let $f : \mathbb{Z}^+ \to \mathbb{R}$ by $f(n) = n$ for all $n \in \mathbb{Z}^+$. Let $g : \mathbb{Z}^+ \to \mathbb{R}$ by $g(n) = g$ for all $n \in \mathbb{Z}^+$.

Now consider this sentence:

There are real numbers c and N such that for all positive integers n, [if $n \geq N$ then $f(n) \leq c \cdot g(n)$].

a) Write this last sentence in mathematical notation.

b) Write the negation of this statement both mathematically and in English.