CS 1050 - Proofs
Homework 6
Assigned Friday, October 1
Due Thursday, October 7

1. a) Prove that for all $k, n \in \mathbb{Z}, 10^n = 10^k (\text{mod} 9)$.

b) Using part (a), prove that if you choose any number N base 10, and then rearrange the digits to get $f(N)$, that $N - f(N) \cong 0(\text{mod} 9)$.

c) Prove that if an integer is a multiple of 9, then its digits must add up to a multiple of 9.

Now go back to "digicc.com/fido." Can you figure out the secret now?

2. In class we saw a proof by contradiction showing that there are an infinite number of primes. The proof said that if there were a finite number of primes, the largest of which was n, then $n! + 1$ would have to be prime too. Does this mean that if n is prime, then $n! + 1$ is always a prime? (Notice that this would give a way of finding new primes! Does this work?) Explain your answer (and don't just look for a counterexample, think about what goes wrong if we try to extend the proof in this context).

3. Consider a set of four boys (a,b,c,d) and four girls (1,2,3,4) with preferences shown in figure 1.

a) Run the traditional marriage algorithm on this instance. Show each stage of the algorithm, and give the reesulting matching, expressed as a set of boy-girl pairs. You can do this by hand.

b) The matching you found is boy-optimal. Find now a girl-optimal stable matching by modifying the algorithm appropriately. Compare the two matchings.

c) We know that there can be no more than n^2 stages of the algorithm. Can you construct an instance with n boys and n girls so that $c \cdot n^2$ stages are resuired, for some respectably

boy	preferences		girl	preferences
a	$1 > 2 > 3 > 4$		1	$d > b > c > a$
b	$2 > 1 > 4 > 3$		2	$a > d > b > c$
c	$1 > 3 > 2 > 4$		3	$a > b > c > d$
d	$2 > 1 > 3 > 4$		4	$d > c > a > b$

Figure 1: Prefernces for the stable marriage problem. ($1 > 2$ means that person 1 is preferred to person 2.)

large constant c? (We are looking for a *general pattern* here that will result in cn^2 stages for any n.)

d) Let M, M' denote two stable matchings, and let $M * M'$ denote the configuration where each girl is married to the better of her two partners in M and M' (according to the girl's preference list). Suppose that $M * M'$ is a matching. Is $M * M'$ guaranteed to be stable if M and M' are? Prove your answer.

4. Chocolate often comes in rectangular bars marked off into smaller squares. It is easy to break a larger rectangle into two smaller rectangles along any of the horizontal or vertical lines between the squares. Suppose I have a bar containing k squares and wish to break it down into its individual squares. Prove that *no matter which way I break it*, it will take exactly $k - 1$ breaks to do this.

5. Suppose I have an axe and a $3 \times 3 \times 3$ Rubics Cube and I want to break the large cube into 27 small cubes. In this problem, assume that in one blow of the axe I can cut through several pieces at once with one slice if I wanted. Show that it will require at least 6 cuts with the axe.

6. a) Write a sentence in English that, when translated to mathematics, would have the structure "for all xxxx there exists yyyy ...." Now negate this sentence in English. (Your sentence should not sound stilted like it came from the mathematical statement. It should sound as normal as possible!)

b) Now write a sentence of the form "there exists xxxx such that there exists yyyyy," also in normal English. Negate this sentence and write it in normal English as well.