# CS 1050 Homework 9 Solutions

**1.** Consider $c = 3$, then we have,

$c \cdot g(n) = 3 \cdot 2^n > 2^{n+1}$ for all $n \geq 1$.

$\Leftrightarrow f(n) < c \cdot g(n)$ for all $n \geq 1$.

$\Leftrightarrow f(n) = O(g(n))$

**2.a** We have,

$$\frac{f(n)}{g(n)} = \frac{(n+1)^2}{n^2}$$

$$= \frac{n^2 + 2n + 1}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2}$$

$$\Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} 1 + \frac{2}{n} + \frac{1}{n^2}$$

$$\Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{2}{n} + \lim_{n \to \infty} \frac{1}{n^2}$$

Now, since $\lim_{n \to \infty} \frac{2}{n} = 0$ and $\lim_{n \to \infty} \frac{1}{n^2} = 0$, we get

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

$\Leftrightarrow f(n) = O(g(n))$

**b.** Consider $c = 4$, and we have,

$c \cdot g(n) = 4n^2 = n^2 + 3n^2$

Since we know that $1 \leq n \leq n^2$ for all $n \geq 1$, we have,

$c \cdot g(n) \geq n^2 + 2n + 1 = (n+1)^2$ for all $n \geq 1$

$\Leftrightarrow f(n) \leq c \cdot g(n)$ for all $n \geq 1$

$\Leftrightarrow f(n) = O(g(n))$.

**3.** We will prove this without using the limit. Clearly,

$n \geq (200)^2 + 50$ for all $n \geq (200)^2 + 50$.

$\Leftrightarrow n^3 \geq (200n)^2 + 50n^2$ for all $n \geq (200)^2 + 50$.

$\Leftrightarrow n^3 \geq (200n)^2 + 50$ for all $n \geq (200)^2 + 50$ (using $50n^2 \geq 50$).

$\Leftrightarrow g(n) \geq f(n)$ for all $n \geq (200)^2 + 50$.

$\Leftrightarrow f(n) = O(g(n))$.

**4.** We have,

$g(n) - 3f(n) = n^2 + 5n - 6 = (n-1)(n+6) \geq 0$ for all $n \geq 1$

$\Leftrightarrow g(n) \geq 3f(n)$ for all $n \geq 1$

$\Leftrightarrow f(n) \leq \frac{1}{3}g(n)$ for all $n \geq 1$

$\Leftrightarrow f(n) = O(g(n))$

**5. Proof:** Consider $N = 2A^2$. Now for any $n \geq N = 2A^2$,

$n! = A^2!(A^2+1)\ldots 2A^2 \ldots n$

Now $A^2! \geq 1$ (since $A$ is a positive integer) and each term on the RHS after $A^2!$ is greater than $A^2$ (there $n - A^2$ such terms). Therefore,

$n! > (A^2)^{n-A^2} = A^{2(n-A^2)}$

Now since $n \geq 2A^2$, we have that $n - \frac{n}{2} \geq A^2$ and so $n - A^2 \geq \frac{n}{2}$. Therefore,

$n! > A^{2(\frac{n}{2})}$

$\Leftrightarrow n! > A^n$

So, $N = 2A^2$ suffices.

**6a.** We need to prove that $f$ is not $O(g)$. We prove it by contradiction. Suppose that $f = O(g)$. That means that there exists constants $c$ and $n_0$ such that,

$c \cdot g(n) \geq f(n)$ for all $n \geq n_0$

Now pick $n_1$ to be any even number greater than $n_0$ and $c$, i.e. $n_1 > \max\{n_0, c\}$ and $n_1$ is even. Clearly such an even number can be picked since $c$ and $n_0$ are constants. We have,

$f(n_1) = n_1^2 > c\, n_1 = c \cdot g(n_1)$

which is a contradiction to the fact that $c \cdot g(n) \geq f(n)$ for all $n \geq n_0$ (since

$n_1 > n_0$ and $n_1 > c$)
Therefore $f(n)$ cannot be $O(g(n))$.