Worksheet 4: Chapter 2/3 (ε-δ defn, limits of the secant/DQ)

1. From the definition of

$$\lim_{x\to 3} x^2 = 9$$

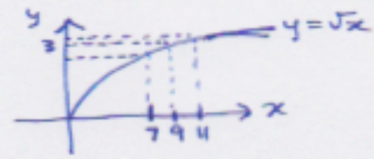
Find the **largest** $\delta > 0$ such that if $|x-3| < \delta$, then $|x^2 - 9| < \varepsilon$, where $\varepsilon = 2$.

$$|x^2-9| < 2 \iff -2 < x^2 - 9 < 2 \iff 7 < x^2 < 11$$

$$\iff \sqrt{7} < x < \sqrt{11} \iff \sqrt{7} - 3 < x - 3 < \sqrt{11} - 3$$

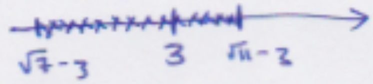
$$\approx -.35 < x - 3 < \underline{.31}$$

take the smaller one.



$\boxed{\text{Set } \delta = \sqrt{11} - 3}$

x must be in this range.



2. Let $f(x) = \sqrt{x}$. Find

$$L = \lim_{x\to 16} f(x).$$

Then, find the largest $\delta > 0$ such that

$$|x - 16| < \delta \implies |f(x) - L| < \varepsilon$$

for $\varepsilon = 4$.

$$|\sqrt{x} - 4| < 4 \iff -4 < \sqrt{x} - 4 < 4 \iff 0 < \sqrt{x} < 8$$

$$\iff |x| < 64 \iff -64 < x < 64$$

$$\iff -64 - 16 < x - 16 < 64 - 16$$

$$\iff -80 < x - 16 < 58$$

take the smaller one.

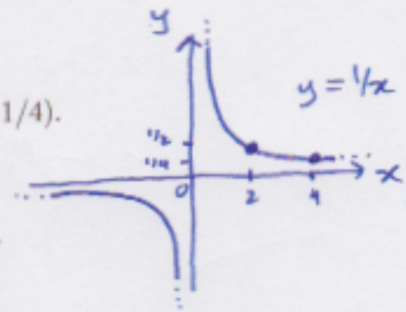
Set $\delta = 58$ to ensure that

$|x - 16| < \delta$ forces $-80 < x - 16 < 58$

3. Consider the function $f(x) = \frac{1}{x}$, and we will examine a **secant line** of $f(x)$ as well as $f'(2)$ which is the derivative of f at $x = 2$, the limit of the slopes of the secant lines near $x = 2$:

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}.$$

(a) Find the slope of the secant line of the points $(2, 1/2)$ and $(4, 1/4)$.



$$m = \frac{\text{rise}}{\text{run}} = \frac{f(4) - f(2)}{4 - 2} = \frac{1/4 - 1/2}{4 - 2} = \frac{-1/4}{2} = \frac{-1}{8}$$

$$\boxed{\text{Slope} = -1/8}$$

(b) Guess the limit of the slopes of the secant lines at $x = 2$ by either creating a table (in a spreadsheet) or using graphing software (like GeoGebra).

Use GeoGebra File on website
under "Worksheet 4 GeoGebra FILE"

Guess probably limit is -0.25.

(c) Find the *instantaneous rate of change* at $x = 2$. That is, find

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}.$$

Note that "instantaneous rate of change" and "limit of the slopes of the secant lines" and "the derivative" and "limit of the difference quotient" are all (essentially) synonyms.

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{1/x - 1/2}{x - 2} = \lim_{x \to 2} \frac{\frac{2 - x}{2x}}{x - 2}$$

$$= \lim_{x \to 2} \frac{1}{x - 2} \cdot \frac{2 - x}{2x}$$

$$= \lim_{x \to 2} \frac{1}{\cancel{x - 2}} \cdot \frac{-1\cancel{(x - 2)}}{2x} = \lim_{x \to 2} \frac{-1}{2x} = \frac{-1}{4}.$$

4. On a walk with my dog my distance from home after t minutes is given by

$$f(t) = x^3/8 - x^2 + 2x.$$

What is my average speed in the first 30 seconds of my walk? *(1)* What is my average speed in the first 2 minutes? *(2)* What is the *average rate of change* of the function $f(x)$ on the interval $[0, 4]$ and *(3)* how can you phrase this question to be similarly phrased as the previous two? *(4)* Set up but **do not solve** a limit which could be evaluated to find $f'(2)$, and finally *(5)* interpret the meaning of $f'(2)$ in this example.

Average speed = average rate of change

(3) = slope of the secant line

$$= \frac{\text{rise}}{\text{run.}}$$

(1) First two minutes between time $t=0$ and $t=2$.

$$\frac{f(2)-f(0)}{2-0} = \frac{(2^3/8 - 2^2 + \cancel{4}) - (0^3/8 - 0^2 + 0)}{2}$$

$$= \tfrac{1}{2}\,(1 - 4 + \cancel{4}) = +1/2.$$

(2)

$$\frac{f(4)-f(0)}{4-0} = \frac{4^3/8 - 4^2 + 2\cdot 4}{4} = \frac{8-16+8}{4} = 0.$$

How is the average speed 0!?
(Hint: look at the graph!)

(4)

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2} \frac{f(x)-1}{x-2} = \text{speed at time } t=2$$

(5) !!