Exam 3

1. Find the value of the series. Clearly show all steps for full credit. (10 pts.)

$$\sum_{n=2}^{\infty} \frac{3-2^n}{3^{n-1}}$$

$$\sum_{n=2}^{\infty} \frac{3-2^n}{3^{n-1}} = \sum_{n=2}^{\infty} \frac{3}{3^{n-1}} - \sum_{n=2}^{\infty} \frac{2^n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{3}{3^{n+1}} - \sum_{n=0}^{\infty} \frac{2^{n+2}}{3^{n+1}}$$

$$= 1\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$$

$$= \frac{1}{1-1/3} - \frac{4}{3}\,\frac{1}{1-2/3}$$

$$= \frac{1}{2/3} - \frac{4}{3}\,\frac{1}{1/3} = \frac{3}{2} - \frac{4}{\cancel{3}}\cdot\cancel{3} = -(4-1.5) = \boxed{-2.5}$$

2. Solve the first order linear differential equation

$$xy' - y = 2x\ln x$$

and find the particular solution $y = y(x)$ satisfying $y = 2$ when $x = 1$. (10 pts.)

tricky?

$$y' - \frac{1}{x}y = 2\ln x$$

use r(x) if necessary

$$\left(-\frac{1}{x}y' + \frac{1}{x^2}y\right) = \frac{-2\ln x}{x}$$

$$\left(\frac{1}{x^2}y\right)' = -\frac{2}{x}\ln x$$

$$\frac{1}{x^2}y = \int -\frac{2}{x}\ln x\,dx$$

1

$$y = x^2 \cdot -2\int \ln x \cdot \frac{1}{x}\,dx$$

$$y = -\frac{\cancel{2}x^2(\ln x)^2}{\cancel{2}} + C$$

$$2 = -1\cdot 0 + C$$

$$\Rightarrow C = 2$$

$$\boxed{y = -x^2(\ln x)^2 + 2}$$

3. Determine whether the series converges or diverges. Give clear justification for your answer for full credit. (8 pts. each)

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$a_n = \frac{3^n}{n!}$

$\boxed{\text{Converges by ratio test}}$

$\frac{a_{n+1}}{a_n} = 3^{n+1}/(n+1)! * \frac{n!}{3^n} = \frac{3}{n+1} \to 0$

(b) $\sum_{n=1}^{\infty} \frac{4^n+3}{2^{2n+1}}$

$\boxed{\text{Diverges}}$ by n-th term test

~~Converges by geometric series test~~

diverges

$\sum a_n = (\sum \frac{1}{2}) + \frac{3}{2} \sum (\frac{1}{4})^n$

converges

$a_n = \frac{4^n+3}{2^{2n+1}} = \frac{4^n}{2 \cdot 4^n} + \frac{3}{2 \cdot 4^n} = \frac{1}{2} + \frac{3}{2 \cdot 4^n}$

(c) $\sum_{n=2}^{\infty} \frac{n^n}{(2n)^2} = \sum \frac{n^n}{4n^2}$

$\boxed{\text{diverges by root test?}}$

$\sqrt[n]{a_n} = \frac{n}{\sqrt[n]{n^2}} \to \infty$

b/c $\sqrt[n]{n^2} = (\sqrt[n]{n})^2 \to (\lim \sqrt[n]{n})^2 = 1^2 = 1$

4. Short answer section. Find the limit of the sequence. Put the value of the limit in the box if it exists, otherwise write DNE in the box. Your work may be graded for partial credit if your answer is incorrect, otherwise just show reasonable work for full credit with a correct answer. (5 pts. each)

(a) $\left(1-\frac{2}{n}\right)^{3n}$

$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{3n} =$

$a_n=\left(1-\frac{2}{n}\right)^{3n}$

$\ln(a_n) = 3n\ln\left(1-\frac{2}{n}\right)$

$= \frac{\ln\left(1-\frac{2}{n}\right)}{1/3n} \to \lim \frac{\left(\frac{2/\cancel{n^2}}{1-2/n}\right)}{-1/3\cancel{n^2}} = 3/2$

$\boxed{e^{3/2}}$

(b) $(-1)^n\left(1-\frac{1}{n}\right)$

$\boxed{\text{DNE}}$ b/c alt. between ± 1 in limit

(c) $\frac{2n+1}{1+\sqrt{n}}$

$\frac{\text{BIG}}{\text{big}}$ or use L'Hop.

$\boxed{\text{DNE}}$

(d) $\left(\frac{3}{n}\right)^{1/n}$

btw * can also use $(n)^{1/n} \to 1$ and $(3)^{1/n} \to 1$ to solve...

$a_n = \left(\frac{3}{n}\right)^{1/n}$

$\ln(a_n) = \frac{1}{n}\ln\left(\frac{3}{n}\right)$

$\boxed{1}$

$= \frac{\ln(3/n)}{n} \to \lim_{n\to\infty}\frac{\frac{-3/n^2}{3/n}}{1} = \frac{-n}{n^2} \to 0$

$\ln(a_n)\to 0$ so $a_n \to 1$ since $\ln(1)=0$

5. Determine whether the each sequence converges or diverges. Clearly give full justification for your answer. (10 pts. each)

6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Integral test

$\boxed{\text{Converges}}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$$

$$= \int_{*}^{*} \frac{1}{u^2}\, du$$

$$= \left.-\frac{1}{u}\right|_{*}^{*} = \left.\frac{-1}{\ln(x)}\right|_2^{\infty} = -0 - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$ converges

7. $\sum_{n=1}^{\infty} \frac{(n+2)^n}{n^n}$

$\boxed{\text{diverges}}$

$\frac{(n+2)^n}{n^n} \geq \frac{n^n}{n^n}$ so by direct comparison

~~is inconclusive~~

Since $\sum \frac{n^n}{n^n} = \sum 1$ diverges

$\sum \frac{(n+2)^n}{n^n}$ also diverges.

8. True and False section. Circle TRUE if the statement is always true in all cases. Otherwise, circle false. (2 pts. each)

(a) The l.u.b. of the sequence $a_n = 1 - \frac{1}{n}$, $n \geq 1$, is 1. TRUE FALSE

limit

$a_1 = 0$ $a_2 = \frac{1}{2}$ ... $a_{20} = 19/20$ ...

(b) The sequence $a_n = |\sin(n)|$ is monotone. TRUE FALSE

(c) The sequence $a_n = \frac{n+1}{n-1}$ converges. TRUE FALSE

to 1.

(d) The series $\sum_{n=2}^{\infty} \frac{n^2+1}{2n^3-3n+1}$ converges. ~~TRUE~~ FALSE

try limit comparison w/ $1/n$ ...

no. harmonic series $1/n$ diverges

& this one is similar in limit.

(e) The series $\sum_{n=2}^{\infty} e^{-n}$ converges. TRUE FALSE

$\int e^{-x}\,dx$ converges so by integral test ✓

(f) The sequence $a_n = \frac{1-(-1)^n}{\sqrt{n}}$ diverges. TRUE FALSE

Converges to zero.

(g) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges. TRUE FALSE

e.g. $a_n = 1/n \to 0$

but $\sum 1/n$ diverges

(h) If $a_{n+1}/a_n = 1$ for all $n \geq 1$, then the series $\sum a_n$ diverges. TRUE FALSE

Thats true.

If $\frac{a_{n+1}}{a_n} = 1$ then $a_n = a_{n+1}$ $\forall n$

and neither is zero.

So $a_n =$ constant & $\sum c$ diverges