Math 1552
Summer 2023
Exam 3 GLC
July 20 @ 3:30pm
Time limit: 75 Minutes

Name (Print): _____

Canvas email: _____

Teaching Assistant/Section: _____

Key

GT ID:

By signing here, you agree to abide by the **Georgia Tech Honor Code**: *I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech Community.*

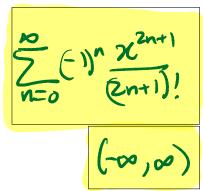Sign Your Name: _____

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.

- **Organize your work** in a reasonably neat and coherent way.

- **Simplify your answers** unless explicitly stated otherwise.

- **Fill in circles** completely. Do not use check marks, X's, or any other marks.

- **Place a box** around your final answer for full credit.

- Calculators, notes, cell phones, books are not allowed.

- Use dark and clear writing: your exam will be scanned into a digital system.

- Exam pages are double sided. Be sure to complete both sides.

- Leave a 1 inch border around the edges of exams.

- The last page is for scratch work. Please use it if you need extra space.

- This exam has 6 pages of questions.

1. (3 points) Which of the following statements are true? *You do not have to show your work.*

true	false	
○	●	The alternating series $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sin^2 n}{n^2}$ diverges.
○	●	The radius of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{2^n}{3^n}(x-1)^n$ is $R = \dfrac{2}{3}$.
●	○	If the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

2. (5 points) State the Taylor series at $x = 0$ for the function and give the interval of convergence. You do not have to show work for this problem. *Hint: this is a common Taylor series.*

$f(x) = \sin(x)$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$(-\infty, \infty)$$

3. (8 points) Find the Taylor series at $x = 0$ for the given function. If you use a common Taylor series state the series you are using.

$$f(x) = \frac{3}{2+x}$$

$$\frac{3}{2+x} = 3 \cdot \frac{1}{2(1+\frac{x}{2})} = \frac{3}{2} \cdot \frac{1}{1-\frac{-x}{2}} \overset{\bigstar}{=} \frac{3}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n$$

$\bigstar$ Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1.$$

So

$$\frac{3}{2+x} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2^{n+1}} x^n}$$

4. (8 points) Determine if the given series converges or diverges. Show your work and clearly justify your answer for credit. *Hint: try a comparison test.*

$$\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n+1}}$$

Direct comparison

$$a_n = \frac{2}{n\sqrt{n+1}} \leq b_n = \frac{2}{n\cdot\sqrt{n}} = \frac{2}{n^{3/2}}$$

Now, $\sum b_n$ converges by p-test w/ $p = 3/2 > 1$.

Since $\sum a_n \leq \sum b_n$, by the direct comparison test $\boxed{\sum a_n \text{ converges}}$

5. (8 points) Determine if the given series converges or diverges. Show your work and clearly justify your answer for credit. *Hint: try the root test.*

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

root test

$$\left(a_n\right)^{1/n} = \left(\left(1 - \frac{1}{n}\right)^{n^2}\right)^{1/n} = \left(1 - \frac{1}{n}\right)^{n^2/n} = \left(1 - \frac{1}{n}\right)^{n}$$

$$\lim_{n \to \infty} \left(a_n\right)^{1/n} = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n} = e^{-1} = \frac{1}{e} = L$$

Since $L < 1$, by the root test the series $\boxed{\sum a_n \text{ converges}}$

6. (8 points) Consider the function $f(x) = e^x$.

(a) Find the Taylor polynomial $P_4(x)$ of degree $N = 4$ expanded at $x = 1$ (in powers of $x - 1$).

$$f(x) = e^x \implies f^{(n)}(x) = e^x, \quad \text{so} \quad f^{(n)}(1) = e.$$

And so,

$$P_4(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$= \boxed{e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4}$$

(b) Use the formula below to find the maximum error of using $P_4(1.1)$, the Taylor polynomial evaluated at $x = 1.1$, to approximate $f(1.1) = e^{1.1}$.

$$|R_N(x)| \le \max |f^{(N+1)}(c)| \frac{|x-a|^{N+1}}{(N+1)!}$$

Hint: $e^{1.1} < 4$.

$$f(x) = e^x$$
$$\text{so } f^{(5)}(x) = e^x$$

and

$$|f^{(5)}(x)| \le e^{1.1} \text{ for all } x \text{ in } [1, 1.1], \text{ and } e^{1.1} < 4$$

$$\text{So} \quad |R_4(x)| \le 4 \cdot \frac{|1.1 - 1|^5}{5!} = 4 \cdot \frac{(0.1)^5}{120} = \frac{1}{30}\left(\frac{1}{10}\right)^5 = \boxed{\frac{1}{3,000,000}}$$

7. (10 points) Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} x^n$.

(a) Find the interval of convergence of the power series $f(x)$.
Hint: don't forget to test the endpoints separately.

Ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n x^n} = -x \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$$

So $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = -x \cdot \sqrt{1} = -x = L$

Now $|L| < 1 \implies -1 < -x < 1$
$$\implies -1 < x < 1$$

$\boxed{(-1, 1]}$

@ $x = 1$
$$\sum \frac{(-1)^n}{\sqrt{n}}$$ converges by alt serg test

@ $x = -1$
$$\sum \frac{(-1)^n}{\sqrt{n}} (-1)^n = \sum \frac{1}{\sqrt{n}}$$
diverges by p-test
w/ $p = \frac{1}{2} < 1$

(b) Find a power series for $\int f(x)\,dx$.

$$\int f(x)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} x^n \, dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \int x^n \, dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \frac{1}{n+1} \cdot x^{n+1} + C$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}(n+1)} x^{n+1} + C}$$

*This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **not be detached** from your exam booklet at any time.*