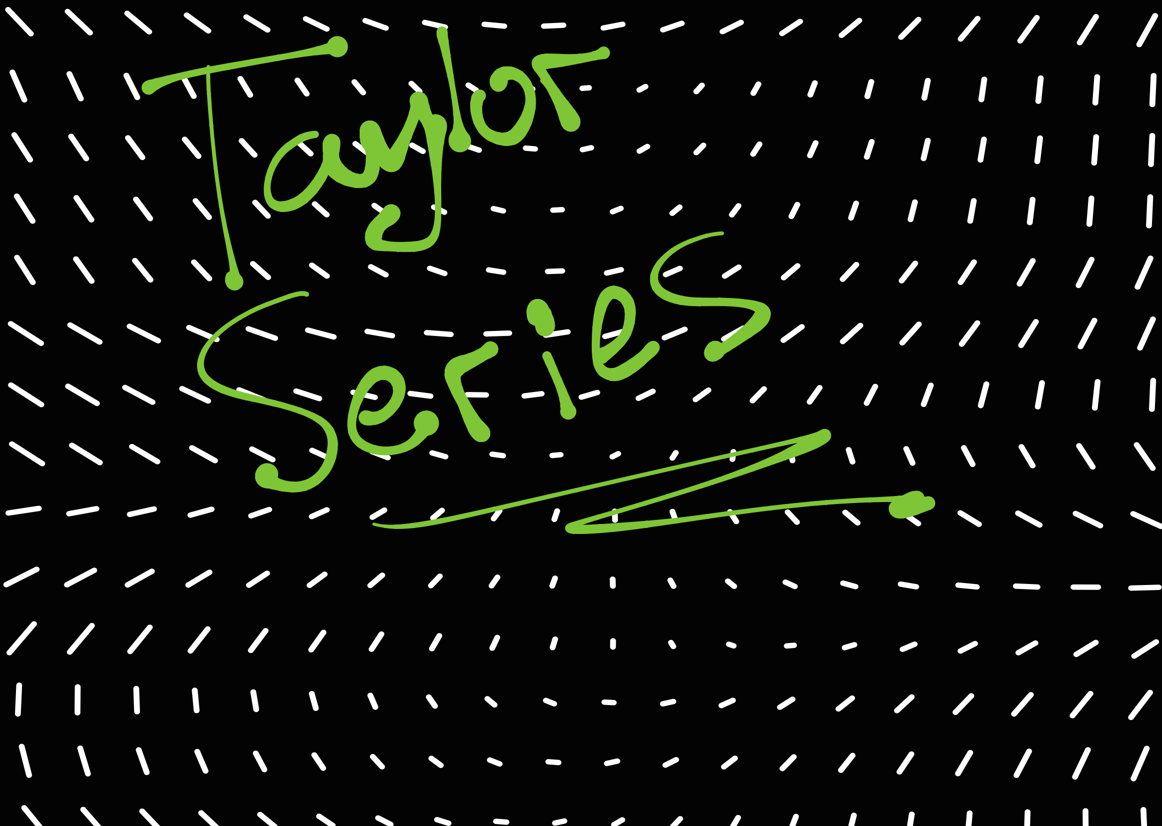
WEEK 10

Taylor Series



MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

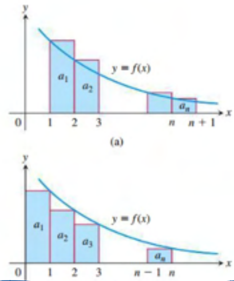
Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 **Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)**	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series
8	Jul 3 **NO CLASS** Independence Day	Jul 4 **NO CLASS** Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	Jul 6 WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 **Test #3 (10.4-10.9)**	Jul 21 Section 6.1: Volumes by Disks
11	Jul 24 Section 6.1: Volumes by Cylindrical Shells Final Review	Jul 25 WS 6.1-6.2 *Last day for MML homework*	Jul 26 Reading Day	Jul 27	Jul 28 **FINAL EXAM** **11:20 AM – 2:10 PM**
12	Jul 31	Aug 1	Aug 2	Aug 3	Aug 4

Geometric Series Formula

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r| < 1}$$



THM: Divergence test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ surely diverges.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

The Integral test

$$\int_1^{\infty} f(x)\,dx \quad \text{vs.} \quad \sum_{k=1}^{\infty} f(k)$$

THEOREM 10—Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

THEOREM 11—Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(a) If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L = 1$, then the test is *INCONCLUSIVE!!!!*

Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $R = \lim_{n\to\infty} \sqrt[n]{a_n}$.

(a) If $R < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $R > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $R = 1$, then the test is *INCONCLUSIVE!!!!*

Alternating Series Test

Let $\sum_k a_k$ be an alternating series.

(a) If $\sum_k |a_k|$ converges, then the series *converges absolutely*.

(b) If (a) fails, then if :

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty} |a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

Estimating the Sum

Let $\sum_k a_k$ be a convergent alternating series with a sum of L.

Then : $|s_n - L| < |a_{n+1}|$.

Section 10.7: 3, 9, 11, 15, 17, 27, 31, 41, 43, 50 (extra practice: 13, 23)

EXERCISES 10.7

Intervals of Convergence

In Exercises 1–36, **(a)** find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

1. $\sum_{n=0}^{\infty} x^n$

2. $\sum_{n=0}^{\infty} (x+5)^n$

3. $\sum_{n=0}^{\infty} (-1)^n(4x+1)^n$

4. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

6. $\sum_{n=0}^{\infty} (2x)^n$

7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{n}$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$

10. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

12. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

13. $\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$

14. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

16. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$

17. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

18. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

19. $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$

20. $\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$

21. $\sum_{n=1}^{\infty} (2+(-1)^n)\cdot(x+1)^{n-1}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}(x-2)^n}{3n}$

23. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

24. $\sum_{n=1}^{\infty} (\ln n)x^n$

25. $\sum_{n=1}^{\infty} n^n x^n$

26. $\sum_{n=0}^{\infty} n!(x-4)^n$

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$

28. $\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$

29. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 10.3, Exercise 61.

30. $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n}$ Get the information you need about $\sum 1/(n\ln n)$ from Section 10.3, Exercise 60.

31. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$

32. $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

33. $\sum_{n=1}^{\infty} \frac{1}{2\cdot 4\cdot 6\cdots(2n)}x^n$

34. $\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{n^2\cdot 2^n}x^{n+1}$

35. $\sum_{n=1}^{\infty} \frac{1+2+3+\cdots+n}{1^2+2^2+3^2+\cdots+n^2}x^n$

36. $\sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n})(x-3)^n$

In Exercises 37–40, find the series' radius of convergence.

37. $\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9\cdots 3n}x^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2\cdot 4\cdot 6\cdots(2n)}{2\cdot 5\cdot 8\cdots(3n-1)}\right)^2 x^n$

39. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!}x^n$

40. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

(*Hint:* Apply the Root Test.)

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function

45. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2}-1\right)^n$

46. $\sum_{n=0}^{\infty} (\ln x)^n$

47. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$

48. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

Using the Geometric Series

49. In Example 2 we represented the function $f(x)=2/x$ as a power series about $x=2$. Use a geometric series to represent $f(x)$ as a power series about $x=1$, and find its interval of convergence.

50. Use a geometric series to represent each of the given functions as a power series about $x=0$, and find their intervals of convergence.

a. $f(x)=\frac{5}{3-x}$ **b.** $g(x)=\frac{3}{x-2}$

51. Represent the function $g(x)$ in Exercise 50 as a power series about $x=5$, and find the interval of convergence.

52. a. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8}{4^{n+2}}x^n.$$

b. Represent the power series in part (a) as a power series about $x=3$ and identify the interval of convergence of the new series. (Later in the chapter you will understand why the new interval of convergence does not necessarily include all of the numbers in the original interval of convergence.)

Theory and Examples

53. For what values of x does the series

$$1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

54. If you integrate the series in Exercise 53 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

55. The series

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$$

converges to $\sin x$ for all x.

a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

56. The series

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

41. $\sum_{n=0}^{\infty} 3^n x^n$

42. $\sum_{n=0}^{\infty} (e^x - 4)^n$

43. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

44. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

answer with the answer in part (b).

56. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

a. Find a series for $(d/dx)e^x$. Do you get the series for e^x? Explain your answer.

b. Find a series for $\int e^x\,dx$. Do you get the series for e^x? Explain your answer.

c. Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

57. The series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of the series for $\ln|\sec x|$. For what values of x should the series converge?

b. Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?

c. Check your result in part (b) by squaring the series given for $\sec x$ in Exercise 58.

58. The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of a power series for the function $\ln|\sec x + \tan x|$. For what values of x should the series converge?

b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?

c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 57.

59. Uniqueness of convergent power series

a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n. (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)

b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n.

60. The sum of the series $\sum_{n=0}^{\infty}(n^2/2^n)$ To find the sum of this series, express $1/(1-x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

61. The sum of the alternating harmonic series This exercise will show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

and

$$\lim_{n\to\infty} (h_{2n} - \ln 2n) = \gamma,$$

where γ is Euler's constant.

c. Use these facts to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} s_{2n} = \ln 2.$$

62. Assume that the series $\sum a_n x^n$ converges for $x = 4$ and diverges for $x = 7$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = -4$

b. Diverges for $x = 5$

c. Converges absolutely for $x = -8.5$

d. Converges for $x = -2$

e. Diverges for $x = 8$

f. Diverges for $x = -6$

g. Converges absolutely for $x = 0$

h. Converges absolutely for $x = -7.1$

63. Assume that the series $\sum a_n(x-2)^n$ converges for $x = -1$ and diverges for $x = 6$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = 1$

b. Diverges for $x = -6$

c. Diverges for $x = 2$

d. Converges for $x = 0$

e. Converges absolutely for $x = 5$

f. Diverges for $x = 4.9$

g. Diverges for $x = 5.1$

h. Converges absolutely for $x = 4$

64. Proof of Theorem 21 Assume that $a = 0$ in Theorem 21 and that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $-R < x < R$. Let $g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$. This exercise will prove that $f'(x) = g(x)$, that is, $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = g(x)$.

a. Use the Ratio Test to show that $g(x)$ converges for $-R < x < R$.

b. Use the Mean Value Theorem to show that

$$\frac{(x+h)^n - x^n}{h} = nc_n^{n-1}$$

for some c_n between x and $x + h$ for $n = 1, 2, 3, \ldots$.

c. Show that

Section 10.8: 3, 9, 15 (extra practice: 5, 7, 33)

EXERCISES 10.8

Finding Taylor Polynomials

In Exercises 1–10, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

1. $f(x) = e^{2x}, \quad a = 0$

2. $f(x) = \sin x, \quad a = 0$

3. $f(x) = \ln x, \quad a = 1$

4. $f(x) = \ln(1 + x), \quad a = 0$

5. $f(x) = 1/x, \quad a = 2$

6. $f(x) = 1/(x + 2), \quad a = 0$

7. $f(x) = \sin x, \quad a = \pi/4$

8. $f(x) = \tan x, \quad a = \pi/4$

9. $f(x) = \sqrt{x}, \quad a = 4$

10. $f(x) = \sqrt{1 - x}, \quad a = 0$

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercises 11–24.

11. e^{-x}

12. xe^x

13. $\dfrac{1}{1 + x}$

14. $\dfrac{2 + x}{1 - x}$

15. $\sin 3x$

16. $\sin \dfrac{x}{2}$

17. $7 \cos(-x)$

18. $5 \cos \pi x$

19. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

20. $\sinh x = \dfrac{e^x - e^{-x}}{2}$

21. $x^4 - 2x^3 - 5x + 4$

22. $\dfrac{x^2}{x + 1}$

23. $x \sin x$

24. $(x + 1) \ln(x + 1)$

Finding Taylor and Maclaurin Series

In Exercises 25–34, find the Taylor series generated by f at $x = a$.

25. $f(x) = x^3 - 2x + 4, \quad a = 2$

26. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

27. $f(x) = x^4 + x^2 + 1, \quad a = -2$

28. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

29. $f(x) = 1/x^2, \quad a = 1$

30. $f(x) = 1/(1 - x)^3, \quad a = 0$

31. $f(x) = e^x, \quad a = 2$

32. $f(x) = 2^x, \quad a = 1$

33. $f(x) = \cos(2x + (\pi/2)), \quad a = \pi/4$

34. $f(x) = \sqrt{x + 1}, \quad a = 0$

In Exercises 35–38, find the first three nonzero terms of the Maclaurin series for each function and the values of x for which the series converges absolutely.

35. $f(x) = \cos x - (2/(1 - x))$

36. $f(x) = (1 - x + x^2)e^x$

37. $f(x) = (\sin x) \ln(1 + x)$

38. $f(x) = x \sin^2 x$

39. $f(x) = x^4 e^{x^2}$

40. $f(x) = \dfrac{x^3}{1 + 2x}$

Theory and Examples

41. Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a\left[1 + (x - a) + \frac{(x - a)^2}{2!} + \cdots\right].$$

42. (*Continuation of Exercise 41.*) Find the Taylor series generated by e^x at $x = 1$. Compare your answer with the formula in Exercise 41.

43. Let $f(x)$ have derivatives through order n at $x = a$. Show that the Taylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = a$.

44. Approximation properties of Taylor polynomials Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x - a) + \cdots + b_n(x - a)^n$ is a polynomial of degree n with constant coefficients $b_0, \ldots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

i) $E(a) = 0$ — The approximation error is zero at $x = a$.

ii) $\displaystyle\lim_{x\to a} \frac{E(x)}{(x - a)^n} = 0,$ — The error is negligible when compared to $(x - a)^n$.

then

$$g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x - a)^n$.

Quadratic Approximations The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the *quadratic approximation* of f at $x = a$. In Exercises 45–50, find the **(a)** linearization (Taylor polynomial of order 1) and **(b)** quadratic approximation of f at $x = 0$.

45. $f(x) = \ln(\cos x)$

46. $f(x) = e^{\sin x}$

47. $f(x) = 1/\sqrt{1 - x^2}$

48. $f(x) = \cosh x$

49. $f(x) = \sin x$

50. $f(x) = \tan x$

Section 10.9: 7, 10, 13, 21, 45 (extra practice: 17, 25, 39, 41, 43)

EXERCISES 10.9

Finding Taylor Series

Use substitution (as in Example 4) to find the Taylor series at $x = 0$ of the functions in Exercises 1–12.

1. e^{-5x} **2.** $e^{-x/2}$ **3.** $5\sin(-x)$

4. $\sin\left(\frac{\pi x}{2}\right)$ **5.** $\cos 5x^2$ **6.** $\cos\left(x^{2/3}/\sqrt{2}\right)$

7. $\ln(1 + x^2)$ **8.** $\tan^{-1}(3x^4)$ **9.** $\frac{1}{1 + \frac{3}{4}x^3}$

10. $\frac{1}{2 - x}$ **11.** $\ln(3 + 6x)$ **12.** $e^{-x^2 + \ln 5}$

Use power series operations to find the Taylor series at $x = 0$ for the functions in Exercises 13–30.

13. xe^x **14.** $x^2 \sin x$ **15.** $\frac{x^2}{2} - 1 + \cos x$

16. $\sin x - x + \frac{x^3}{3!}$ **17.** $x\cos \pi x$ **18.** $x^2\cos(x^2)$

19. $\cos^2 x$ (*Hint:* $\cos^2 x = (1 + \cos 2x)/2$.)

20. $\sin^2 x$ **21.** $\frac{x^2}{1 - 2x}$ **22.** $x\ln(1 + 2x)$

23. $\frac{1}{(1 - x)^2}$ **24.** $\frac{2}{(1 - x)^3}$ **25.** $x\tan^{-1} x^2$

26. $\sin x \cdot \cos x$ **27.** $e^x + \frac{1}{1 + x}$ **28.** $\cos x - \sin x$

29. $\frac{x}{3}\ln(1 + x^2)$ **30.** $\ln(1 + x) - \ln(1 - x)$

Find the first four nonzero terms in the Maclaurin series for the functions in Exercises 31–38.

31. $e^x \sin x$ **32.** $\frac{\ln(1 + x)}{1 - x}$ **33.** $(\tan^{-1} x)^2$

34. $\cos^2 x \cdot \sin x$ **35.** $e^{\sin x}$ **36.** $\sin(\tan^{-1} x)$

37. $\cos(e^x - 1)$ **38.** $\cos\sqrt{x} + \ln(\cos x)$

Error Estimates

39. Estimate the error if $P_3(x) = x - (x^3/6)$ is used to estimate the value of $\sin x$ at $x = 0.1$.

40. Estimate the error if $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ is used to estimate the value of e^x at $x = 1/2$.

41. For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.

42. If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large, or too small? Give reasons for your answer.

43. How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$?

44. The estimate $\sqrt{1 + x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.

45. The approximation $e^x = 1 + x + (x^2/2)$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

46. (*Continuation of Exercise 45.*) When $x < 0$, the series for e^x is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing e^x by $1 + x + (x^2/2)$ when $-0.1 < x < 0$. Compare your estimate with the one you obtained in Exercise 45.

Theory and Examples

47. Use the identity $\sin^2 x = (1 - \cos 2x)/2$ to obtain the Maclaurin series for $\sin^2 x$. Then differentiate this series to obtain the Maclaurin series for $2\sin x\cos x$. Check that this is the series for $\sin 2x$.

48. (*Continuation of Exercise 47.*) Use the identity $\cos^2 x = \cos 2x + \sin^2 x$ to obtain a power series for $\cos^2 x$.

49. Taylor's Theorem and the Mean Value Theorem Explain how the Mean Value Theorem (Section 4.2, Theorem 4) is a special case of Taylor's Theorem.

50. Linearizations at inflection points Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = a$, then the linearization of f at $x = a$ is also the quadratic approximation of f at $x = a$. This explains why tangent lines fit so well at inflection points.

51. The (second) second derivative test Use the equation

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c_2)}{2}(x - a)^2$$

to establish the following test.

Let f have continuous first and second derivatives and suppose that $f'(a) = 0$. Then

a. f has a local maximum at a if $f'' \le 0$ throughout an interval whose interior contains a;

b. f has a local minimum at a if $f'' \ge 0$ throughout an interval whose interior contains a.

52. A cubic approximation Use Taylor's formula with $a = 0$ and $n = 3$ to find the standard cubic approximation of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the error in the approximation when $|x| \le 0.1$.

53. a. Use Taylor's formula with $n = 2$ to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ (k a constant).

b. If $k = 3$, for approximately what values of x in the interval $[0, 1]$ will the error in the quadratic approximation be less than $1/100$?

54. Improving approximations of π

a. Let P be an approximation of π accurate to n decimals. Show that $P + \sin P$ gives an approximation correct to $3n$ decimals. (*Hint:* Let $P = \pi + x$.)

T **b.** Try it with a calculator.

55. The Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $\sum_{n=0}^{\infty} a_n x^n$ A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ has a Taylor series that converges to the function at every point of $(-R, R)$. Show this by showing that the Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $\sum_{n=0}^{\infty} a_n x^n$ itself.

An immediate consequence of this is that series like

$$x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

Section 10.10: 15, 49

EXERCISES 10.10

Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1–10.

1. $(1+x)^{1/2}$ **2.** $(1+x)^{1/3}$

3. $(1-x)^{-3}$ **4.** $(1-2x)^{1/2}$

5. $\left(1+\frac{x}{2}\right)^{-2}$ **6.** $\left(1-\frac{x}{3}\right)^{4}$

7. $(1+x^3)^{-1/2}$ **8.** $(1+x^2)^{-1/3}$

9. $\left(1+\frac{1}{x}\right)^{1/2}$ **10.** $\frac{x}{\sqrt[3]{1+x}}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1+x)^4$ **12.** $(1+x^2)^3$

13. $(1-2x)^3$ **14.** $\left(1-\frac{x}{2}\right)^{4}$

Approximations and Nonelementary Integrals

T In Exercises 15–18, use series to estimate the integrals' values with an error of magnitude less than 10^{-5}. (The answer section gives the integrals' values rounded to seven decimal places.)

15. $\int_0^{0.6} \sin x^2\, dx$ **16.** $\int_0^{0.4} \frac{e^{-x}-1}{x}\, dx$

17. $\int_0^{0.5} \frac{1}{\sqrt{1+x^4}}\, dx$ **18.** $\int_0^{0.35} \sqrt[3]{1+x^2}\, dx$

T Use series to approximate the values of the integrals in Exercises 19–22 with an error of magnitude less than 10^{-8}.

19. $\int_0^{0.1} \frac{\sin x}{x}\, dx$ **20.** $\int_0^{0.1} e^{-x^2}\, dx$

21. $\int_0^{0.1} \sqrt{1+x^4}\, dx$ **22.** $\int_0^{1} \frac{1-\cos x}{x^2}\, dx$

23. Estimate the error if $\cos t^2$ is approximated by $1-\frac{t^4}{2}+\frac{t^8}{4!}$ in the integral $\int_0^1 \cos t^2\, dt$.

24. Estimate the error if $\cos\sqrt{t}$ is approximated by $1-\frac{t}{2}+\frac{t^2}{4!}-\frac{t^3}{6!}$ in the integral $\int_0^1 \cos\sqrt{t}\, dt$.

In Exercises 25–28, find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

25. $F(x)=\int_0^x \sin t^2\, dt,\quad [0,1]$

26. $F(x)=\int_0^x t^2 e^{-t^2}\, dt,\quad [0,1]$

27. $F(x)=\int_0^x \tan^{-1} t\, dt,\quad$ **(a)** $[0, 0.5]$ **(b)** $[0,1]$

28. $F(x)=\int_0^x \frac{\ln(1+t)}{t}\, dt,\quad$ **(a)** $[0, 0.5]$ **(b)** $[0,1]$

Indeterminate Forms

Use series to evaluate the limits in Exercises 29–40.

29. $\lim_{x\to 0} \frac{e^x-(1+x)}{x^2}$ **30.** $\lim_{x\to 0} \frac{e^x-e^{-x}}{x}$

31. $\lim_{t\to 0} \frac{1-\cos t-(t^2/2)}{t^4}$ **32.** $\lim_{\theta\to 0} \frac{\sin\theta-\theta+(\theta^3/6)}{\theta^5}$

33. $\lim_{y\to 0} \frac{y-\tan^{-1} y}{y^3}$ **34.** $\lim_{y\to 0} \frac{\tan^{-1} y-\sin y}{y^3\cos y}$

35. $\lim_{x\to\infty} x^2\left(e^{-1/x^2}-1\right)$ **36.** $\lim_{x\to\infty} (x+1)\sin\frac{1}{x+1}$

37. $\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x}$ **38.** $\lim_{x\to 2} \frac{x^2-4}{\ln(x-1)}$

39. $\lim_{x\to 0} \frac{\sin 3x^2}{1-\cos 2x}$ **40.** $\lim_{x\to 0} \frac{\ln(1+x^3)}{x\cdot\sin x^2}$

Using Table 10.1

In Exercises 41–52, use Table 10.1 to find the sum of each series.

41. $1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

42. $\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\left(\frac{1}{4}\right)^6+\cdots$

43. $1-\frac{3^2}{4^2\cdot 2!}+\frac{3^4}{4^4\cdot 4!}-\frac{3^6}{4^6\cdot 6!}+\cdots$

44. $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$

45. $\frac{\pi}{3}-\frac{\pi^3}{3^3\cdot 3!}+\frac{\pi^5}{3^5\cdot 5!}-\frac{\pi^7}{3^7\cdot 7!}+\cdots$

46. $\frac{2}{3}-\frac{2^3}{3^3\cdot 3}+\frac{2^5}{3^5\cdot 5}-\frac{2^7}{3^7\cdot 7}+\cdots$

47. $x^3+x^4+x^5+x^6+\cdots$

48. $1-\frac{3^2x^2}{2!}+\frac{3^4x^4}{4!}-\frac{3^6x^6}{6!}+\cdots$

49. $x^3-x^5+x^7-x^9+x^{11}-\cdots$

50. $x^2-2x^3+\frac{2^2x^4}{2!}-\frac{2^3x^5}{3!}+\frac{2^4x^6}{4!}-\cdots$

51. $-1+2x-3x^2+4x^3-5x^4+\cdots$

52. $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\cdots$

Theory and Examples

53. Replace x by $-x$ in the Taylor series for $\ln(1+x)$ to obtain a series for $\ln(1-x)$. Then subtract this from the Taylor series for $\ln(1+x)$ to show that for $|x|<1$,

$$\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right).$$

54. How many terms of the Taylor series for $\ln(1+x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}? Give reasons for your answer.

Math 1552
Sections 10.8 and 10.9

Taylor Polynomials and Taylor Series

$$f \quad \infty$$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

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9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 **Test #3 (10.4-10.9)**	Jul 21 Section 6.1: Volumes by Disks
11	Jul 24 Section 6.1: Volumes by Cylindrical Shells Final Review	Jul 25 WS 6.1-6.2 *Last day for MML homework*	Jul 26 Reading Day	Jul 27	Jul 28 **FINAL EXAM** **11:20 AM – 2:10 PM**
12	Jul 31	Aug 1	Aug 2	Aug 3	Aug 4

Review Question:

The series: $\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k^2+1}}$

A. Converges absolutely
B. Converges conditionally
C. Diverges

Defn.
Taylor series for $f(x)$ at $\underline{x=0}$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

Power Series

A *power series* is an *infinite polynomial* and a *function* of *x*:

Power series in x: $f(x)=\sum_{k=0}^{\infty}a_k x^k$

Power series in x-c: $f(x)=\sum_{k=0}^{\infty}a_k (x-c)^k$

Learning Goals

- Understand the process to finding a Taylor polynomial for a given function and center
- Estimate a function value using Taylor Polynomials and a specified error range
- Recognize standard formulas for basic MacLaurin series
- Manipulate the standard series to find MacLaurin series for other functions
- Appropriately use error terms for alternating and non-alternating Taylor series

Ex. Find the Taylor series for $f(x) = \sin(x)$ at $x=0$.

Common MacLaurin Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, x \in \Re$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, x \in \Re$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, x \in \Re$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, |x|<1$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, |x|<1$$

$$f(x) = \frac{\sin(3x)}{x}$$

$$g(x) = \frac{2x}{3+x}$$

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Taylor Series at $x=0$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

Taylor Series at $\underline{x=a}$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ex. Compute the Taylor Series expansion of $f(x)=\frac{1}{x}$ at $x=2$

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Example 1:

Find the third degree Taylor polynomial of the function

$$f(x) = \sqrt{x}$$

in powers of x-1.

Taylor POLYNOMIALS

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

at $x=0$

at $x=a$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

"Like a Taylor series, but not infinite!"

Example 2:

Find the maximum error when $\sqrt{1.5}$ is approximated using a 3rd degree Taylor polynomial to the function $f(x) = \sqrt{2-x}$.

Taylor Remainder Term

The remainder term for P_n, where c is some number between a and x, is given by:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We can find an upper bound for the remainder using the formula:

$$|R_n(x)| \le \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$$

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$$

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

REVIEW

Find a MacLaurin series for

$$f(x) = \cos(2x)$$

1. $2\sum_k (-1)^k \frac{x^{2k}}{(2k)!}$
2. $\sum_k (-1)^k \frac{x^{2k+2}}{k!}$
3. $\sum_k (-1)^k \frac{2^k x^{2k}}{(2k)!}$
4. $\sum_k (-1)^k \frac{4^k x^{2k}}{(2k)!}$

Common MacLaurin Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, x \in \Re$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, x \in \Re$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, x \in \Re$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, |x| < 1$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, |x| < 1$$