MATH 1552

Chapter 10

sequences/series integral test

Math 1552

Section 10.2
Infinite Series

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 Test #1 (4.8, 5.1-5.6, 8.2-8.3)	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

Review Question:

Which of the following sequences converge?

(A) $\left\{\frac{2n+1}{1-3n}\right\}$ $\rightarrow$ $n = 1{,}000{,}000$: $\frac{2{,}000{,}001}{-2{,}999{,}999} \approx \frac{-2}{3}$

(B) $\{(-1)^n\}$ $\quad 1, -1, 1, -1, 1, -1, \ldots$

(C) $\left\{\frac{2^n}{n!}\right\}$ $\rightarrow$ converges to zero.

(D) $\left\{\left(1+\frac{4}{n}\right)^n\right\}$ $\rightarrow e^4$

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4! = 5\cdot 24 = 120$

Infinite Series

An *infinite series* is a *sum* of infinite terms:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

Which of these series do you think converges?

diverges. $\rightarrow$ (A) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

✓ (B) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$

✓ (C) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$

(D) None of these

Ex 1: $\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$

$= .3 + .03 + .003 + .0003 + \cdots$

$= .3333\ldots$

$= 1/3$ ✓

$= \frac{3}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right)$

$= \frac{3}{10}\left(\frac{1}{1-1/10}\right) = \frac{3}{10} * \frac{1}{9/10} = \frac{3}{\cancel{10}} \cdot \frac{\cancel{10}}{9}$

$= \frac{3}{9} = \boxed{1/3}$

$$\boxed{1 + x + x^2 + \cdots = \frac{1}{1-x}}$$

$(1-x)(1 + x + x^2 + x^3 + \cdots)$?? $= 1$

$= (1-x) + (1-x)x + (1-x)x^2 + \cdots$

$= 1 - x + x - x^2 + x^2 - x^3 + \cdots$

Geometric Series Formula

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r| < 1}$$

Ex 2: $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$

$= \frac{1}{1-\frac{2}{3}} = \frac{1}{1/3} = \boxed{3}$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$

Hint How would you deal with $\displaystyle\int_1^{\infty} \frac{1}{(x+1)(x+3)}\,dx$?

Partial fraction decomp.

$$\frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} = \frac{+1/2}{n+1} + \frac{-1/2}{n+3}$$

$$= \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

$A(n+3) + B(n+1) = 1$

$\Rightarrow n(A+B) + (3A+B) = 1$

$A+B=0 \qquad A=-B$

$3A+B=1 \qquad -3B+B=1$

$-2B=1$

$B=-1/2 \quad A=+1/2$

$$\sum_{n=1}^{N} \frac{1}{(n+1)(n+3)} = \sum_{n=1}^{N} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \frac{1}{2}\Big[\underbrace{\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right)}_{n=1} + \underbrace{\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right)}_{n=2} + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right)$$

$$+ \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cdots + \underbrace{\left(\cancel{\frac{1}{N+1}} - \frac{1}{N+3}\right)}_{n=N}$$

5/6

??

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3}\right] - \underbrace{\frac{1}{N+2} - \frac{1}{N+3}}_{N\to\infty}$$ go to zero

In the limit

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \boxed{\frac{5}{12}} - 0 - 0$$

```
clc
k=5;
N=10;
A=0;
for i=1:k
   N=10^i;
  for n=2:N
   A=A+log(n/(n+1));
  end
N
A
end
```

```
N =
   10.00
A =
   -1.70
N =
   100.00
A =
   -5.63
N =
   1000.00
A =
   -11.84
N =
   10000.00
A =
   -20.36
N =
   100000.00
A =
   -31.18
>>
```

Ex. $\displaystyle\sum_{k=2}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$$\sum_{k=2}^{N} \ln\left(\frac{n}{n+1}\right) = \sum_{k=2}^{N} \ln(n) - \ln(n+1)$$

$$= \underbrace{\ln(2) - \cancel{\ln(3)}}_{k=2} + \underbrace{\cancel{\ln(3)} - \cancel{\ln(4)}}_{k=3} + \cancel{\ln(4)} - \cancel{\ln(5)} + \cdots + \underbrace{\cancel{\ln(N)} - \ln(N+1)}_{k=N}$$

Ex.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r|<1$$

$$\sum_{n=0}^{\infty} \frac{5^{n-1}+3\cdot 2^{3n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{5^{n-1}}{9^n} + \frac{3\cdot 2^{3n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{5^{n-1}}{9^n} + 3\cdot \sum_{n=0}^{\infty} \frac{2^{3n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{5^n\cdot 5^{-1}}{9^n} + 3 \sum_{n=0}^{\infty} \frac{(2^3)^n}{9^n}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty} \left(\frac{5}{9}\right)^n + 3\cdot \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n$$

$$= \frac{1}{5}\left(\frac{1}{1-5/9}\right) + 3\left(\frac{1}{1-8/9}\right)$$

$$= \frac{1}{5}\cdot\frac{1}{4/9} + 3\cdot\frac{1}{1/9}$$

$$\boxed{27.45}$$

$$= \frac{1}{5}\cdot\frac{9}{4} + 3\cdot\frac{9}{1} = 27 + \frac{9}{20} = \boxed{\frac{549}{20}}$$

Ex.

$$\sum_{n=2}^{\infty} \frac{5^{n-1}+3\cdot 2^{3n}}{9^n} = \dots =$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r|<1$$

$$= \frac{1}{5}\sum_{n=2}^{\infty}\left(\frac{5}{9}\right)^n + 3\cdot\sum_{n=2}^{\infty}\left(\frac{8}{9}\right)^n$$

$$\sum_{n=2}^{\infty} r^n = \boxed{r^2+r^3+r^4+\cdots}$$

Opt 1

$$= \frac{1}{5}\left(\underline{\frac{1}{1-5/9}-1-\frac{5}{9}}\right) + 3\left(\frac{1}{1-8/9}-1-\frac{8}{9}\right)$$

So $1+r+\sum_{n=2}^{\infty} r^n = \frac{1}{1-r}$

So $\sum_{n=2}^{\infty} r^n = \frac{1}{1-r}-1-r$

$= r^2(1+r+r^2+\cdots)$

$= r^2\left(\frac{1}{1-r}\right)$

Opt 2

$$= \frac{1}{5}\cdot\left(\frac{5}{9}\right)^2\frac{1}{1-5/9} + 3\cdot\left(\frac{8}{9}\right)^2\left(\frac{1}{1-8/9}\right)$$

$$= \frac{1}{5}\cdot\frac{5^2}{9^2}\cdot\frac{1}{4/9} + 3\cdot\frac{8^2}{9^2}\cdot\frac{1}{1/9}$$

$$= \frac{5}{9^2}\cdot\frac{9}{4} + 3\cdot\frac{8^2}{\cancel{9^2}}\cdot\cancel{9}$$

$$= \frac{5}{28} + \frac{8}{3} = \frac{15+8^2\cdot 28}{3\cdot 28} = \boxed{\frac{1807}{84}}$$

General Geometric series Starting at other than $n=0$

$\boxed{\text{If } |r|<1}$

$$\sum_{n=\#}^{\infty} r^n = r^{\#}\left(\frac{1}{1-r}\right)$$

$$\text{or} = \left(\frac{1}{1-r}-1-r-r^2-\dots-r^{\#}\right)$$

THM: Divergence test.

If $\lim_{n\to\infty} a_n \neq 0$, then

$\sum_{n=1}^{\infty} a_n$ <u>surely diverges</u>.

Ex. $a_n = n$

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\cdots$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$$

both DIVERGE b/c terms $a_n \not\to 0$.

Ex. Determine if $a_n \to 0$ or not to see if the divergence test applies in each case.

(a) $\sum_{k=1}^{\infty} \left(1+\frac{3}{k}\right)^k$ $\qquad a_k = \left(1+\frac{3}{k}\right)^k$ converges to e^2

(b) $\sum_{k=2}^{\infty} \frac{3k}{5k-7}$ $\qquad a_k = \frac{3k}{5k-7}$ converges to $\frac{3}{5}$

(c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+6}}$ $\qquad a_k = \frac{1}{\sqrt{k^3+6}}$ converges to 0.

Q: If $\lim_{n\to\infty} a_n = 0$, then do we know for sure that $\sum_{n=1}^{\infty} a_n$ <u>converges</u>?

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3}+\frac{1}{4}}_{} + \underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\text{bigger than } \frac{1}{2}} + \underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{16}}_{\text{bigger than } \frac{1}{2}}+\cdots$$

$\frac{1}{2}$ smaller

bigger

bigger than $\frac{1}{2}$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x}\,dx$$

$$= \lim_{N\to\infty} \ln(x)\Big|_1^N = \lim_{N\to\infty} \ln(N) - \cancel{\ln(1)}$$

Which statement is always true?

If $\lim_{n\to\infty} a_n = 0$, then:

A. The series converges.
B. The sequence converges.
C. The sequence of partial sums converges.
D. The series diverges.

Ex. Find the limit of a_k as $k \to \infty$.

$$a_k = \left(1 + \frac{3}{k}\right)^k$$

Soln.

$\ln(a_k) = b_k$ analyze b_k instead.

$$\lim_{k\to\infty} \ln(a_k) = \lim_{k\to\infty} \ln\left(\left(1+\frac{3}{k}\right)^k\right)$$

$\infty * 0$

$$= \lim_{k\to\infty} k \cdot \ln\left(1+\frac{3}{k}\right)$$

L'Hop "$\frac{0}{0}$"

$$= \lim_{k\to\infty} \frac{\ln\left(1+\frac{3}{k}\right)}{1/k} = \lim_{k\to\infty} \frac{\frac{1}{1+\frac{3}{k}}\left(1+\frac{3}{k}\right)'}{-1/k^2}$$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{3}{k}} \cdot \frac{-3}{\cancel{k^2}} \cdot -\cancel{k^2} = +3$$

$$\lim_{k\to\infty} \ln(a_k) \to 3 \qquad a_k \to \boxed{e^3} \qquad \ln(\cancel{*}) = 3$$

e^3

$f(x)$ is continuous at $x = a$

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right)$$

$$= f(a)$$

EXERCISES 10.2

Section 10.2: 7, 29, 45, 49, 59, 63, 79, 95
(extra practice: 65, 67, 71, 77, 84, 94, 96)

Finding nth Partial Sums

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}\, 2^{n-1} + \cdots$

5. $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$

10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

19. $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$

20. $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$

21. $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$

22. $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23. $0.\overline{23} = 0.23\ 23\ 23\ldots$

24. $0.\overline{234} = 0.234\ 234\ 234\ldots$

25. $0.\overline{7} = 0.7777\ldots$

26. $0.\overline{d} = 0.dddd\ldots,$ where d is a digit

27. $0.0\overline{6} = 0.06666\ldots$

28. $1.\overline{414} = 1.414\ 414\ 414\ldots$

29. $1.24\overline{123} = 1.24\ 123\ 123\ 123\ldots$

30. $3.\overline{142857} = 3.142857\ 142857\ldots$

Using the nth-Term Test

In Exercises 31–38, use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31. $\sum_{n=1}^{\infty} \frac{n}{n+10}$

32. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

33. $\sum_{n=0}^{\infty} \frac{1}{n+4}$

34. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

35. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$

36. $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

37. $\sum_{n=1}^{\infty} \ln\frac{1}{n}$

38. $\sum_{n=0}^{\infty} \cos n\pi$

Telescoping Series

In Exercises 39–44, find a formula for the nth partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

39. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

40. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

41. $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

43. $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$

44. $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

Find the sum of each series in Exercises 45–52.

45. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

46. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

47. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

48. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

49. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

50. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$

51. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

52. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

69. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

70. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

71. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

72. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

73. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3}\right)$

74. $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

75. $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$

76. $\sum_{n=0}^{\infty} (\ln(4e^n - 1) - \ln(2e^n + 1))$

Geometric Series with a Variable x

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

77. $\sum_{n=0}^{\infty} (-1)^n x^n$

78. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

79. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

80. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

In Exercises 81–86, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

81. $\sum_{n=0}^{\infty} 2^n x^n$

82. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

83. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

84. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

Convergence or Divergence

Which series in Exercises 53–76 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

53. $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

54. $\sum_{n=0}^{\infty}(\sqrt{2})^n$

55. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$

56. $\sum_{n=1}^{\infty}(-1)^{n+1}n$

57. $\sum_{n=0}^{\infty}\cos\left(\frac{n\pi}{2}\right)$

58. $\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

59. $\sum_{n=0}^{\infty}e^{-2n}$

60. $\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$

61. $\sum_{n=1}^{\infty}\frac{2}{10^n}$

62. $\sum_{n=0}^{\infty}\frac{1}{x^n},\quad |x|>1$

63. $\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$

64. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$

65. $\sum_{n=0}^{\infty}\frac{n!}{1000^n}$

66. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

67. $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$

68. $\sum_{n=1}^{\infty}\frac{2^n+4^n}{3^n+4^n}$

85. $\sum_{n=0}^{\infty}\sin^n x$

86. $\sum_{n=0}^{\infty}(\ln x)^n$

Theory and Examples

87. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\quad\text{and}\quad\sum_{n=-1}^{\infty}\frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with **(a)** $n=-2$, **(b)** $n=0$, **(c)** $n=5$.

88. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty}\frac{5}{n(n+1)}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with **(a)** $n=-1$, **(b)** $n=3$, **(c)** $n=20$.

89. Make up an infinite series of nonzero terms whose sum is

a. 1 **b.** -3 **c.** 0.

90. (*Continuation of Exercise 89.*) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

91. Show by example that $\Sigma(a_n/b_n)$ may diverge even though Σa_n and Σb_n converge and no b_n equals 0.

92. Find convergent geometric series $A=\Sigma a_n$ and $B=\Sigma b_n$ that illustrate the fact that $\Sigma a_n b_n$ may converge without being equal to AB.

93. Show by example that $\Sigma(a_n/b_n)$ may converge to something other than A/B even when $A=\Sigma a_n$, $B=\Sigma b_n\neq 0$, and no b_n equals 0.

94. If Σa_n converges and $a_n>0$ for all n, can anything be said about $\Sigma(1/a_n)$? Give reasons for your answer.

95. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

96. If Σa_n converges and Σb_n diverges, can anything be said about their term-by-term sum $\Sigma(a_n+b_n)$? Give reasons for your answer.

97. Make up a geometric series Σar^{n-1} that converges to the number 5 if

a. $a=2$ **b.** $a=13/2$.

98. Find the value of b for which

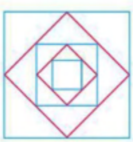
$$1+e^b+e^{2b}+e^{3b}+\cdots=9.$$

99. For what values of r does the infinite series

$$1+2r+r^2+2r^3+r^4+2r^5+r^6+\cdots$$

converge? Find the sum of the series when it converges.

100. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



101. **Drug dosage** A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k=0.12$.

a. How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?

b. In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?

102. Show that the error $(L-s_n)$ obtained by replacing a convergent geometric series with one of its partial sums s_n is $ar^n/(1-r)$.

103. **The Cantor set** To construct this set, we begin with the closed interval $[0,1]$. From that interval, remove the middle open interval $(1/3, 2/3)$, leaving the two closed intervals $[0,1/3]$ and $[2/3,1]$. At the second step we remove the open middle third interval from each of those remaining. From $[0,1/3]$ we remove the open interval $(1/9, 2/9)$, and from $[2/3,1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0,1/9]$, $[2/9,1/3]$, $[2/3,7/9]$, and $[8/9,1]$. At the next step, we remove the middle open third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0,1/9]$, leaving the closed intervals $[0,1/27]$ and $[2/27,1/9]$; $(7/27, 8/27)$ is removed from $[2/9,1/3]$, leaving behind $[2/9,7/27]$ and $[8/27,1/3]$, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval $[0,1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

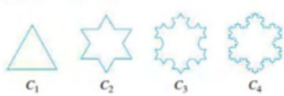
a. The Cantor set contains infinitely many numbers in $[0,1]$. List 12 numbers that belong to the Cantor set.

b. Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from $[0,1]$ is equal to 1.

104. **Helga von Koch's snowflake curve** Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

a. Find the length L_n of the nth curve C_n and show that $\lim_{n\to\infty}L_n=\infty$.

b. Find the area A_n of the region enclosed by C_n and show that $\lim_{n\to\infty}A_n=(8/5)A_1$.



105. The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semicircles of diminishing size. What is the sum of the areas of all of the circles?

Math 1552

Section 10.3

The Integral test

$$\int_1^\infty f(x)\,dx \quad \text{vs.} \quad \sum_{k=1}^{\infty} f(k)$$

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 Test #1 (4.8, 5.1-5.6, 8.2-8.3)	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 NO CLASS Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

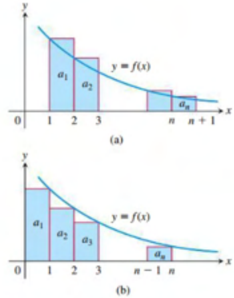


FIGURE 10.12 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Thm:

Integral TEST

IF $y=f(x)$ is positive and decreasing for all $x \geq 1$, THEN

The series

$$\sum_{k=1}^{\infty} f(k) \text{ converges}$$

If and only if The integral

$$\int_1^{\infty} f(x)\,dx \text{ converges}$$

So For example

$\int_1^{\infty} \frac{1}{x^2}\,dx$ tells us about the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$

AND $\int_1^{\infty} \frac{1}{x}\,dx$ tells us about $\sum_{k=1}^{\infty} \frac{1}{k}$ (etc.)

Ex. Use the integral test to determine if (a) $\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2}}$ $\boxed{\text{converges}}$ or diverges.

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$\underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{\text{improper integral}} = \underbrace{\lim_{N\to\infty} \int_1^{N} \frac{1}{x^2}\,dx}_{\text{is a limit of definite integrals}}$$

$$= \lim_{N\to\infty} -\frac{1}{x}\Big|_1^N = \lim_{N\to\infty} -\frac{1}{N} - \frac{-1}{1}$$

$$= 0 + 1 = 1$$

Since the improper integral CONVERGES. So too the series converges

(b) $\sum_{k=1}^{\infty} \frac{1}{k}$ The Harmonic series

(6) $\sum_{k=1}^{\infty} \frac{1}{k}$ does it converge or diverge?

$\boxed{10^{80}}$

$$\int_1^N \frac{1}{x}\,dx = \ln x \Big|_1^N$$

$$= \ln N - \cancel{\ln(1)}^{0}$$

$\ln(10{,}000) \approx 9.2103$

$e^{9.2103} \approx 10{,}000$

$\ln(e^{100{,}000}) = 100{,}00$

So the improper integral $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N \to \infty} \ln N = +\infty$ DNE

Since the improper integral DIVERGES so too the series DIVERGES by the integral test.

Ex

$\sum r^n$ geometric

$\downarrow$ is a number.

$$\boxed{\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}} \quad \boxed{\text{diverges}}$$

$$\boxed{\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = \boxed{2}$$

$$\int_2^N \frac{1}{x\ln x}\,dx = \int_*^* \frac{1}{u}\,du$$

lim $N\to\infty$ later

u-sub box

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$= \ln u \Big|_*^*$$

$$= \ln(\ln x)\Big|_2^N$$

$$= \ln(\ln N) - \ln(\ln 2)$$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$ $\quad \sum \frac{1}{k}$ diverges

$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$ $\quad \sum \frac{1}{k^2}$ converges

$\sum \frac{1}{k\ln k}$ diverges

$\sum \frac{1}{k^2+1}$ converges

$$\lim_{N\to\infty} \int_2^N \frac{1}{x\ln x}\,dx = \lim_{N\to\infty} \ln(\ln N) - \ln(\ln 2) = \infty \text{ DNE}$$

Ex.

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2+1}} \quad \boxed{\text{Converges}}$$

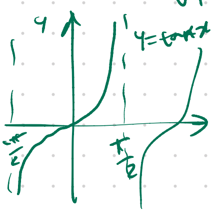
$$\boxed{\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta\,d\theta \end{aligned}}$$ trig sub.

integrate this.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{N\to\infty} \boxed{\int_1^N \frac{1}{x^2+1}\,dx}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$$

$$\theta = \pi/4 = 45^\circ$$



$$\int_1^N \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_1^N$$

$$= \tan^{-1}(N) - \tan^{-1}(1)$$

$$= \tan^{-1}(N) - \frac{\pi}{4}$$

So $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx$

$$= \lim_{N\to\infty} \tan^{-1}(N) - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Ex.

$$\boxed{\sum_{k=1}^{\infty} k e^{-k^2}} \quad \boxed{\text{Converges}}$$

$$1e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16} + \cdots$$

$$= \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} + \cdots$$

First integrate then take limit as $N\to\infty$

$$\int_1^N x e^{-x^2}\,dx = -\frac{1}{2}\int_*^* e^u\,du = -\frac{1}{2}e^u\Big|_*^* = -\frac{1}{2}e^{-x^2}\Big|_1^N$$

u-sub

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \\ -\tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$$= -\frac{1}{2}e^{-N^2} - -\frac{1}{2}e^{-1} = \frac{1}{2e} - \frac{1}{2e^{N^2}}$$

$$\text{So } \int_1^{\infty} x e^{-x^2}\,dx = \lim_{N\to\infty} \frac{1}{2e} - \frac{1}{2e^{N^2}} = \boxed{\frac{1}{2e}}$$

0

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

$\sum \frac{1}{n^{5/2}}$

$p = 5/2$

$\sum \frac{1}{\sqrt{n}}$ vs. $\sum \frac{1}{n}$ vs $\sum \frac{1}{n^2}$

$p = 1/2$ $\quad$ $p = 1$ $\quad$ $p = 2$

diverges $\quad$ converges

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

$$\int_1^N \frac{1}{x^p}\, dx = \int_1^N x^{-p}\, dx = \left. \frac{x^{-p+1}}{-p+1} \right|_1^N$$

$$= \frac{1}{-p+1} \left. \boxed{x^{-p+1}} \right|_1^N$$

If $p > 1$ then x^{-p+1} is "in the denominator"

$p \leq 1$ then x^{-p+1} is "in the numerator"

$$\sum \frac{1}{n^3+n^5} \overset{?}{\leq} \sum \frac{1}{n^3+n^3} = \sum \frac{1}{2n^3}$$

$n^5 \geq n^3$ ✓ $(n \geq 1)$

? $n^5 + n^3 \geq n^3 + n^3$

$$\frac{1}{n^5+n^3} \leq \frac{1}{n^3+n^3}$$

Section 10.3: 3, 6, 9 (extra practice: 21, 25, 26, 27, 37, 43, 51)

EXERCISES 10.3

Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–12 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

4. $\sum_{n=1}^{\infty} \frac{1}{n+4}$

5. $\sum_{n=1}^{\infty} e^{-2n}$

6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

8. $\sum_{n=2}^{\infty} \frac{\ln (n^2)}{n}$

9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

10. $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

11. $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$

12. $\sum_{n=2}^{\infty} \frac{1}{5n+10\sqrt{n}}$

Determining Convergence or Divergence

Which of the series in Exercises 13–46 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

13. $\sum_{n=1}^{\infty} \frac{1}{10^n}$ geo converge

14. $\sum_{n=1}^{\infty} e^{-n}$ geo converge

15. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by div test.

16. $\sum_{n=1}^{\infty} \frac{5}{n+1}$ integral test diverge

17. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ p=1/2 diverges

18. $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$ p=1.5 converges

19. $\sum_{n=1}^{\infty} -\frac{1}{8^n}$ geo. converges.

20. $\sum_{n=1}^{\infty} \frac{-8}{n}$ p=1 div.

21. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ integral test. diverge.

22. ~~$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$~~

23. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ geo. conv.

24. $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

25. $\sum_{n=0}^{\infty} \frac{-2}{n+1}$

26. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

27. $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ divergence test diverges.

28. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

29. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

30. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

31. $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

32. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

33. $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$

34. $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

35. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

36. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

37. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

38. $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$

39. $\sum_{n=1}^{\infty} \frac{e^n}{10+e^n}$

40. $\sum_{n=1}^{\infty} \frac{e^n}{(10+e^n)^2}$

41. $\sum_{n=2}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}\sqrt{n+2}}$

42. $\sum_{n=3}^{\infty} \frac{7}{\sqrt{n+1}\ln\sqrt{n+1}}$

43. $\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$

44. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

45. $\sum_{n=1}^{\infty} \operatorname{sech} n$

46. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum s_n be today, assuming a 365-day year?

50. Are there any values of x for which $\sum_{n=1}^{\infty}(1/nx)$ converges? Give reasons for your answer.

51. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every n? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

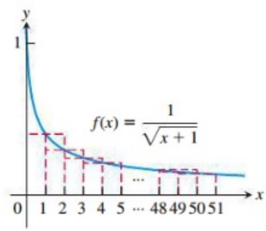
52. (*Continuation of Exercise 51.*) Is there a "largest" convergent series of positive numbers? Explain.

53. **$\sum_{n=1}^{\infty}\left(1/\sqrt{n+1}\right)$ diverges**

   a. Use the accompanying graph to show that the partial sum $s_{50} = \sum_{n=1}^{50}\left(1/\sqrt{n+1}\right)$ satisfies

$$\int_1^{51} \frac{1}{\sqrt{x+1}}\,dx < s_{50} < \int_0^{50} \frac{1}{\sqrt{x+1}}\,dx.$$

   Conclude that $11.5 < s_{50} < 12.3$.



   b. What should n be in order that the partial sum

$$s_n = \sum_{i=1}^{n}\left(1/\sqrt{i+1}\right) \text{ satisfy } s_n > 1000?$$

54. **$\sum_{n=1}^{\infty}(1/n^4)$ converges**

   a. Use the accompanying graph to find an upper bound for the error if $s_{30} = \sum_{n=1}^{30}(1/n^4)$ is used to estimate the value of $\sum_{n=1}^{\infty}(1/n^4)$.

Does the series converge or diverge.

Justify fully for full credit.

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n \qquad \sum \frac{1}{n^p}$$

Soln. ~~Using the divergence test.~~

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

↑ Formula from class.

So $\sum a_n$ diverges b/c $a_n \not\to 0$.

by the divergence test.

Converge.

$$\boxed{\sum_{n=1}^{\infty} e^{-n}} = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \left(\frac{1}{e}\right)\left(\frac{1}{1-1/e}\right) = \frac{1}{e} \frac{1}{\frac{e}{e}-\frac{1}{e}} = \frac{1}{e} \cdot \frac{1}{(e-1)/e}$$

$$= \frac{1}{e} \cdot \frac{e}{e-1}$$

$$= \boxed{\frac{1}{e-1}}$$