WEEK 7

Direct & Limit Comparison

Ratio test, root test

Alternating series test

Math 1552: Sections 10.3, 10.4, 10.5

Convergence Tests
for Infinite Series

7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series
8	Jul 3 **NO CLASS** Independence Day	Jul 4 **NO CLASS** Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 Test #3 (10.4-10.9)	Jul 21 Section 6.1: Volumes by Disks

EXAM 2

on Thursday

in Studio/QUP 24hr.

OR GCC 12:30 - 3:30

THEOREM 10—Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

THEOREM 11—Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EX. Compare $a_n = \frac{1}{1+2^k}$ with something bigger b_n such that $\sum b_n$ converges.

$$\sum_{k=1}^{\infty} \frac{1}{1+2^k} \le 1$$

Notice:

$$a_n = \frac{1}{1+2^k} \le \frac{1}{2^k} = b_n$$

$$\frac{3}{5} \le \frac{4}{5}$$

$$\frac{3}{5} \le \frac{3}{4}$$

AND $\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \boxed{1}$

↑ Converges by Geometric series w/ $r = \frac{1}{2}$, $|r| < 1$.

So $\sum a_n \le \sum b_n < \infty$

so $\boxed{\sum a_n \text{ converges}}$ by direct comparison w/ $\sum b_n$.

Ex. $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1}$ ← guess diverge.

$\int_2^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \le 1$.

Try to compare $\sum \frac{1}{\sqrt{k}-1}$ to $\sum \frac{1}{\sqrt{k}}$

$$a_n = \frac{1}{\sqrt{k}-1} \overset{?}{\ge} b_n = \frac{1}{\sqrt{k}}$$

$$\sqrt{k} \overset{?}{\ge} \sqrt{k}-1$$

$$\Rightarrow \frac{1}{\sqrt{k}} \le \frac{1}{\sqrt{k}-1}$$

Since $\sum b_n$ diverges by p-series test w/ $p = 1/2$, $(p < 1)$

AND $\sum a_n \ge \sum b_n$, by direct comparison

$\boxed{\sum a_n \text{ also diverges.}}$

Ex. Determine if the series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ vs. (b) $\sum_{n=1}^{\infty} \frac{5}{5n+1}$

② comparison

Harmonic series.

analyze ③ $\sum b_n$ convergence

$a_n = \frac{5}{5n-1} \geq b_n = \frac{1}{n}$

$\sum b_n = \sum \frac{1}{n}$ p-series w/ $p=1$. $(p \leq 1)$ diverges.

$\frac{5}{5n-1} \overset{?}{\geq} \frac{5}{5n} = \frac{1}{n}$ true ✓

So $\boxed{\sum a_n \text{ diverges}}$ b/c $\sum a_n \geq \sum b_n$ & $\sum b_n$ diverges.

① answer – ④ justification

(b) $a_n = \frac{5}{5n+1} \overset{?}{\geq} b_n = \frac{1}{n}$

Check $\frac{5}{5n+1} \leq \frac{5}{5n}$ ✓

~~Since $\sum a_n \leq \sum b_n$ and $\sum b_n$ diverges, so too must $\boxed{\sum a_n \text{ diverge}}$~~ ??? doesn't work.

$\frac{5}{5n+1} \overset{?}{\geq} \frac{5}{5n+n} = \frac{5}{6n} = b_n$

$\sum b_n = \sum \frac{5}{6n} = \frac{5}{6} \cdot \sum \frac{1}{n}$ diverges by p-series w/ $p=1$, $(p \leq 1)$.

So $\boxed{\sum a_n \text{ diverges}}$ by direct comparison w/ $\sum b_n = \sum \frac{5}{6n}$ which diverges.

THEOREM 11—Limit Comparison Test
Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

EX. Converge or diverge

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1} \geq \sum_{n=1}^{\infty} \frac{2n}{n^2+2n^2+n^2} = \sum_{n=1}^{\infty} \frac{2n}{4n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n}}_{\text{diverges}}$$

Limit Comparison w/ $b_n = \frac{2}{n}$ (or $\frac{1}{n}$)

$a_n = \frac{2n+1}{n^2+2n+1}$ $\quad b_n = \frac{2}{n}$

$a_n * \frac{1}{b_n}$

$$\frac{a_n}{b_n} = \frac{\left(2n+1 / n^2+2n+1\right)}{(2/n)} = \frac{2n+1}{n^2+2n+1} * \frac{n}{2} = \frac{2n^2+n}{2n^2+4n+2} \xrightarrow[n\to\infty]{} 1$$

So $\boxed{\sum a_n \text{ diverges}}$ by limit comparison w/ $\sum b_n$, which $\sum b_n$ diverges by p-series w/ $p=1$ ($p \leq 1$)

EX.

$$\sum \frac{2n+1}{n^3+2n+1}$$

$\boxed{\sum a_n \text{ converges}}$ by limit comparison w/ $\sum b_n$

Compare $a_n = \frac{2n+1}{n^3+2n+1}$ w/ $b_n = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges

$p=2$
$(p>1)$

limit comparison

$$\frac{a_n}{b_n} = a_n * \frac{1}{b_n} = \frac{2n+1}{n^3+2n+1} \cdot \frac{n^2}{1}$$

$$= \frac{2n^3+n^2}{n^3+2n+1} \xrightarrow[n\to\infty]{} 2$$

Ex. does the series converge or diverge.

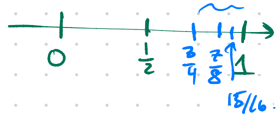
$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\left(\frac{1}{1-1/2}\right) = 1$$

compare to

$$1 = \underbrace{\underbrace{\underbrace{\frac{1}{2} + \frac{1}{4}}_{} + \frac{1}{8}}_{7/8} + \frac{1}{16}}_{15/16} + \frac{1}{32} + \cdots$$

$\Sigma b_n = \Sigma (\frac{1}{2})^n$ converges by Geometric series test w/ $r = \frac{1}{2}$, $|r| < 1$.



$$z^n = \left(e^{\ln(z)}\right)^n = e^{\ln z \cdot n}$$

$$(z^n)' = \ln z \cdot e^{\ln z \cdot n}$$

$$= \ln z \cdot z^n \checkmark$$

Try Limit comparison of Σa_n w/ Σb_n.

$$a_n = \frac{1}{2^n - 1}, \quad b_n = \frac{1}{2^n} \qquad \frac{a_n}{b_n} = \frac{1}{2^n - 1} * 2^n = \frac{2^n}{2^n - 1}$$

$$\lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{(2^n)'}{(2^n - 1)'} = \lim_{n\to\infty} \frac{\cancel{\ln(2)}\, \cancel{2^n}}{\cancel{\ln 2}\, \cancel{2^n} - 0}$$

$$= \lim_{n\to\infty} 1 = 1 = L$$

$$0 < L < \infty \checkmark$$

So by limit comparison $\boxed{\Sigma a_n \text{ also converges.}}$

Ex. Converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}} \quad \text{compare w/} \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

converges by p-series w/ $p = 3/2 > 1$.

Direct comparison

$$a_n = \frac{\cos^2 n}{n^{3/2}} \overset{?}{\leq} \frac{1}{n^{3/2}} = b_n$$

$-1 \leq \cos n \leq 1$

so $0 \leq \cos^2 n \leq 1$.

Since $\Sigma a_n \leq \Sigma b_n$ & Σb_n converges so too does $\boxed{\Sigma a_n \text{ converge}}$

Section 10.4: 5, 9, 17, 21, 25, 34, 41
(extra practice: 13, 18, 23, 31, 39, 51, 58)

EXERCISES 10.4

Direct Comparison Test

In Exercises 1–8, use the Direct Comparison Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$
2. $\sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2}$
3. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$
4. $\sum_{n=2}^{\infty} \frac{n+2}{n^2 - n}$
5. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$
6. $\sum_{n=1}^{\infty} \frac{1}{n3^n}$
7. $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4 + 4}}$
8. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}}$

Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty} (1/n^2)$)
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2 + 2}}$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty} (1/\sqrt{n})$)
11. $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2 + 1)(n-1)}$
12. $\sum_{n=1}^{\infty} \frac{2^n}{3 + 4^n}$
13. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\, 4^n}$
14. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$
15. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
   (*Hint:* Limit Comparison with $\sum_{n=2}^{\infty} (1/n)$)
16. $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty} (1/n^2)$)

Determining Convergence or Divergence

Which of the series in Exercises 17–56 converge, and which diverge? Use any method, and give reasons for your answers.

17. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$
18. $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$
19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$
20. $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$
21. $\sum_{n=1}^{\infty} \frac{2n}{3n - 1}$
22. $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$
23. $\sum_{n=1}^{\infty} \frac{10n + 1}{n(n+1)(n+2)}$
24. $\sum_{n=3}^{\infty} \frac{5n^3 - 3n}{n^2(n-2)(n^2 + 5)}$
25. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$
26. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2}}$
27. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$
28. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$
29. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n}$
30. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$
31. $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$
32. $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$
33. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$
34. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$
35. $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$
36. $\sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n}$
37. $\sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 1}$
38. $\sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^n}$
39. $\sum_{n=1}^{\infty} \frac{n+1}{n^2 + 3n} \cdot \frac{1}{5n}$
40. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$
41. $\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n}$
42. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$
43. $\sum_{n=2}^{\infty} \frac{1}{n!}$
   (*Hint:* First show that $(1/n!) \leq (1/n(n-1))$ for $n \geq 2$.)
44. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$
45. $\sum_{n=1}^{\infty} \sin \frac{1}{n}$
46. $\sum_{n=1}^{\infty} \tan \frac{1}{n}$
47. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$
48. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$
49. $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$
50. $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$
51. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$
52. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$
53. $\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n}$
54. $\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}$
55. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$
56. $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$

Theory and Examples

57. Prove **(a)** Part 2 and **(b)** Part 3 of the Limit Comparison Test.
58. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} (a_n/n)$? Explain.
59. Suppose that $a_n > 0$ and $b_n > 0$ for $n \geq N$ (N an integer). If $\lim_{n\to\infty} (a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.
60. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.
61. Suppose that $a_n > 0$ and $\lim_{n\to\infty} a_n = \infty$. Prove that $\sum a_n$ diverges.
62. Suppose that $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.

10.5 Absolute Convergence; The Ratio and Root Tests

7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 **Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)**	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series
8	Jul 3 **NO CLASS** Independence Day	Jul 4 **NO CLASS** Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	Jul 6 WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 **Test #3 (10.4-10.9)**	Jul 21 Section 6.1: Volumes by Disks

Exam 2
Tomorrow
Thurs.

Recap of last class:

- *Divergence test*: if the limit is not 0, the series diverges
- *Comparison test*: find a bigger series that converges or a smaller series that diverges
- *Integral test*: use with a function that has an "easy" antiderivative

DROP DEADLINE is SAT

Recap of last class:

- *Limit Comparison test*: pick a series that you know converges or diverges.

If the limit of the ratio of terms in your series to the given series approaches a finite, positive number, then both series either converge or diverge.

$\boxed{L=0}$

Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $\boxed{L} = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ $\qquad \frac{a_n}{a_{n+1}} \to \frac{1}{L}$

(a) If $L<1$, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $L>1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L=1$, then the test is *INCONCLUSIVE!!!*

Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $L = \lim_{k\to\infty} \sqrt[k]{a_k}$ $= \lim_{n\to\infty} (a_n)^{1/n}$

(a) If $L<1$, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $L>1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L=1$, then the test is *INCONCLUSIVE!!!*

$$\sum_{k=1}^{\infty} \frac{3^k}{k^2}$$

$$\sum_{k=1}^{\infty} \left(1-\frac{2}{k}\right)^{k^2}$$

$$\sum_{k=1}^{\infty} \frac{k\cdot 3^k}{(2k)!}$$

Topics on Exam 2

* trig sub
* partial fractions
* L'Hop
* Improper integrals
* sequences
* Convergence tests for series
  - ↳ integral test
  - ↳ geometric series
  - ↳ telescoping series
* Divergence test

Value of the series

Ex.

$$\sum_{k=1}^{\infty} \frac{3^k}{k^2}$$

If $\lim a_n \neq 0$ then $\sum a_n$ diverges ← the divergence test

$\boxed{\sum \frac{3^k}{k^2}}$

Ratio test

$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}/(k+1)^2}{3^k/k^2} = \frac{k^2}{3^k}\cdot\frac{3^{k+1}}{(k+1)^2} = \frac{k^2\cdot 3\cdot \cancel{3^k}}{\cancel{3^k}\cdot(k+1)^2}$$

$$= 3\cdot\frac{k^2}{(k+1)^2} = 3\cdot\frac{k^2}{k^2+2k+1} \xrightarrow[k\to\infty]{} 3\cdot 1 = 3$$

Since $L = 3 > 1$, by the RATIO TEST, the series $\boxed{\sum a_n \text{ diverges}}$

$\frac{3^k}{k^2} \approx$ BIG when $k \gg 0$

$$\lim_{k\to\infty}\frac{3^k}{k^2} = \lim_{k\to\infty}\frac{\ln 3\cdot 3^k}{2k} = \lim_{k\to\infty}\frac{(\ln 3)^2\, 3^x}{2} = \text{too DNE}$$

L'Hop

$$(3^x)' = \ln 3\cdot 3^x$$

So by divergence test $\boxed{\sum a_n \text{ diverges}}$

← option 1 | option 2 →

Ex.

$$\sum_{k=1}^{\infty}\frac{k\cdot 3^k}{(2k)!}$$

$$\frac{5!}{3!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}$$

$$a_k = \frac{k\cdot 3^k}{(2k)!}$$

$$\frac{(2k)!}{(2k+2)!} = \frac{\cancel{2k}\cdot\cancel{(2k-1)}\cancel{(2k-2)}\cdots\cancel{3\cdot 2\cdot 1}}{(2k+2)(2k+1)\cancel{2k}\cancel{(2k-1)}\cancel{(2k-2)}\cdots\cancel{3\cdot 2\cdot 1}} = \frac{1}{(2k+2)(2k+1)}$$

$$a_{k+1} = \frac{(k+1)\,3^{k+1}}{(2(k+1))!}$$

FALSE

$$\cancel{\frac{a_{k+1}}{a_k} \boxed{=} \frac{(2k+2)!}{(k+1)\,3^{k+1}} * \frac{k\cdot 3^k}{(2k)!} = \frac{3^k}{3^{k+1}}\cdot\frac{k}{k+1}\cdot\frac{(2k+2)!}{(2k)!}}$$

$$\cancel{\cancel{a_{k+1}}\,\frac{1}{a_k} * \cancel{a_k}\, a_{k+1} = \frac{1}{3}\cdot\frac{k}{k+1}}$$

So $\frac{a_{k+1}}{a_k} \to 3\cdot 1\cdot 0 = \underline{0}$

$$\frac{a_{k+1}}{a_k} = \frac{(2k)!}{k\cdot 3^k}\cdot\frac{(k+1)\,3^{k+1}}{(2k+2)!} = \frac{3^{k+1}}{3^k}\cdot\frac{k+1}{k}\cdot\frac{(2k)!}{(2k+2)!} = 3\cdot\frac{k+1}{k}\cdot\frac{1}{(2k+2)(2k+1)}$$

Ex.

$$\sum_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{k^2}$$

Compare to ?

$$\left(1+\frac{2}{k}\right)^{k^2} \geq \left(1+\frac{2}{k}\right)^{k} \geq \left(1+\frac{2}{k}\right)$$

$$\boxed{\sum\left(1+\frac{2}{k}\right)}$$

$$a_k = \left(1+\frac{2}{k}\right)^{k^2}$$

$$(a_k)^{1/k} = \left[\left(1+\frac{2}{k}\right)^{k^2}\right]^{1/k}$$

$$= \left(1+\frac{2}{k}\right)^{k^2/k} = \left(1+\frac{2}{k}\right)^{k} \xrightarrow[k\to\infty]{} e^2 = L > 1$$

So by root test $\boxed{\sum a_n \text{ diverges}}$

Ex. Use Root test.

$$\sum_{n=1}^{\infty}\frac{4^n}{(3n)^n}$$

$\lim_{n\to\infty}(a_n)^{1/n} = L$

$L<1$ then $\sum a_n$ converges

$L>1$ then $\sum a_n$ diverges

$L=1$ INCONCLUSIVE.

$$\int_1^{\infty}\frac{4^x}{(3x)^x}\,dx$$

$$a_n = \frac{4^n}{(3n)^n}$$

$$(a_n)^{1/n} \stackrel{?}{=} \left(\frac{4^n}{(3n)^n}\right)^{1/n} = \frac{(4^n)^{1/n}}{((3n)^n)^{1/n}} = \frac{4}{3n} \xrightarrow[n\to\infty]{} 0 = L$$

Since $L<1$ the series $\boxed{\sum a_n \text{ converge}}$

Ex.

$$\sum_{n=1}^{\infty}\left(\frac{4n+1}{3n+1}\right)^n$$

$$a_n = \left(\frac{4n+1}{3n+1}\right)^n$$

$$(a_n)^{1/n} = \frac{4n+1}{3n+1} \longrightarrow \frac{4}{3} = L$$

Since $L>1$ the series $\boxed{\sum a_n \text{ diverges}}$

Section 10.5: 7, 13, 15, 17, 18, 31, 34, 57, 63, 67
(extra practice: 7, 17, 18, 19, 21, 25, 37, 42, 59)

EXERCISES 10.5

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges absolutely or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$
3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$
5. $\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$
6. $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$
7. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!\, 3^{2n}}$
8. $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges absolutely or diverges.

9. $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$
10. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$
11. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$
12. $\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$
13. $\sum_{n=1}^{\infty} \frac{-8}{(3+(1/n))^{2n}}$
14. $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$
15. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$

    (*Hint:* $\lim_{n\to\infty} (1 + x/n)^n = e^x$)
16. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$

Determining Convergence or Divergence

In Exercises 17–46, use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$
18. $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$
19. $\sum_{n=1}^{\infty} n!(-e)^{-n}$
20. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$
21. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$
22. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$
23. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$
24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$
25. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$
26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$
27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$
28. $\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$
29. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$
30. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$
31. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$
32. $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n}$
33. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$
34. $\sum_{n=1}^{\infty} e^{-n}(n^3)$
35. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$
36. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$
37. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$
38. $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$
39. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$
40. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$
41. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$
42. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$
43. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$
44. $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$
45. $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$
46. $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

55. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$
56. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

Convergence or Divergence

Which of the series in Exercises 57–64 converge, and which diverge? Give reasons for your answers.

57. $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$
58. $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!}$
59. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$
60. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$
61. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$
62. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$
63. $\sum_{n=1}^{\infty} \frac{1\cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$
64. $\sum_{n=1}^{\infty} \frac{1\cdot 3 \cdot \cdots \cdot (2n-1)}{[2\cdot 4 \cdot \cdots \cdot (2n)](3^n+1)}$
65. Assume that b_n is a sequence of positive numbers converging to $4/5$. Determine if the following series converge or diverge.

    a. $\sum_{n=1}^{\infty} (b_n)^{1/n}$ b. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n (b_n)$

    c. $\sum_{n=1}^{\infty} (b_n)^n$ d. $\sum_{n=1}^{\infty} \frac{1000^n}{n! + b_n}$
66. Assume that b_n is a sequence of positive numbers converging to $1/3$. Determine if the following series converge or diverge.

    a. $\sum_{n=1}^{\infty} \frac{b_{n+1}b_n}{n4^n}$ b. $\sum_{n=1}^{\infty} \frac{n^n}{n!\, b_1^2 b_2^2 \cdots b_n^2}$

Theory and Examples

67. Neither the Ratio Test nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

Ideas for Exam 2 review.

① Go over the series stuff at a high level

- ↳ integral test
- ↳ divergence test ✓ →
- ↳ geometric series ✓
- ↳ telescoping series.

Like, How to recognize when to do what.

If not Q then not P
If not P then not Q

If P then Q
If Q then P
If not P then Q
If not Q then P

Divergence test Question

[If $\lim a_n \neq 0$ then $\sum a_n$ diverges]

T/F Suppose $\sum a_n$ converges then $\lim a_n = 0$ — T

T/F Suppose $\sum a_n$ diverges then $\lim a_n \neq 0$ — F

Consider $\boxed{\sum \frac{1}{n}}$

T/F If $\lim a_n = 0$ then $\sum a_n$ diverges — F

Geometric series.

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r| \geq 1. \end{cases}$$

① (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{1}{2/3} = \frac{3}{2} = \boxed{1.5}$ w/ $r = 1/3$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n = \boxed{\text{DNE}}$ b/c $r=-1$ and $|r| \geq 1$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^{2n}} = \sum_{n=0}^{\infty} \frac{3^n}{(2^2)^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = \boxed{4}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{2^n \cdot 2}{3^n} = 2 \cdot \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

$$= 2 \cdot \frac{2}{3} \cdot \frac{1}{1-2/3} = 2 \cdot \frac{2}{3} \cdot \frac{1}{1/3}$$

$$= 2 \cdot \frac{2}{\cancel{3}} \cdot \cancel{3}$$

$$= \boxed{4}$$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots + \left(\frac{2}{3}\right)^n + \dots$$

$$= \frac{2}{3}\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^{n-1} + \dots\right)$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} \cdot \frac{1}{1-2/3}$$

$$\sum_{n=\#}^{\infty} r^n = r^{\#} \frac{1}{1-r}$$

Section 10.2: 7, 29, 45, 49, 59, 63, 79, 95
(extra practice: 65, 67, 71, 77, 84, 94, 96)

EXERCISES 10.2

Finding *n*th Partial Sums

In Exercises 1–6, find a formula for the *n*th partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$

5. $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$

10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

19. $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$

20. $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$

21. $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$

22. $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23. $0.\overline{23} = 0.23\ 23\ 23\ldots$

24. $0.\overline{234} = 0.234\ 234\ 234\ldots$

25. $0.\overline{7} = 0.7777\ldots$

26. $0.\overline{d} = 0.dddd\ldots,$ where d is a digit

27. $0.0\overline{6} = 0.06666\ldots$

28. $1.\overline{414} = 1.414\ 414\ 414\ldots$

29. $1.24\overline{123} = 1.24\ 123\ 123\ 123\ldots$

30. $3.\overline{142857} = 3.142857\ 142857\ldots$

Using the *n*th-Term Test

In Exercises 31–38, use the *n*th-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31. $\sum_{n=1}^{\infty} \frac{n}{n+10}$

32. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

33. $\sum_{n=0}^{\infty} \frac{1}{n+4}$

34. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

35. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$

36. $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

37. $\sum_{n=1}^{\infty} \ln\frac{1}{n}$

38. $\sum_{n=0}^{\infty} \cos n\pi$

Telescoping Series

In Exercises 39–44, find a formula for the *n*th partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

39. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

40. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

41. $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

43. $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$

44. $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

Find the sum of each series in Exercises 45–52.

45. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

46. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

47. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

48. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

49. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

50. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$

51. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

52. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

69. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

70. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

71. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

72. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

73. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3}\right)$

74. $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

75. $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$

76. $\sum_{n=0}^{\infty} (\ln(4e^n - 1) - \ln(2e^n + 1))$

Geometric Series with a Variable *x*

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

77. $\sum_{n=0}^{\infty} (-1)^n x^n$

78. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

79. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

80. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

In Exercises 81–86, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

81. $\sum_{n=0}^{\infty} 2^n x^n$

82. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

83. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

84. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \sqrt{n+4} - \sqrt{n+3}$$

Does diverge or converge?

$$\sum_{n=1}^{N} \sqrt{n+4} - \sqrt{n+3} = \underbrace{\cancel{\sqrt{5}} - \sqrt{4}}_{n=1} + \underbrace{\cancel{\sqrt{6}} - \cancel{\sqrt{5}}}_{n=2} + \underbrace{\cancel{\sqrt{7}} - \cancel{\sqrt{6}}}_{n=3}$$

$$+ \dots + \underbrace{\cancel{\sqrt{N+3}} - \cancel{\sqrt{N+2}}}_{n=N-1} + \underbrace{\boxed{\sqrt{N+4}} - \cancel{\sqrt{N+3}}}_{n=N}$$

$$= -\sqrt{4} + \sqrt{N+4} \xrightarrow[N\to\infty]{} \cancel{-\sqrt{4} + \infty}$$

$$\boxed{+\infty \text{ DNE}}$$

The telescoping series DIVERGES.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

vs.

(b) $\sum_{n=1}^{\infty} \sqrt{n+2} - \sqrt{n+1}$

Diverge or converge?

(a) $$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{\sqrt{2}}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right)}_{n=2}$$

$$+ \underbrace{\left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{4}}}\right)}_{n=3} + \underbrace{\left(\cancel{\frac{1}{\sqrt{4}}} - \cancel{\frac{1}{\sqrt{5}}}\right)}_{n=4}$$

$$+ \cdots$$

$$+ \underbrace{\left(\cancel{\frac{1}{\sqrt{N-1}}} - \cancel{\frac{1}{\sqrt{N}}}\right)}_{n=N-1} + \underbrace{\left(\cancel{\frac{1}{\sqrt{N}}} - \frac{1}{\sqrt{N+1}}\right)}_{n=N}$$

$$= 1 - \frac{1}{\sqrt{N+1}}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \boxed{\sqrt{n+2}-\sqrt{n+1}}$

$$= \cancel{\sqrt{3}} - \sqrt{2} + \cancel{\sqrt{4}} - \cancel{\sqrt{3}} + \cancel{\sqrt{5}} - \cancel{\sqrt{4}} + \cdots$$

$$= -\sqrt{2}$$

$$\sum_{n=1}^{N} \sqrt{n+2}-\sqrt{n+1} = \cancel{\sqrt{3}} \boxed{-\sqrt{2}} + \cancel{\sqrt{4}} - \cancel{\sqrt{3}} + \cdots$$

$$\overbrace{+ \cancel{\sqrt{N+1}} - \cancel{\sqrt{N}}}^{n=N-1} \overbrace{+ \boxed{\sqrt{N+2}} - \cancel{\sqrt{N+1}}}^{n=N}$$

$$= -\sqrt{2} + \sqrt{N+2}$$

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Section 10.6

Alternating Series

7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series
8	Jul 3 NO CLASS Independence Day	Jul 4 NO CLASS Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	Jul 6 WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 Test #3 (10.4-10.9)	Jul 21 Section 6.1: Volumes by Disks

Alternating Series Test

Let $\sum_k a_k$ be an alternating series.

(a) If $\sum_k |a_k|$ converges, then the series *converges absolutely*.

(b) If (a) fails, then if :

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty} |a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

$$f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{\sqrt{k+4}} \leftarrow \text{a } \sout{\text{number}}$$

$0! = 1$ $\quad 4! = 4\cdot3\cdot2\cdot1 = 24$

$1! = 1$ $\quad 5! = 5\cdot24 = 120$

$2! = 2\cdot1 = 2$

$3! = 3\cdot2\cdot1 = 6$

Looking ahead.

$$g(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$$

NEW! $\quad g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

!!

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

$$\int e^{x^2}\,dx = ???$$

Conditional convergence

Ex. Converge or diverge?

(a) $\boxed{\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k+4}}}$ $\boxed{\text{converges}}$

$a_k = \frac{1}{\sqrt{k+4}}$

$\lim_{k\to\infty} \frac{1}{\sqrt{k+4}} = 0.$ ✓

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{3^k}$$

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+4}}$ $\boxed{\text{diverges}}$

$\lim_{k\to\infty} (-1)^k \cdot \frac{1}{\sqrt{k+4}} \overset{?}{=} 0$

integral test

$$\int_1^{\infty} \frac{1}{\sqrt{x+4}}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{\sqrt{x+4}}\,dx$$

u-sub
$u = x+4$
$du = dx$

Since $a_k \to 0$, $\left(\begin{matrix} a_k \geq 0 \\ \text{decreasing} \end{matrix}\right)$

$\sum (-1)^k a_k$ converges.

alternating series test

$$= \lim_{N\to\infty} \int_*^* \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{N\to\infty} 2\sqrt{u}\,\Big|_*^* = \lim_{N\to\infty} 2\sqrt{x+4}\,\Big|_1^N$$

$$= \lim_{N\to\infty} 2\sqrt{N+4} - 2\sqrt{1+4}$$

$$= +\infty \text{ DNE}$$

Converge or diverge? $a_k = \frac{k}{3^k}$

(a) $\sum_{k=1}^{\infty} (-1)^k \frac{k}{3^k}$ $\boxed{\text{converges}}$

(b) $\sum_{k=1}^{\infty} \frac{k}{3^k}$ $\boxed{\text{converges}}$

by ratio test

p-series

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p>1$

diverges $p \le 1$.

ratio test

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{3^{k+1}} \cdot \frac{3^k}{k}$$

$$= \frac{k+1}{k} \cdot \frac{1}{3} \xrightarrow[k\to\infty]{} \frac{1}{3} = L$$

$L < 1 \Rightarrow \sum a_k$ converges.

$\boxed{\text{Absolute convergence}}$

$\sum a_n$ vs. $\sum (-1)^k a_k$

alternating series.

$\boxed{\text{If } a_n \geq 0, \text{ decreasing sequence, then}}$

TRUTH (a) $\sum a_n$ converges $\Longrightarrow$ $\sum (-1)^n a_n$ also converges.

(b) $\sum (-1)^n a_n$ diverge $\Rightarrow$ $\sum a_n$ also diverges.

FAKE TRUTHS

$\sum (-1)^n a_n$ converges $\not\Rightarrow$ $\sum a_n$ converges

$\sum a_n$ diverges $\not\Rightarrow$ $\sum (-1)^n a_n$ diverges.

eg. $\sum \frac{1}{\sqrt{k+4}}$ vs $\sum \frac{(-1)^k}{\sqrt{k+4}}$

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^3}{k^3 + 2k + 1}$ $\boxed{\text{diverges}}$

$a_k = \frac{k^3}{k^3+2k+1} \longrightarrow 1$

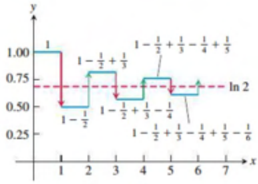
$\lim_{k\to\infty} a_k = 0$?? FALSE!

divergence test

$\boxed{\text{If } \lim a_n \neq 0 \text{ then } \sum a_n \text{ diverges.}}$

EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$



(b)

Estimating the Sum

Let $\sum_k a_k$ be a convergent alternating series with a sum of L.

Then : $|s_n - L| < |a_{n+1}|$.

Alternating harmonic series. Converge or diverge?

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

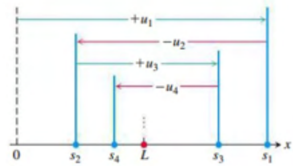
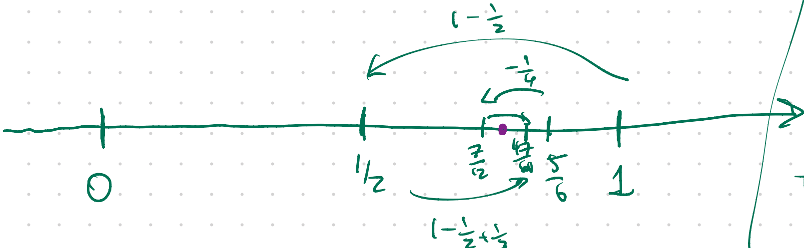


FIGURE 10.15 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

$a_n = \frac{1}{n}$, decreasing non-negative

So $\sum (-1)^n a_n$ converges by alternating series test.



$$L = \sum_{n=1}^{\infty}(-1)^n a_n$$

$$S_N = \sum_{n=1}^{N}(-1)^n a_n$$

Then $|L - S_N| \leq a_{N+1}$

Example:

Estimate the sum of the series below within an error range of 0.001.

$$L = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)!}$$

$a_k = \frac{1}{(2k+1)!} \to 0$

so alt. series test

$\sum (-1)^k a_k$ converges.

$k=1 \quad a_1 = \frac{1}{(2\cdot 1+1)!} = \frac{1}{3!} = \frac{1}{6} \overset{?}{\leq} 0.001 = \frac{1}{1{,}000}$

$$L \approx \sum_{k=0}^{N}(-1)^k\frac{1}{(2k+1)!} = S_N$$

$|L - S_N| \leq a_{N+1} \leq 0.001$

So question is what value of k makes

$$\frac{1}{(2k+1)!} \leq 0.001$$

Need.

$(2k+1)! \geq 1{,}000$

$k=2 \quad (2\cdot 2+1)! = 5! = 120$

$k=3 \quad (2\cdot 3+1)! = 7! = 5040$

So $\sum_{k=0}^{2}(-1)^k\frac{1}{(2k+1)!}$ is within 0.001 of L.

Does the alternating series converge

absolutely?, conditionally? or diverge?

#15.

$$\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$$ $\boxed{\text{absolute.}}$

Notice $a_n = (0.1)^n$ then $\sum a_n = \sum (0.1)^n$ is

a geometric series w/ $r = 0.1 = \frac{1}{10}$

& $|r| < 1$ so the series $\sum a_n$ converges.

#16 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$ So let's consider the non-alternating version FIRST.

$\boxed{\text{absolute}}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(0.1)^n}{n} \qquad \frac{(0.1)^n}{n} \leq (0.1)^n \quad \text{if } n \geq 1.$$

So $\sum a_n$ converges by direct comparison

to $\sum (0.1)^n$ (which converges by geo. series test)

Cond.? Abs? Diverges?

#19. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$

root test on $\sum_{n=1}^{\infty} \left(\frac{n}{10}\right)^n$ $\quad a_n = \left(\frac{n}{10}\right)^n$

$a_1 = \frac{1}{10}$

$a_2 = \left(\frac{2}{10}\right)^2$

$a_3 = \left(\frac{3}{10}\right)^3$

$\vdots$

$a_{10} = \left(\frac{10}{10}\right)^{10} = 1$

$a_{11} = \left(\frac{11}{10}\right)^{11} \overset{??}{>} 1$

$\lim_{n\to\infty} (a_n)^{1/n} = L$

If $L<1$ conv
$L>1$ div.
$L=1$ inconclusive

So $(a_n)^{1/n} = \left(\left(\frac{n}{10}\right)^n\right)^{1/n} = \frac{n}{10}$

$\lim_{n\to\infty} \frac{n}{10} = \infty$ DNE

Ex. #50.

$L = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$

If we add up the First Four terms, how close are we to the limit L?

$\approx (-1)^{1+1} \frac{1}{10^1} +$

$= \frac{1}{10} - \frac{1}{100} + \frac{1}{1{,}000} - \frac{1}{10{,}000} =$

$= .1 - .01 + .001 - .0001 = 0.0909$

$\frac{1}{100{,}000} = \boxed{.00001}$

Section 10.6: 6, 19, 21, 23, 25, 27, 34, 35, 37, 47
(extra practice: 20, 22, 27, 31, 91)

EXERCISES 10.6

Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$
3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n3^n}$
4. $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$
5. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$
6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+5}{n^2+4}$
7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$
8. $\sum_{n=1}^{\infty}(-1)^{n}\frac{10^n}{(n+1)!}$
9. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

root test.

10. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$
11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$
12. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$
14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute and Conditional Convergence

Which of the series in Exercises 15–48 converge absolutely, which converge, and which diverge? Give reasons for your answers.

✓15. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$
✓16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$
17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$
19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$
20. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$
21. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$
22. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$
23. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$
24. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$
25. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$
26. $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt[n]{10})$
27. $\sum_{n=1}^{\infty}(-1)^{n}n^2(2/3)^n$
28. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$
29. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\tan^{-1}n}{n^2+1}$
30. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n-\ln n}$
31. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$
32. $\sum_{n=1}^{\infty}(-5)^{-n}$
33. $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$
34. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$
35. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$
36. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$
37. $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$
38. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$

Error Estimation

In Exercises 49–52, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

49. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$
50. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$
51. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$ As you will see in Section 10.7, the sum is ln (1.01).
52. $\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n, \quad 0<t<1$

In Exercises 53–56, determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

53. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^2+3}$
54. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$
55. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n+3\sqrt{n})^3}$
56. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\ln(\ln(n+2))}$

In Exercises 57–82, use any method to determine whether the series converges or diverges. Give reasons for your answer.

57. $\sum_{n=1}^{\infty}\frac{3^n}{n^n}$
58. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$
59. $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$
60. $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)$
61. $\sum_{n=0}^{\infty}(-1)^{n}\frac{(n+2)!}{(2n)!}$
62. $\sum_{n=2}^{\infty}\frac{(3n)!}{(n!)^3}$
63. $\sum_{n=1}^{\infty}n^{-2/\sqrt{5}}$
64. $\sum_{n=2}^{\infty}\frac{3}{10+n^{4/3}}$
65. $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n^2}$
66. $\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n$
67. $\sum_{n=1}^{\infty}\frac{n-2}{n^2+3n}\left(-\frac{2}{3}\right)^n$
68. $\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n$

39. $\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{2^n n! n}$

40. $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^2 3^n}{(2n+1)!}$

41. $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2+n}-n\right)$

43. $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

44. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$

45. $\sum_{n=1}^{\infty}(-1)^n \operatorname{sech} n$

46. $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n$

47. $\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12}-\frac{1}{14}+\cdots$

48. $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}+\cdots$

69. $\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\cdots$

70. $1-\frac{1}{8}+\frac{1}{64}-\frac{1}{512}+\frac{1}{4096}-\cdots$

71. $\sum_{n=3}^{\infty}\sin\left(\frac{1}{\sqrt{n}}\right)$

72. $\sum_{n=1}^{\infty}\tan(n^{1/n})$

73. $\sum_{n=2}^{\infty}\frac{n}{\ln n}$

74. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

75. $\sum_{n=2}^{\infty}\ln\left(\frac{n+2}{n+1}\right)$

76. $\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^3$

77. $\sum_{n=2}^{\infty}\frac{1}{1+2+2^2+\cdots+2^n}$

78. $\sum_{n=2}^{\infty}\frac{1+3+3^2+\cdots+3^{n-1}}{1+2+3+\cdots+n}$

79. $\sum_{n=0}^{\infty}(-1)^n\frac{e^n}{e^n+e^{n^2}}$

80. $\sum_{n=0}^{\infty}\frac{(2n+3)(2^n+3)}{3^n+2}$

81. $\sum_{n=1}^{\infty}\frac{n^2 3^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

82. $\sum_{n=1}^{\infty}\frac{4\cdot 6\cdot 8\cdots(2n)}{5^{n+1}(n+2)!}$

T Approximate the sums in Exercises 83 and 84 with an error of magnitude less than 5×10^{-6}.

83. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$ As you will see in Section 10.9, the sum is cos 1, the cosine of 1 radian.

84. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ As you will see in Section 10.9 the sum is e^{-1}.

Theory and Examples

85. a. The series

$$\frac{1}{3}-\frac{1}{2}+\frac{1}{9}-\frac{1}{4}+\frac{1}{27}-\frac{1}{8}+\cdots+\frac{1}{3^n}-\frac{1}{2^n}+\cdots$$

does not meet one of the conditions of Theorem 14. Which one?

b. Use Theorem 17 to find the sum of the series in part (a).

90. Show that if $\sum_{n=1}^{\infty}a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty}a_n\right|\le\sum_{n=1}^{\infty}|a_n|.$$

91. Show that if $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge absolutely, then so do the following.

a. $\sum_{n=1}^{\infty}(a_n+b_n)$

b. $\sum_{n=1}^{\infty}(a_n-b_n)$

c. $\sum_{n=1}^{\infty}ka_n$ (k any number)

92. Show by example that $\sum_{n=1}^{\infty}a_nb_n$ may diverge even if $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge.

93. If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges.

94. Does the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$$

converge or diverge? Justify your answer.

T 95. In the alternating harmonic series, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new