1552

With Sal!

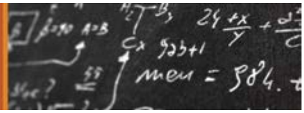
CHAPTER 5&8

* Calc 1 review
* Riemann Sums
* The Definite Integral
* Area between curves

* U-sub
* IBP
* trig integrals

Math 1552

CALCULUS 1 REVIEW



Things you probably will need to know...

Special Cases for Limits at Infinity

If the degree of the numerator is greater than that of the denominator:

$$\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right)=\pm\infty$$

If the degrees are equal:

$$\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right)=\frac{\text{leading coeff. of } P}{\text{leading coeff. of } Q}$$

If the degree of the numerator is smaller than that of the denominator:

$$\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right)=0$$

... eventually.

Example:

Evaluate the following limits.

$(a)\ \lim_{x\to\infty}\left(\frac{3+x^2}{9-5x^2}\right)$

$(b)\ \lim_{x\to\infty}\left(\frac{3x^2+5x^3}{7x^4-2x^3+4x^2}\right)$

(a) $\lim_{x\to\infty}\frac{3+x^2}{9-5x^2}$

(b) $\lim_{x\to\infty}\frac{3x^2+5x^3}{7x^4-2x^3+4x^2}$

Derivative Rules

Power Rule: $\frac{d}{dx}\left[x^n\right]=nx^{n-1}$

Ex. (a) $\frac{d}{dx}x^3 =$

(b) $\frac{d}{dx}x^{-1} =$

Example:

Differentiate the following functions:

(a) $f(x)=x^3\tan(x)$

(b) $g(x)=\frac{3\sec(x)}{2+x\cos(x)}$

(c) $h(x)=\arctan(\ln(5x))$

(a) $f'(x)=\left(x^3\tan(x)\right)'$

$= ?$

(b) $g(x)=\left(\frac{3\sec x}{2+x\cos x}\right)'$

$= ?$

(c) $h(x)=\left(\arctan(\ln(5x))\right)'$

$= ?$

Remember this?

The Chain Rule

Let $y=f(u)$ and $u=g(x)$.

Then :

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[f(g(x))\right]=\left(f'(g(x))\right)\left(g'(x)\right)$$

OR

$$\frac{d}{dx}\left[f(stuff)\right]=f'(stuff)\cdot(stuff)'$$

$[f(g(x))]' \stackrel{?}{=}$

It's all coming back... ... or not.

Critical Number/Local Extrema

The number c is a *critical number* of f if

$$f'(c)=0 \text{ or } f'(c) \text{ DNE}$$

We say the point (c, $f(c)$) is a *local extrema* if $f(c)$ is the smallest or largest value of f for all x-values "close" to c.

Increasing/Decreasing

A function f is said to be *increasing* on the interval (a,b) if $f'(x)>0$ for all $x \in (a,b)$

A function f is said to be *decreasing* on the interval (a,b) if $f'(x)<0$ for all $x \in (a,b)$

A function f is said to be *constant* on the interval (a,b) if $f'(x)=0$ for all $x \in (a,b)$

Example:

Find all extreme values for the function below.

$$f(x)=(x-1)^2(x-2)^2 \text{ on } [0,4]$$

Soln. Need to find critical values where $f'(x)=0$ & then evaluate each region of $[0,4)$ between the critical values (e.g. use a sign chart).

Ok, remembering now...

Absolute Extreme Values

Let f be continuous on $[a,b]$ and differentiable on (a,b). To find the ***absolute maximum*** and ***absolute minimum*** on $[a,b]$:

1. Find all critical numbers of f on $[a,b]$.
2. Evaluate $f(a)$, $f(b)$, and $f(c)$ for all critical numbers c.
3. The largest value in step 2 is the absolute maximum; the smallest value is the absolute minimum.

Do we need to know this...?

Some Derivative Formulas

$$\frac{d}{dx}\left[e^x\right]=e^x$$

$$\frac{d}{dx}[\sin x]=\cos x \qquad \frac{d}{dx}[\sec x]=\sec x\tan x$$

$$\frac{d}{dx}[\cos x]=-\sin x \qquad \frac{d}{dx}[\csc x]=-\csc x\cot x$$

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\cot x]=-\csc^2 x$$

Oh man, that's a lot of formulas,...

Derivatives of Inverse Functions

$$\frac{d}{dx}\left[\ln|u|\right]=\frac{1}{u}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\left[\sin^{-1}(x)\right]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[a^u\right]=a^u\ln(a)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right]=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\log_a u\right]=\frac{1}{u\ln(a)}\frac{du}{dx} \qquad \frac{d}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{1+x^2}$$

Make it stop 

Is this on the exam?

Product Rule:

$$\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$$

OR

$$\frac{d}{dx}[\text{first}\times\text{second}]$$
$$=\left(\left[\frac{d}{dx}(\text{first})\right]\times\text{second}\right)+\left(\text{first}\times\left[\frac{d}{dx}(\text{second})\right]\right)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

OR

$$\frac{d}{dx}\left[\frac{hi}{lo}\right]=\frac{lo\cdot d(hi)-hi\cdot d(lo)}{lo\cdot lo}$$

Limit Theorems

Let $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$.

Then:

$(i)\ \lim_{x\to a}\left[f(x)\pm g(x)\right]=L\pm M$

$(ii)\ \lim_{x\to a}\alpha f(x)=\alpha L$, where $\alpha\in\Re$

$(iii)\ \lim_{x\to a}\left[f(x)g(x)\right]=LM$

$(iv)\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{L}{M}, M\neq 0$

Welcome to Math 1552

Syllabus Basics

- Your grade will be determined by:
  - **Classwork Points (CP)**
    - Pre-lecture Videos
    - Online homework on MML
    - Studio Quizzes
  - **Exams (*Friday evenings*)**
    - Three 75-minute tests
    - For in-person sections exams are during studio (*QUP set up proctoring locally*)
    - Tests will be in person on June 8, June 29, July 20
  - **Final Examination (not optional)**
    - Thursday, July 28, from 11:20 am – 2:10 pm
    - Held in-person
    - Bonus points from CIOS and extra CP

Classwork Points *In-person ONLY*

Assignment	Maximum Number of Points
Pre-lecture videos	0 CP (36 videos – optional – not for a grade)
Start-of-semester survey and Syllabus Quiz	20 CP (10 points each)
Online problems on MML	37 CP (13 assignments, divide by 4 to convert MML points to CP)
Studio Quizzes	120 CP (6 quizzes, 20 points each)
Maximum total points for 100%:	130 out of 197
Extra credit	Every extra 1 CP over 130 is converted to 0.05 bonus point on the final exam (CIOS bonus an additional 5pt for a max total bonus of 8.35 bonus points on the final exam)

Grading Rubric *In-person ONLY*

Assessment	Weight
Classwork	20%
Midterm exams	55%
Final Exam*	25%

Classwork Points *QUP online ONLY*

Assignment	Maximum Number of Points
Pre-lecture videos	0 CP (36 videos – optional – not for a grade)
Start-of-semester survey, Syllabus Quiz and QUP Gradescope Quiz	30 CP (10 points each)
Online problems on MML	49 CP (13 assignments, divide by 3 to convert MML points to CP)
Studio Quizzes	120 CP (6 quizzes, 20 points each)
Maximum total points for 100%:	130 out of 199
Extra credit	Every extra 1 CP over 130 is converted to 0.05 bonus point on the final exam (CIOS bonus an additional 5pt for a max total bonus of 8.45 bonus points on the final exam)

Grading Rubric *QUP online ONLY*

Assessment	Weight
Classwork	05%
Midterm exams	70%
Final Exam*	25%

QUP section please do the Module 0 quiz in Canvas called "Proctoring Method Survey" by this Friday, May 19

Important Websites

- **Course Information:** canvas.gatech.edu
- **Textbook/Homework Access:**
  Use the "Pearson Access" tool on Canvas
- **On-line Discussions**: www.piazza.com
  (*highly recommended*)
- **Gradescope**:
  Use the "Pearson Access" tool on Canvas

Textbook: What to purchase?

- MyMathLab code is required to complete the online assignments.
  - You may sign up for temporary access for two weeks.
- IMPORTANT: Please register through **CANVAS**, not the MyMathLab site.

Important Policies

- Make-ups
  - NO MAKEUPS on studio quizzes, as there are extra points built into the classwork category
  - Contact me right away if you will miss an exam
  - One make-up policy
- Attendance
  - 1 CP per day of attendance with active participation
- Calculators/websites/phones
  - Not allowed on any quizzes or exams
  - Zero on the assignment for first offense
  - **QUP online-only** students may use calculators for quizzes but not exams

Policies (cont)

- Academic Misconduct
  - Any cases will be submitted to the Dean's office.
- Disability Services
  - Please discuss any accommodations with me.
- Regrades
  - Submit on Gradescope within one week of receiving your graded paper.
  - Indicate which rubric item was not applied correctly

Math 1552

Section 4.8: Antiderivatives

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

Find an anti-derivative for each function below.

(a) $f(x) = \sin(x) + \sqrt{x}$

(b) $g(x) = \frac{1}{4x^3} - \sec^2(x)$

(c) $h(x) = \left(x^3 - \frac{1}{x}\right)^2$

Example

(a) $f(x) = \sin(x) + \sqrt{x}$

So $F(x) = \square + \square$

↑ its derivative is $\sin x$ ↑ its derivative is $\sqrt{x}$

Antiderivatives

Definition: We say the function F is an *antiderivative* of the function f if $F'(x) = f(x)$.

Some useful formulas.

Function	***Antiderivative***
$ax^n, n \neq -1$	$a \cdot \frac{x^{n+1}}{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc^2(x)$	$-\cot(x)$
$\csc(x)\cot(x)$	$-\csc(x)$

Function	***Antiderivative***
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

Example 2:

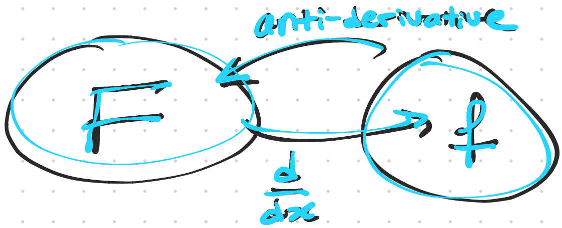
A particle travels with an acceleration, in meters per second squared, given by:

$$a(t) = t - 5t^2.$$

Find the particle's velocity and position at time t=1 second if the initial position is 2 m and the initial velocity is 10 m/s.

$\frac{d}{dx}$ position = velocity

So the anti-derivative of velocity is position



Ex $3x^2 \xrightarrow{d/dx} 6x$ (anti-derivative: $6x \to 3x^2$)

So this is why we needed to learn derivatives ...

Example 3:

Evaluate each indefinite integral.

$$(a)\int\left(e^{-5x}+\sec x(\tan x-\sec x)\right)dx$$

$$(b)\int\left(\frac{1}{\sqrt{16-x^2}}-\frac{2}{x}\right)dx$$

Derivatives of Inverse Functions

$$\frac{d}{dx}\left[\ln|u|\right]=\frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right]=a^u\ln(a)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a u\right]=\frac{1}{u\ln(a)}\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1}(x)\right]=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{1+x^2}$$

Function	***Antiderivative***
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

(a) $\int e^{-5x} + \sec x * (\tan x - \sec x)\, dx$

Wait... what?

1. $\int_1^e x\ln(x^4)\,dx$

Function	*Antiderivative*
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

Ummm...

2. $\int \sin^5(x)cos^2(x)\,dx$

Derivatives of Inverse Functions

$$\frac{d}{dx}[\ln|u|]=\frac{1}{u}\cdot\frac{du}{dx} \qquad \frac{d}{dx}[\sin^{-1}(x)]=\frac{1}{\sqrt{1-x^2}}$$

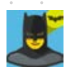
$$\frac{d}{dx}[a^u]=a^u\ln(a)\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1}(x)]=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\log_a u]=\frac{1}{u\ln(a)}\frac{du}{dx} \qquad \frac{d}{dx}[\tan^{-1}(x)]=\frac{1}{1+x^2}$$

you've got to be joking

3. $\int \frac{2}{x^2\sqrt{x^2-1}}\,dx$

not as hard as it looks

3. $\int\left(\frac{e^{\sqrt{2}}+x^{\sqrt{2}}}{\sqrt{x}}\right)dx$ Studio WS on Thursday 

4. $\int\left(\frac{1}{1+9x^2}\right)dx$

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

Find

(1) The anti-derivative

(2) The average value over the interval [1,2]

(3) The definite integral of h(x) over the interval [1,2]

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C,\quad n \neq -1$	**8.** e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.** $\frac{1}{x}$	$\ln\lvert x\rvert + C,\quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	**10.** $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.** $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.** $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.** a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

EXERCISES 4.8

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$

2. a. $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$

3. a. $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$

4. a. $2x^{-3}$ **b.** $\frac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$

5. a. $\frac{1}{x^2}$ **b.** $\frac{5}{x^2}$ **c.** $2 - \frac{5}{x^2}$

6. a. $-\frac{2}{x^3}$ **b.** $\frac{1}{2x^3}$ **c.** $x^3 - \frac{1}{x^3}$

7. a. $\frac{3}{2}\sqrt{x}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \frac{1}{\sqrt{x}}$

8. a. $\frac{4}{3}\sqrt[3]{x}$ **b.** $\frac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

9. a. $\frac{2}{3}x^{-1/3}$ **b.** $\frac{1}{3}x^{-2/3}$ **c.** $-\frac{1}{3}x^{-4/3}$

10. a. $\frac{1}{2}x^{-1/2}$ **b.** $-\frac{1}{2}x^{-3/2}$ **c.** $-\frac{3}{2}x^{-5/2}$

11. a. $\frac{1}{x}$ **b.** $\frac{7}{x}$ **c.** $1 - \frac{5}{x}$

12. a. $\frac{1}{3x}$ **b.** $\frac{2}{5x}$ **c.** $1 + \frac{4}{3x} - \frac{1}{x^2}$

13. a. $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$

14. a. $\pi \cos \pi x$ **b.** $\frac{\pi}{2}\cos\frac{\pi x}{2}$ **c.** $\cos\frac{\pi x}{2} + \pi \cos x$

15. a. $\sec^2 x$ **b.** $\frac{2}{3}\sec^2\frac{x}{3}$ **c.** $-\sec^2\frac{3x}{2}$

16. a. $\csc^2 x$ **b.** $-\frac{3}{2}\csc^2\frac{3x}{2}$ **c.** $1 - 8\csc^2 2x$

17. a. $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc\frac{\pi x}{2}\cot\frac{\pi x}{2}$

18. a. $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec\frac{\pi x}{2}\tan\frac{\pi x}{2}$

19. a. e^{3x} **b.** e^{-x} **c.** $e^{x/2}$

20. a. e^{-2x} **b.** $e^{4x/3}$ **c.** $e^{-x/5}$

21. a. 3^x **b.** 2^{-x} **c.** $\left(\frac{5}{3}\right)^x$

22. a. $x^{\sqrt{3}}$ **b.** x^{π} **c.** $x^{\sqrt{2}-1}$

23. a. $\frac{2}{\sqrt{1-x^2}}$ **b.** $\frac{1}{2(x^2+1)}$ **c.** $\frac{1}{1+4x^2}$

24. a. $x - \left(\frac{1}{2}\right)^x$ **b.** $x^2 + 2^x$ **c.** $\pi^x - x^{-1}$

Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

25. $\int (x + 1)\, dx$ **26.** $\int (5 - 6x)\, dx$

27. $\int \left(3t^2 + \frac{t}{2}\right) dt$ **28.** $\int \left(\frac{t^2}{2} + 4t^3\right) dt$

29. $\int (2x^3 - 5x + 7)\, dx$ **30.** $\int (1 - x^2 - 3x^5)\, dx$

31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$ **32.** $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

33. $\int x^{-1/3}\, dx$ **34.** $\int x^{-5/4}\, dx$

35. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$ **36.** $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$ **38.** $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

39. $\int 2x(1 - x^{-3})\, dx$ **40.** $\int x^{-3}(x + 1)\, dx$

41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$ **42.** $\int \frac{4 + \sqrt{t}}{t^3}\, dt$

43. $\int (-2 \cos t)\, dt$ **44.** $\int (-5 \sin t)\, dt$

45. $\int 7 \sin\frac{\theta}{3}\, d\theta$ **46.** $\int 3 \cos 5\theta\, d\theta$

47. $\int (-3 \csc^2 x)\, dx$ **48.** $\int \left(-\frac{\sec^2 x}{3}\right) dx$

49. $\int \frac{\csc\theta \cot\theta}{2}\, d\theta$ **50.** $\int \frac{2}{5}\sec\theta \tan\theta\, d\theta$

51. $\int (e^{3x} + 5e^{-x})\, dx$ **52.** $\int (2e^x - 3e^{-2x})\, dx$

53. $\int (e^{-x} + 4^x)\, dx$ **54.** $\int (1.3)^x\, dx$

55. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

56. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$

57. $\int (\sin 2x - \csc^2 x)\, dx$ **58.** $\int (2 \cos 2x - 3 \sin 3x)\, dx$

59. $\int \frac{1 + \cos 4t}{2}\, dt$ **60.** $\int \frac{1 - \cos 6t}{2}\, dt$

Math 1552

Sections 5.1-5.3:
Area under the Curve
The Definite Integral

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

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Day 1 Learning Goals

- Understand how to partition an interval
- Draw a picture to approximate the area under the curve with a given number of rectangles
- Compute the Upper and Lower sums
- Calculate the midpoint estimate

Riemann Sums

- Idea: Find the area bounded by a function $f(x)$, the lines $x=a$, $x=b$, and the x-axis.
- Procedure: Break the interval $[a,b]$ into n subintervals, and draw a rectangle in each subinterval.
- Summing the areas of the rectangles will approximate the area under the curve.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

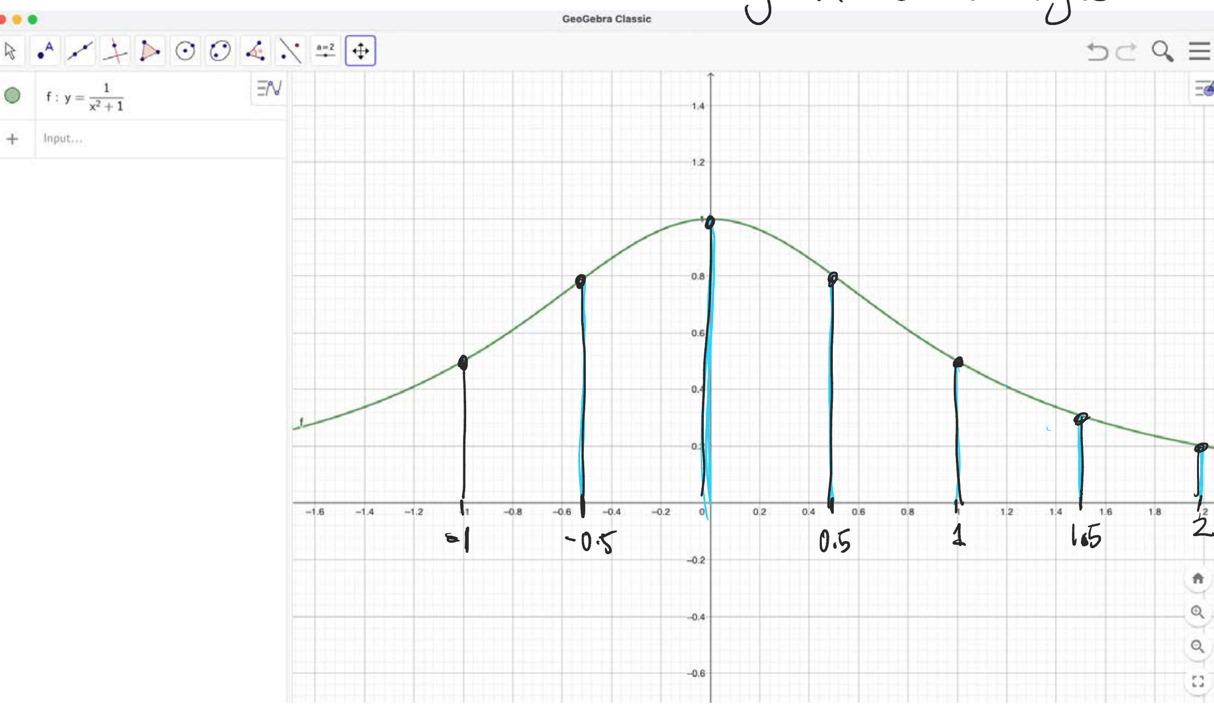
Example 1:

Find the upper and lower sums for the function

$$f(x) = \frac{1}{x^2+1}$$

on the interval [-1,2] with n=6 subintervals.

Ex $f(x) = \frac{1}{1+x^2}$ over $[-1,2]$ using $n=6$ rectangles.



Over-estimate / underestimate / left / right

Midpoint Estimate

Plug in the midpoint of each subinterval.

On the subinterval $[x_{i-1}, x_i]$,

the midpoint is: $\frac{x_{i-1}+x_i}{2}$

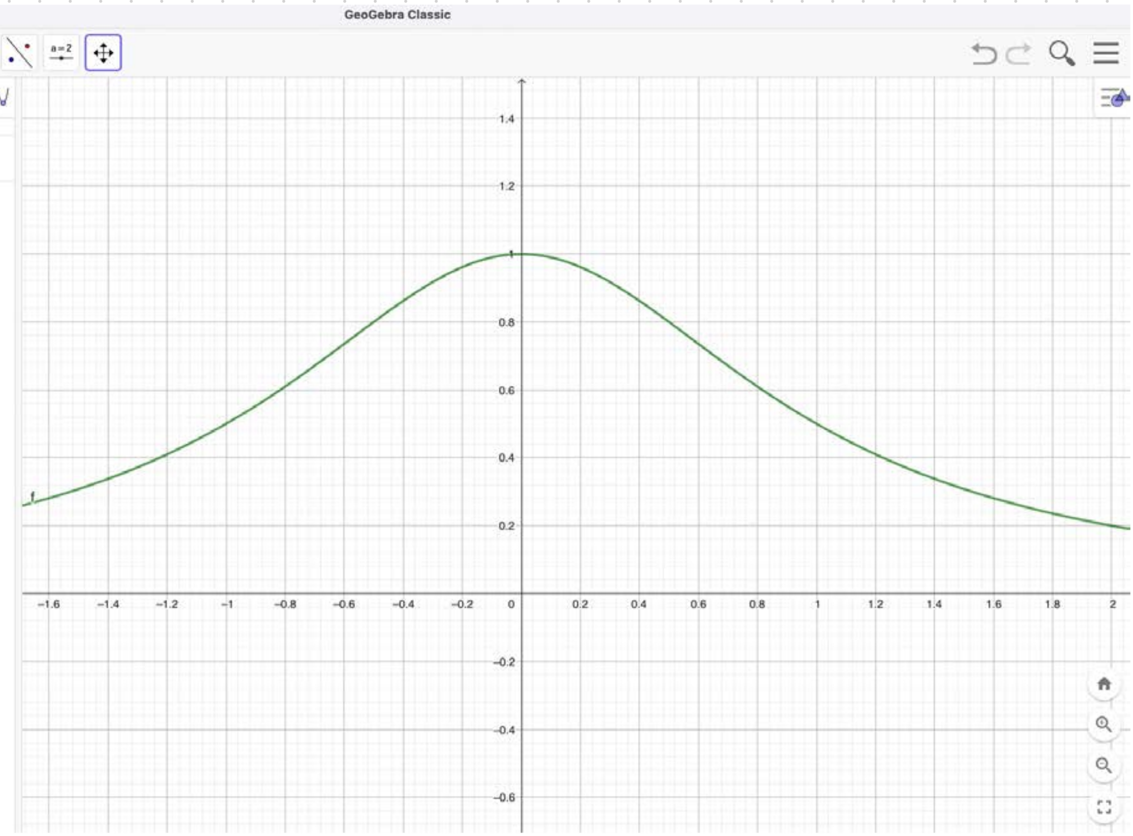
and the midpoint sum is:

$$M_f = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

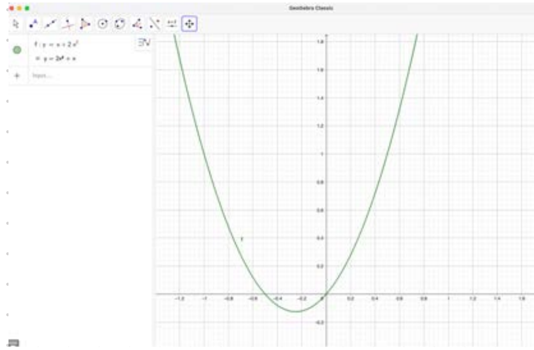
Average Value

The ***average value*** of *f* on [a,b] is the *y*-value that would generate a rectangle with the same area as *f* on [a,b].

$$AV = \frac{Area}{b-a}$$

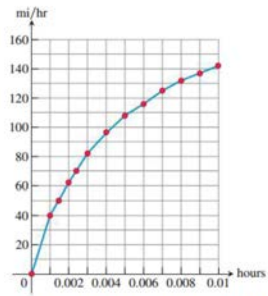


2. Consider the function $f(x) = x + 2x^2$ on the interval $[0,2]$. Using a midpoint estimate with $n = 4$ subintervals, estimate the *average value* of f.



12. Distance from velocity data The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

EXERCISES 5.1

Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.

b. a lower sum with four rectangles of equal width.

c. an upper sum with two rectangles of equal width.

d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

6. $f(x) = x^3$ between $x = 0$ and $x = 1$.

7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. Distance traveled The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (cm/sec)	Time (sec)	Velocity (cm/sec)
0	0	6	28
1	30	7	15
2	56	8	5
3	25	9	15
4	38	10	0
5	33		

10. Distance traveled upstream You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

5.2 Sigma Notation and Limits of Finite Sums

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

Days 2 & 3 Learning Goals

- Be able to find the equation for a general Riemann Sum
- Take the limit of your answer to find the actual area beneath the curve
- Understand the definition of the definite integral
- Understand key properties of the definite integral

EXAMPLE 1

A sum in sigma notation

$\sum_{k=1}^{5} k$

$\sum_{k=1}^{3} (-1)^k k$

$\sum_{k=1}^{2} \frac{k}{k+1}$

$\sum_{k=4}^{5} \frac{k^2}{k-1}$

Warm-up

General Riemann Sum

Partition the interval $[a,b]$ into n equal pieces:

$a = x_0 < x_1 < x_2 < .... < a_n = b$

Let x_i^* be an arbitrary point in the interval $[x_{i-1}, x_i]$.

Then we can estimate the area under the curve between $x = a$ and $x = b$ with the formula:

$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$

Note that: $L_f \le A \le U_f$

What is x_i*?

A. The left-hand endpoint of the subinterval.
B. The right-hand endpoint of the subinterval.
C. The midpoint of the subinterval.
D. Any value on the subinterval.

First example of the day

Ex

A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$C(t) = 5t - t^2,$$

where t is time in weeks and the number of customers is given in thousands.
Using the general form of the definite integral,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*),$$

calculate the **average** number of customers gained during the three-week campaign.

Example 2:

Use the method of Riemann Sums to evaluate the following definite integral. Choose x_i^* to be the right-hand endpoint of each subinterval.

$$\int_{-1}^{2} (x+1)^2\,dx$$

The Definite Integral

We define the definite integral to be the limit of the Riemann Sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Helpful Summation Formulas

$$\sum_{i=1}^{n} 1 = n$$

① $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ ← how to deal w/ x

② $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$ ← how to deal w/ x^2

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

If more practice is needed.

Example 3:

In a memory experiment, the rate of memorization is measured by the function:

$f(t) = -0.006\,t^2 + 0.2\,t$,

where t is the time in minutes, and $f(t)$ is the number of words per minute.

(a) How many words are memorized in the first 20 minutes (from t=0 to t=20)? USE RIEMANN SUMS.

(b) What is the average number of words memorized each minute?

$f(t) = -0.006t^2 + 0.2t$

So here is the theory

The Definite Integral

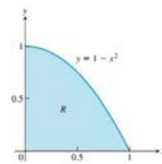
We define the definite integral to be the limit of the Riemann Sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Check your understanding

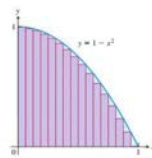
True or False?

- The definite integral represents the total area bounded by the function, the lines x=a and x=b, and the x-axis.



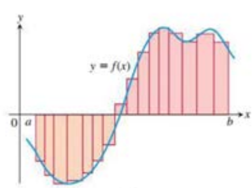
The Definite Integral and Area

If the function is always non-negative on [a,b], we have found *TOTAL AREA* under the curve.



(a)

If the function takes on negative values, then we have found the *NET AREA* under the curve.



(a)

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^{2} \frac{6k}{k+1}$
2. $\sum_{k=1}^{3} \frac{k-1}{k}$
3. $\sum_{k=1}^{4} \cos k\pi$
4. $\sum_{k=1}^{5} \sin k\pi$
5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$
6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

 a. $\sum_{k=1}^{6} 2^{k-1}$ b. $\sum_{k=0}^{5} 2^{k}$ c. $\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

 a. $\sum_{k=1}^{6} (-2)^{k-1}$ b. $\sum_{k=0}^{5} (-1)^{k} 2^{k}$ c. $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

 a. $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ b. $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ c. $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

10. Which formula is not equivalent to the other two?

 a. $\sum_{k=1}^{4} (k-1)^2$ b. $\sum_{k=-1}^{3} (k+1)^2$ c. $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$
12. $1 + 4 + 9 + 16$
13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
14. $2 + 4 + 6 + 8 + 10$
15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$
16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

 a. $\sum_{k=1}^{n} 3a_k$ b. $\sum_{k=1}^{n} \frac{b_k}{6}$ c. $\sum_{k=1}^{n} (a_k + b_k)$

 d. $\sum_{k=1}^{n} (a_k - b_k)$ e. $\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

 a. $\sum_{k=1}^{n} 8a_k$ b. $\sum_{k=1}^{n} 250b_k$

 c. $\sum_{k=1}^{n} (a_k + 1)$ d. $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. a. $\sum_{k=1}^{10} k$ b. $\sum_{k=1}^{10} k^2$ c. $\sum_{k=1}^{10} k^3$

20. a. $\sum_{k=1}^{13} k$ b. $\sum_{k=1}^{13} k^2$ c. $\sum_{k=1}^{13} k^3$

21. $\sum_{k=1}^{7} (-2k)$

22. $\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\sum_{k=1}^{6} (3 - k^2)$

24. $\sum_{k=1}^{6} (k^2 - 5)$

25. $\sum_{k=1}^{5} k(3k + 5)$

26. $\sum_{k=1}^{7} k(2k + 1)$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

29. a. $\sum_{k=1}^{7} 3$ b. $\sum_{k=1}^{500} 7$ c. $\sum_{k=3}^{264} 10$

30. a. $\sum_{k=9}^{36} k$ b. $\sum_{k=3}^{17} k^2$ c. $\sum_{k=18}^{71} k(k-1)$

31. a. $\sum_{k=1}^{n} 4$ b. $\sum_{k=1}^{n} c$ c. $\sum_{k=1}^{n} (k-1)$

32. a. $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ b. $\sum_{k=1}^{n} \frac{c}{n}$ c. $\sum_{k=1}^{n} \frac{k}{n^2}$

33. $\sum_{k=1}^{50} [(k+1)^2 - k^2]$

34. $\sum_{k=2}^{20} [\sin(k-1) - \sin k]$

35. $\sum_{k=7}^{30} (\sqrt{k-4} - \sqrt{k-3})$

36. $\sum_{k=1}^{40} \frac{1}{k(k+1)}$ $\left(\textit{Hint: } \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}\right)$

Riemann Sums

In Exercises 37–42, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

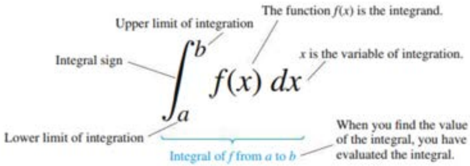
37. $f(x) = x^2 - 1$, $[0, 2]$
38. $f(x) = -x^2$, $[0, 1]$
39. $f(x) = \sin x$, $[-\pi, \pi]$
40. $f(x) = \sin x + 1$, $[-\pi, \pi]$
41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.
42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

43. $f(x) = 1 - x^2$ over the interval $[0, 1]$.
44. $f(x) = 2x$ over the interval $[0, 3]$.
45. $f(x) = x^2 + 1$ over the interval $[0, 3]$.
46. $f(x) = 3x^2$ over the interval $[0, 1]$.
47. $f(x) = x + x^2$ over the interval $[0, 1]$.
48. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.
49. $f(x) = 2x^3$ over the interval $[0, 1]$.
50. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

5.3 The Definite Integral



DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon.$$

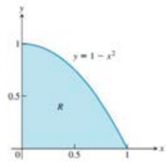
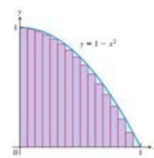


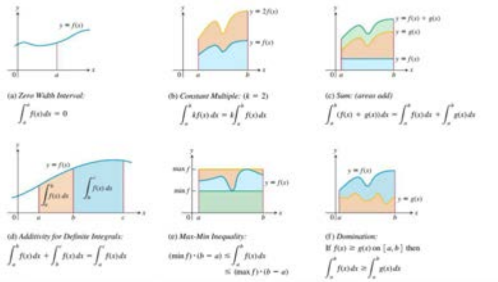
FIGURE 5.1 The area of a region R cannot be found by a simple formula.

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \qquad (1)$$



(a)



(a) Zero Width Interval: $\int_a^a f(x)\,dx = 0$

(b) Constant Multiple: $(k = 2)$ $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(c) Sum: (areas add) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(d) Additivity for Definite Integrals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) Max-Min Inequality: $(\min f)\cdot(b - a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b - a)$

(f) Domination: If $f(x) \ge g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then
$$(\min f)\cdot(b - a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b - a).$$
7. *Domination:* If $f(x) \ge g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
   If $f(x) \ge 0$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge 0$. — Special case

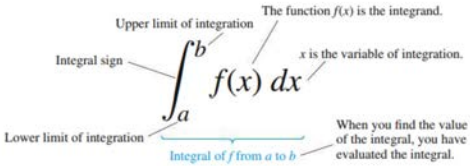
EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

1. $\int_4^1 f(x)\,dx$

2. $\int_{-1}^{1} [\,2f(x) + 3h(x)\,]\,dx =$

3. $\int_{-1}^{4} f(x)\,dx =$



DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon.$$

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \qquad (1)$$

EXERCISES 5.3

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\,dx$ **b.** $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d.** $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ **f.** $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3} f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

Example 6:

Given that $\int_1^3 2f(x)dx = 4$ and $\int_1^0 f(x)dx = -1$,

find $\int_0^3 f(x)dx$.

A. -3
B. 1
C. 3
D. 5

Ex. Evaluate the definite integral. (Hint: the functions are odd)

(a) $\int_{-1}^{1} x + x^3 \, dx$

(b) $\int_{-\pi}^{\pi} \sin(3x) \, dx$

4. $f(x)$ is an even function. If $\int_0^4 f(x)dx = 3$ and $\int_4^6 f(x)dx = 5$, find $\int_{-4}^{6} f(x)dx$.

Properties of the Definite Integral

Let $f(x)$ be continuous on $[a,b]$.

(1) $\int_a^b c\,dx = c(b-a)$

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(3) $\int_a^a f(x)dx = 0$

(4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in [a,b]$

(5) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(6) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Some More Integral Properties

(1) If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$.

(2) If $f(x) \geq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

(3) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

(4) If f is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0.$$

(5) If f is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

So now, we can combine the ideas

Find

(1) The anti-derivative

(2) The average value over the interval [a,b]

(3) The definite integral of f(x) over the interval [a,b]

$$\int f(x)\,dx = F(x) + C$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$

57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

5.4 The Fundamental Theorem of Calculus

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

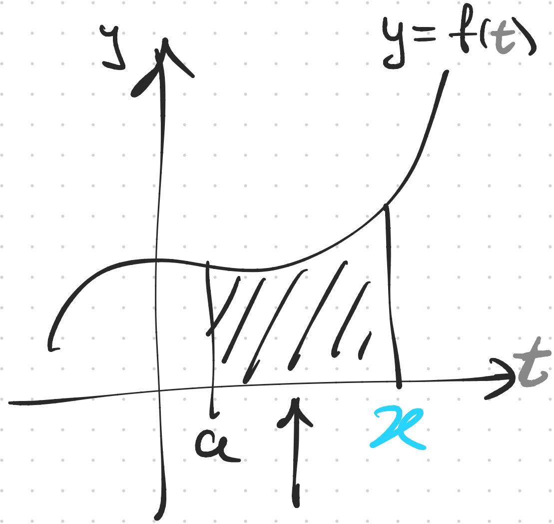
Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

Today's Learning Goals

- Know the statements of the FTC and the Second FTC
- Apply the FTC to evaluating definite integrals using the formulas from Section 4.8
- Apply the Second FTC to differentiate an integral

$$F(x) = \int_a^x f(t)\, dt. \qquad (1)$$



Area =

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$

Week	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

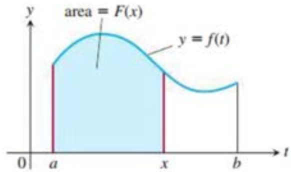
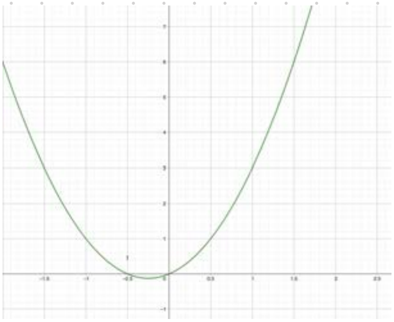


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

y=2x^2+x over [0,2]



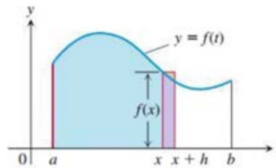


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x+h)$ is the area to the left of $x+h$. The difference quotient $[F(x+h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

$$F'(x) = f(x).$$

$$\frac{F(x+h) - F(x)}{h} \approx f(x).$$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

THEOREM 4—The Fundamental Theorem of Calculus, Part 1
If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

EXAMPLE Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$

(b) $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$

(d) $y = \int_{1+3x^2}^{4} \frac{1}{2 + e^t}\,dt$

Example : Find F′(2).

$$F(x) = \int_{1}^{x} \frac{t}{t^3 + 3}\,dt$$

A. 2/7
B. 2/11
C. ¼
D. 3/44

If time…

Example : Extension to 2nd FTC

Use this extension:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

to find $F'(x)$ if $F(x) = \int_{3x}^{\cos x} \frac{1}{1+t}\,dt$.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(a) $\displaystyle\int_0^{\pi} \cos x\,dx$

(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\,dx =$

(c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx =$

(d) $\displaystyle\int_0^1 \frac{dx}{x + 1} =$

(e) $\displaystyle\int_0^1 \frac{dx}{x^2 + 1} =$

Check your understanding

Example : Evaluate.

$$\int_1^3 \frac{1}{x^2}\,dx$$

A. 2/3
B. 4/3
C. 26/9
D. 26/81

Example :

The percent of toxin in a lake, where time is in years, is given by the function:

$$f(t) = 50\left(\frac{1}{4}\right)^t.$$

Find the average amount of toxin in the lake between years 1 and 3.

THEOREM 3—The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

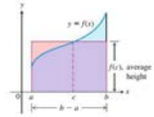


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Example 5:

Find the average value of the function:

$$f(x) = 1 - x^2, -1 \leq x \leq 3.$$

Then find a c that satisfies the MVT for integration.

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

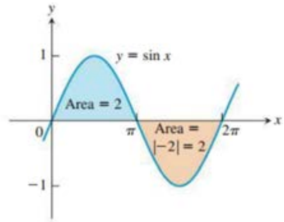


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

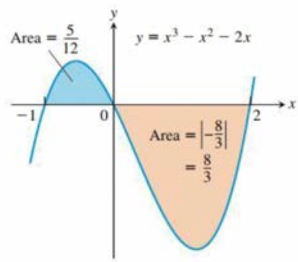


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x-3)\,dx$

2. $\int_{-1}^1 (x^2 - 2x + 3)\,dx$

3. $\int_{-2}^2 \frac{3}{(x+3)^4}\,dx$

4. $\int_{-1}^1 x^{299}\,dx$

5. $\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$

6. $\int_{-2}^3 (x^3 - 2x + 3)\,dx$

7. $\int_0^1 \left(x^2 + \sqrt{x}\right) dx$

8. $\int_1^{32} x^{-6/5}\,dx$

9. $\int_0^{\pi/3} 2\sec^2 x\,dx$

10. $\int_0^{\pi} (1 + \cos x)\,dx$

11. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$

12. $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$

13. $\int_{\pi/2}^0 \frac{1 + \cos 2t}{2}\,dt$

14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$

15. $\int_0^{\pi/4} \tan^2 x\,dx$

16. $\int_0^{\pi/6} (\sec x + \tan x)^2\,dx$

17. $\int_0^{\pi/8} \sin 2x\,dx$

18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

19. $\int_1^{-1} (r+1)^2\,dr$

20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt$

21. $\int_{\sqrt{2}}^1 \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$

22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\,dy$

23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\,ds$

24. $\int_1^8 \frac{(x^{1/3}+1)(2 - x^{2/3})}{x^{1/3}}\,dx$

25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

26. $\int_0^{\pi/3} (\cos x + \sec x)^2\,dx$

27. $\int_{-4}^4 |x|\,dx$

28. $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$

29. $\int_0^{\ln 2} e^{3x}\,dx$

30. $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx$

31. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}}\,dx$

32. $\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2}$

33. $\int_2^4 x^{\pi-1}\,dx$

34. $\int_{-1}^0 \pi^{x-1}\,dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 xe^{x^2}\,dx$

36. $\int_1^2 \frac{\ln x}{x}\,dx$

37. $\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$

38. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

a. by evaluating the integral and differentiating the result.
b. by differentiating the integral directly.

39. $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$

40. $\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt$

41. $\frac{d}{dt}\int_0^{t^4} \sqrt{u}\,du$

42. $\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy$

43. $\frac{d}{dx}\int_0^{x^3} e^{-t}\,dt$

44. $\frac{d}{dt}\int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx$

Find dy/dx in Exercises 45–56.

45. $y = \int_0^x \sqrt{1+t^2}\,dt$

46. $y = \int_1^x \frac{1}{t}\,dt, \quad x > 0$

47. $y = \int_{\sqrt{x}}^0 \sin(t^2)\,dt$

48. $y = x\int_2^{x^2} \sin(t^3)\,dt$

49. $y = \int_{-1}^x \frac{t^2}{t^2+4}\,dt - \int_3^x \frac{t^2}{t^2+4}\,dt$

50. $y = \left(\int_0^x (t^3+1)^{10}\,dt\right)^3$

51. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$

52. $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$

53. $y = \int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt$

54. $y = \int_{2^x}^1 \sqrt[3]{t}\,dt$

55. $y = \int_0^{\sin^{-1} x} \cos t\,dt$

56. $y = \int_{-1}^{x^{1/\pi}} \sin^{-1} t\,dt$

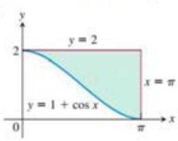
Area

In Exercises 57–60, find the total area between the region and the x-axis.

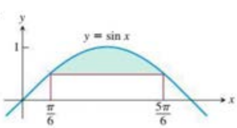
57. $y = -x^2 - 2x, \quad -3 \le x \le 2$
58. $y = 3x^2 - 3, \quad -2 \le x \le 2$
59. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$
60. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 61–64.

61.



62.



5.5 Indefinite Integrals and the Substitution Method

2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution

For which of the functions below can we currently find an antiderivative?

$$f(x) = \sec x$$
$$g(x) = \csc(3x)\cot(3x)$$
$$h(x) = x\sin x$$
$$k(x) = x\cos(x^2)$$

Functions we can integrate:

$$x^n, \sin(ax), \cos(ax)$$
$$\csc(ax)\cot(ax)$$
$$\sec(ax)\tan(ax)$$
$$\sec^2(ax), \csc^2(ax)$$
$$e^{ax}, b^{ax}$$
$$\frac{1}{1+(ax)^2}, \frac{1}{\sqrt{1-(ax)^2}}$$

THEOREM 6—The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du.$$

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

$$\int x^2 e^{x^3}\,dx =$$

$$\int x\sqrt{2x+1}\,dx.$$

Hint: multiply by e^x on top and bottom

(a) $\displaystyle\int \frac{dx}{e^x + e^{-x}} =$

Hint: multiply by sec(x)+tan(x) on top and bottom

(b) $\displaystyle\int \sec x\, dx =$

Integrals of the tangent, cotangent, secant, and cosecant functions

$\displaystyle\int \tan x\, dx = \ln|\sec x| + C$ $\qquad$ $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$\displaystyle\int \cot x\, dx = \ln|\sin x| + C$ $\qquad$ $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$$

Method 1: Substitute $u = z^2 + 1$.

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$(a)\int \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}\,\sin^2\left(\sqrt{t}\right)}\,dt$$

$$(b)\int_2^e \frac{1}{x(\ln x)^3}\,dx$$

$$(c)\int w\sqrt{1+w}\,dw$$

EXERCISES 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx,\quad u = 2x+4$

2. $\int 7\sqrt{7x-1}\,dx,\quad u = 7x-1$

3. $\int 2x(x^2+5)^{-4}\,dx,\quad u = x^2+5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx,\quad u = x^4+1$

5. $\int (3x+2)(3x^2+4x)^4\,dx,\quad u = 3x^2+4x$

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx,\quad u = 1+\sqrt{x}$

7. $\int \sin 3x\,dx,\quad u = 3x$

8. $\int x\sin(2x^2)\,dx,\quad u = 2x^2$

9. $\int \sec 2t\tan 2t\,dt,\quad u = 2t$

10. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt,\quad u = 1-\cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}},\quad u = 1-r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy,\quad u = y^4+4y^2+1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx,\quad u = x^{3/2}-1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx,\quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta\cot 2\theta\,d\theta$

   a. Using $u = \cot 2\theta$ b. Using $u = \csc 2\theta$

16. $\int \frac{dx}{\sqrt{5x+8}}$

   a. Using $u = 5x+8$ b. Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3-2s}\,ds$

18. $\int \frac{1}{\sqrt{5s+4}}\,ds$

19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

20. $\int 3y\sqrt{7-3y^2}\,dy$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x\sec^2 x\,dx$

25. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

26. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

34. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

51. $\int (\cos x)\,e^{\sin x}\,dx$

52. $\int (\sin 2\theta)\,e^{\sin^2\theta}\,d\theta$

53. $\int \frac{1}{\sqrt{x}e^{-\sqrt{x}}}\sec^2(e^{\sqrt{x}}+1)\,dx$

54. $\int \frac{1}{x^2}e^{1/x}\sec(1+e^{1/x})\tan(1+e^{1/x})\,dx$

55. $\int \frac{dx}{x\ln x}$

56. $\int \frac{\ln\sqrt{t}}{t}\,dt$

57. $\int \frac{dz}{1+e^z}$

58. $\int \frac{dx}{x\sqrt{x^4-1}}$

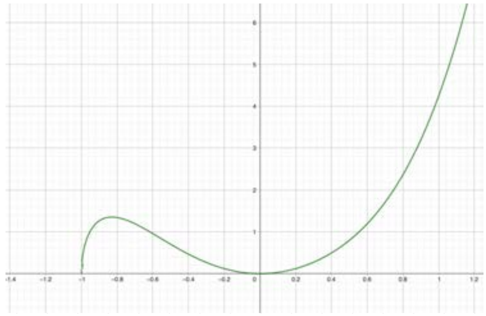
59. $\int \frac{5}{9+4r^2}\,dr$

60. $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta$

5.6 Definite Integral Substitutions and the Area Between Curves

3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx$.



Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx =$$

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx =$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta =$

u=

du=

(b) $\int_{-\pi/4}^{\pi/4} \tan x \, dx =$

u=

du=

Check your understanding

Example 2: Evaluate the integral.

$$\int (\sin 6x) e^{\cos 6x}\, dx$$

(A) $\frac{1}{6} e^{\cos 6x} + C$

(B) $-\frac{1}{6} e^{\cos 6x} + C$

(C) $\frac{1}{6} (\cos 6x) e^{\cos 6x} + C$

(D) $\frac{1}{2} \left(e^{\cos 6x}\right)^2 + C$

Areas Between Curves

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

But how to tell with no picture...?

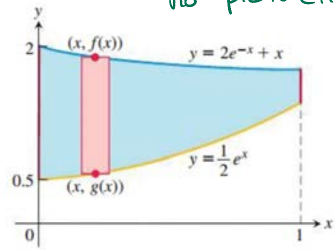


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Area = TOP − BOT

$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ (if $f \geq g$)

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solve
① $f(x) = g(x)$ to find points of intersection

② test each interval

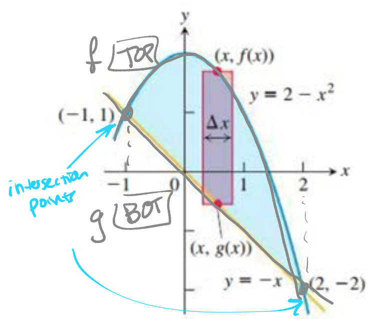


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle from a Riemann sum.

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

1. a. $\int_0^3 \sqrt{y+1}\,dy$ b. $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ b. $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\int_0^1 t^3(1+t^4)^3\,dt$ b. $\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ b. $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ b. $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\int_0^1 t\sqrt{4+5t}\,dt$ b. $\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

 b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

 b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

16. $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

25. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$

26. $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$

27. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$

28. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

29. $\int_1^2 \frac{2\ln x}{x}\,dx$

30. $\int_2^4 \frac{dx}{x\ln x}$

31. $\int_2^4 \frac{dx}{x(\ln x)^2}$

32. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

33. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

34. $\int_{\pi/4}^{\pi/2} \cot t\,dt$

35. $\int_0^{\pi/3} \tan^2\theta\cos\theta\,d\theta$

36. $\int_0^{\pi/12} 6\tan 3x\,dx$

37. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1+(\sin\theta)^2}$

38. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1+(\cot x)^2}$

39. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}}$

40. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1+\ln^2 t)}$

41. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}}$

42. $\int_0^{\sqrt[3]{2}/4} \frac{ds}{\sqrt{9-4s^2}}$

43. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$

44. $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$

45. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}}$

46. $\int_0^3 \frac{y\,dy}{\sqrt{5y+1}}$

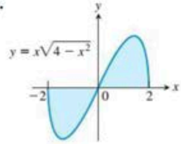
47. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

48. $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1}3x)}{1+9x^2}\,dx$

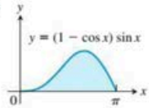
Area

Find the total areas of the shaded regions in Exercises 49–64.

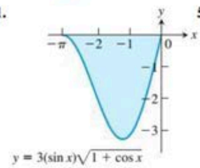
49.



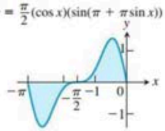
50.



51.



52. $y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))$



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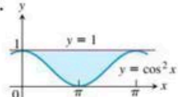
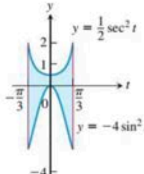


54.



Find the areas of the regions enclosed by the lines and curves in Exercises 65–74.

65. $y = x^2 - 2$ and $y = 2$

66. $y = 2x - x^2$ and $y = -3$

67. $y = x^4$ and $y = 8x$

68. $y = x^2 - 2x$ and $y = x$

69. $y = x^2$ and $y = -x^2 + 4x$

70. $y = 7 - 2x^2$ and $y = x^2 + 4$

71. $y = x^4 - 4x^2 + 4$ and $y = x^2$

72. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

73. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

74. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Math 1552

Section 8.2:
Integration by Parts

3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

Integration by parts is a technique for simplifying integrals of the form

$$\int u(x)\, v'(x)\, dx.$$

Integration by Parts Formula

$$\int u(x)\underbrace{v'(x)\, dx}_{dv} = u(x)\, v(x) - \int v(x)\underbrace{u'(x)\, dx}_{du} \qquad (1)$$

Integration by Parts Formula—Differential Version

$$\int u\, dv = uv - \int v\, du \qquad (2)$$

Order in which to choose u

Choose u according to the *ILATE* rule:

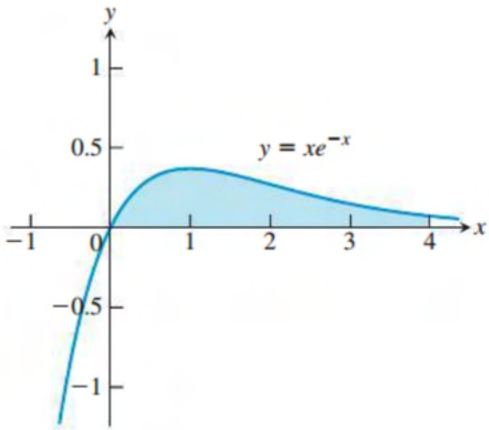
- **I** – Inverse Functions
- **L** – Logarithmic Functions
- **A** – Algebraic Expressions (polynomials, rational functions, etc.)
- **T** – Trigonometric Functions
- **E** – Exponential Functions

$$\int x \cos x\, dx.$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx \qquad (3)$$

$$\int_0^4 xe^{-x}\,dx.$$



$$\int \sin^{-1}(x)dx.$$

$$\int x\sin(x)\cos(x)dx.$$

$$\int \sin[\ln(x)]dx.$$

The difficult ones

$$\int u\,dv = uv - \int v\,du$$

$$\int x^2 e^x\,dx.$$

$$\int \ln x\,dx.$$

$$\int e^x \cos x\,dx.$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta\, d\theta$

3. $\int t^2 \cos t\, dt$

4. $\int x^2 \sin x\, dx$

5. $\int_1^2 x \ln x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

7. $\int xe^x\, dx$

8. $\int xe^{3x}\, dx$

9. $\int x^2 e^{-x}\, dx$

10. $\int (x^2 - 2x + 1)e^{2x}\, dx$

11. $\int \tan^{-1} y\, dy$

12. $\int \sin^{-1} y\, dy$

13. $\int x \sec^2 x\, dx$

14. $\int 4x \sec^2 2x\, dx$

15. $\int x^3 e^x\, dx$

16. $\int p^4 e^{-p}\, dp$

17. $\int (x^2 - 5x)e^x\, dx$

18. $\int (r^2 + r + 1)e^r\, dr$

19. $\int x^5 e^x\, dx$

20. $\int t^2 e^{4t}\, dt$

21. $\int e^\theta \sin \theta\, d\theta$

22. $\int e^{-y} \cos y\, dy$

23. $\int e^{2x} \cos 3x\, dx$

24. $\int e^{-2x} \sin 2x\, dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$

26. $\int_0^1 x\sqrt{1-x}\, dx$

27. $\int_0^{\pi/3} x \tan^2 x\, dx$

28. $\int \ln (x + x^2)\, dx$

29. $\int \sin (\ln x)\, dx$

30. $\int z(\ln z)^2\, dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x \sec x^2\, dx$

32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$

33. $\int x (\ln x)^2\, dx$

34. $\int \frac{1}{x (\ln x)^2}\, dx$

35. $\int \frac{\ln x}{x^2}\, dx$

36. $\int \frac{(\ln x)^3}{x}\, dx$

37. $\int x^3 e^{x^4}\, dx$

38. $\int x^5 e^{x^3}\, dx$

39. $\int x^3\sqrt{x^2+1}\, dx$

40. $\int x^2 \sin x^3\, dx$

41. $\int \sin 3x \cos 2x\, dx$

42. $\int \sin 2x \cos 4x\, dx$

43. $\int \sqrt{x} \ln x\, dx$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

45. $\int \cos \sqrt{x}\, dx$

46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$

48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$

49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\, dt$

50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$

51. $\int x \tan^{-1} x\, dx$

52. $\int x^2 \tan^{-1} \frac{x}{2}\, dx$

Order in which to choose *u*

Choose *u* according to the *ILATE* rule:

- **I** – Inverse Functions
- **L** – Logarithmic Functions
- **A** – Algebraic Expressions (polynomials, rational functions, etc.)
- **T** – Trigonometric Functions
- **E** – Exponential Functions

Math 1552

Section 8.3:
Powers and Products of Trigonometric Functions

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

Review Question: Which integrals can we evaluate *by parts*?

$$\text{(A)} \int \frac{x^2}{1+x^3}\,dx$$

$$\text{(B)} \int \frac{1}{x} e^{\ln x}\,dx$$

$$\text{(C)} \int x^5 e^{x^3}\,dx$$

$$\text{(D)} \int x \tan^{-1}(x)\,dx$$

$$\int \sin^3 x \cos^2 x\,dx.$$

Case 1 If **m is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\,dx$ equal to $-d(\cos x)$.

Case 2 If **n is odd** in $\int \sin^m x \cos^n x\,dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\,dx$ equal to $d(\sin x)$.

Case 3 If **both m and n are even** in $\int \sin^m x \cos^n x\,dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

$$(*)\sin^2 x + \cos^2 x = 1$$

$$(*)1 + \tan^2 x = \sec^2 x$$

$$(*)\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$$

$$(*)\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$$

$$(*)\sin(2x) = 2\sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\int \cos^5 x\,dx.$$

$$\int \sin^2 x \cos^4 x \, dx.$$

(*) $\sin^2 x + \cos^2 x = 1$

(*) $1 + \tan^2 x = \sec^2 x$

(*) $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$

(*) $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$

(*) $\sin(2x) = 2\sin x \cos x$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$$\int \tan^4 x \, dx.$$

$$\int \sec^3 x \, dx.$$

$$\int \tan^4 x \sec^4 x \, dx.$$

$$\int \cos^2(x)\cot(x)dx$$

$$\int \sin^4(x)dx$$

$$\int \tan^3(x)dx$$

Evaluate the integral.

$\int \sin^2(x)\cos^3(x)dx$

$(A)\frac{1}{5}\sin^5(x)+C$

$(B)\frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+C$

$(C)\frac{1}{12}\sin^3(x)\cos^4(x)+C$

$(D)-\frac{1}{3}\cos^3(x)+\frac{1}{5}\cos^5(x)+C$

$$\int \tan^3(x)\sec^3(x)dx$$

$$\int \sec^3(x)dx$$

EXERCISES 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x\, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx$
3. $\int \cos^3 x \sin x\, dx$
4. $\int \sin^4 2x \cos 2x\, dx$
5. $\int \sin^3 x\, dx$
6. $\int \cos^3 4x\, dx$
7. $\int \sin^5 x\, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$
9. $\int \cos^3 x\, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x\, dx$
11. $\int \sin^3 x \cos^3 x\, dx$
12. $\int \cos^3 2x \sin^5 2x\, dx$
13. $\int \cos^2 x\, dx$
14. $\int_0^{\pi/2} \sin^2 x\, dx$
15. $\int_0^{\pi/2} \sin^7 y\, dy$
16. $\int 7 \cos^7 t\, dt$
17. $\int_0^{\pi} 8 \sin^4 x\, dx$
18. $\int 8 \cos^4 2\pi x\, dx$
19. $\int 16 \sin^2 x \cos^2 x\, dx$
20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y\, dy$
21. $\int 8 \cos^3 2\theta \sin 2\theta\, d\theta$
22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}}\, dx$
24. $\int_0^{\pi} \sqrt{1 - \cos 2x}\, dx$
25. $\int_0^{\pi} \sqrt{1 - \sin^2 t}\, dt$
26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta}\, d\theta$
27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}}\, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x}\, dx$

    $\left(\textit{Hint:} \text{ Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}.\right)$
29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}}\, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x}\, dx$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta}\, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2}\, dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x\, dx$
34. $\int \sec x \tan^2 x\, dx$
35. $\int \sec^3 x \tan x\, dx$
36. $\int \sec^3 x \tan^3 x\, dx$
37. $\int \sec^2 x \tan^2 x\, dx$
38. $\int \sec^4 x \tan^2 x\, dx$
39. $\int_{-\pi/3}^{0} 2 \sec^3 x\, dx$
40. $\int e^x \sec^3 e^x\, dx$