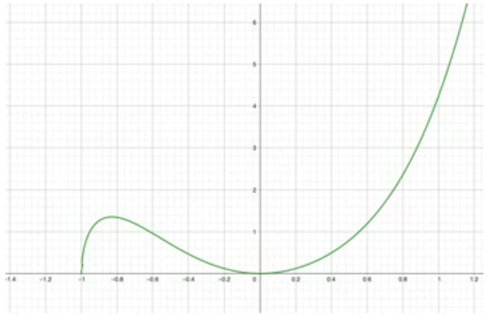
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts

Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.



Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx =$$

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx =$$

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta =$

u=

du=

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x \, dx =$

u=

du=

Check your understanding

Example 2: Evaluate the integral.

$$\int (\sin 6x) e^{\cos 6x} \, dx$$

(A) $\frac{1}{6} e^{\cos 6x} + C$

(B) $-\frac{1}{6} e^{\cos 6x} + C$

(C) $\frac{1}{6} (\cos 6x) e^{\cos 6x} + C$

(D) $\frac{1}{2} \left(e^{\cos 6x}\right)^2 + C$

Areas Between Curves

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

But how to tell with no picture...?

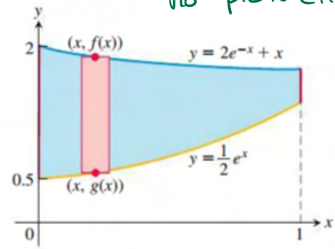


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Area = TOP − BOT

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \quad (\text{if } f \geq g)$$

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solve

① $f(x) = g(x)$ to find points of intersection

② test each interval

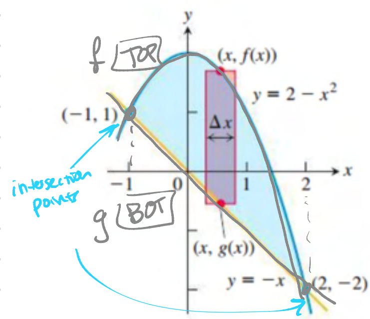


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle from a Riemann sum.

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

1. a. $\int_0^3 \sqrt{y+1}\,dy$ **b.** $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ **b.** $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b.** $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b.** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\int_0^1 t^3(1+t^4)^3\,dt$ **b.** $\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b.** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b.** $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b.** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b.** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\int_0^1 t\sqrt{4+5t}\,dt$ **b.** $\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$ **16.** $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$ **18.** $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$ **20.** $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$ **24.** $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

25. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$ **26.** $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$

27. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$ **28.** $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

29. $\int_1^2 \frac{2\ln x}{x}\,dx$ **30.** $\int_2^4 \frac{dx}{x\ln x}$

31. $\int_2^4 \frac{dx}{x(\ln x)^2}$ **32.** $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

33. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$ **34.** $\int_{\pi/4}^{\pi/2} \cot t\,dt$

35. $\int_0^{\pi/3} \tan^2\theta\cos\theta\,d\theta$ **36.** $\int_0^{\pi/12} 6\tan 3x\,dx$

37. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1+(\sin\theta)^2}$ **38.** $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1+(\cot x)^2}$

39. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}}$ **40.** $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1+\ln^2 t)}$

41. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}}$ **42.** $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}}$

43. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$ **44.** $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$

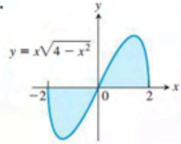
45. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}}$ **46.** $\int_0^3 \frac{y\,dy}{\sqrt{5y+1}}$

47. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$ **48.** $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1}3x)}{1+9x^2}\,dx$

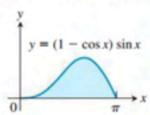
Area

Find the total areas of the shaded regions in Exercises 49–64.

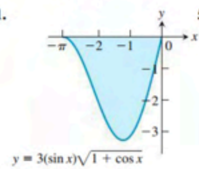
49.



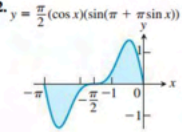
50.



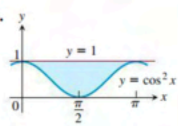
51.



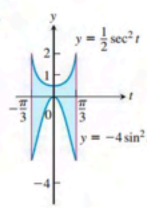
52. $y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))$



53.



54.



Find the areas of the regions enclosed by the lines and curves in Exercises 65–74.

65. $y = x^2 - 2$ and $y = 2$

66. $y = 2x - x^2$ and $y = -3$

67. $y = x^4$ and $y = 8x$

68. $y = x^2 - 2x$ and $y = x$

69. $y = x^2$ and $y = -x^2 + 4x$

70. $y = 7 - 2x^2$ and $y = x^2 + 4$

71. $y = x^4 - 4x^2 + 4$ and $y = x^2$

72. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

73. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

74. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Math 1552

Section 8.2:

Integration by Parts

3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

Integration by parts is a technique for simplifying integrals of the form

$$\int u(x)\,v'(x)\,dx.$$

Integration by Parts Formula

$$\int u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)\,v(x) - \int v(x)\underbrace{u'(x)\,dx}_{du} \qquad (1)$$

Integration by Parts Formula—Differential Version

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Order in which to choose u

Choose u according to the *ILATE* rule:

- **I** – Inverse Functions
- **L** – Logarithmic Functions
- **A** – Algebraic Expressions (polynomials, rational functions, etc.)
- **T** – Trigonometric Functions
- **E** – Exponential Functions

$\boxed{C_2 = -C_1}$

EX $\boxed{\int x \cos x \, dx.} \neq \cancel{\frac{1}{2}x^2 \cdot \sin x + C}$

$\int u\,dv$

$= uv - \int v\,du$

IBP BOX

$u = x \quad dv = \cos x\,dx$

$du = 1\,dx \quad v = \sin x$

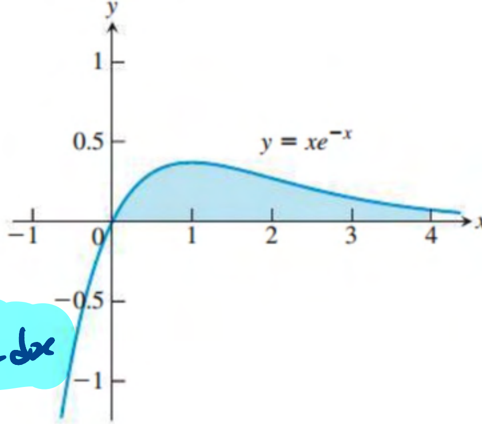
$= x \cdot \sin x - \int \sin x \cdot 1\,dx$

$= x\sin x - (-\cos x + C_1)$

$= x\sin x + \cos x + C_2$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx \qquad (3)$$



$$\int u\,dv = u\cdot v - \int v\,du$$

Ex 2.

$$\int \underbrace{x}_{u}\underbrace{e^{-x}\,dx}_{dv} = x\cdot(-e^{-x}) - \int -e^{-x}\cdot 1\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

IBP BOX

$u = x$	$dv = e^{-x}\,dx$
$du = 1\,dx$	$v = -e^{-x}$

$$= -xe^{-x} - e^{-x} + C$$

$$\int u\,dv + \int v\,du = uv$$

$$\int u\,dv = uv - \underbrace{\int v\,du}$$

Ex. 3

$$\int \underbrace{\sin^{-1}(x)}\,dx = (\sin^{-1}x)(x) - \int x\cdot\frac{1}{\sqrt{\underbrace{1-x^2}_{u}}}\,dx$$

$-\frac{1}{2}du$

IBP BOX

$u = \sin^{-1}x$	$dv = 1\,dx$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$	$v = x$

u-sub Box

$u = 1 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$= x\sin^{-1}(x) + \frac{1}{2}\int\frac{1}{\sqrt{u}}\cdot\cancel{\frac{-1}{2}}\,du$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot 2\sqrt{u} + C$$

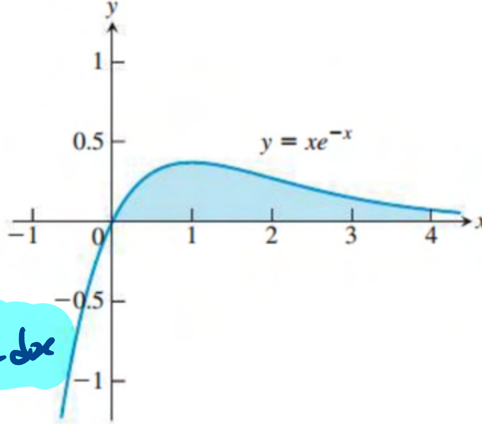
Prod rule

CHAIN.

$$= \underbrace{x\sin^{-1}(x)}_{\text{Prod rule}} + \underbrace{\sqrt{1-x^2}}_{\text{CHAIN.}} + C$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx \qquad (3)$$



$\int u\,dv = u\cdot v - \int v\,du$

Ex 2

$$\int \underbrace{x}_{u}\underbrace{e^{-x}\,dx}_{dv} = x\cdot(-e^{-x}) - \int -e^{-x}\cdot 1\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

IBP BOX

$u = x$	$dv = e^{-x}\,dx$
$du = 1\,dx$	$v = -e^{-x}$

$$= \underbrace{-xe^{-x} - e^{-x} + C}_{F(x)}$$

Ex 2'

$$\int_0^4 xe^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_0^4$$

$$= F(4) - F(0)$$

$$= \left(-4e^{-4} - e^{-4}\right) - \left(-0e^0 - e^0\right)$$

$$= \boxed{-5e^{-4} + 1}$$

Ex 5.

$$\int x\sin(x)\cos(x)dx.$$

IBP Box??

(1)

$u = x$	$dv = \sin x \cos x \, dx$
$du = dx$	$v = ???$

??

u-sub??

IBP Box?

(2)

$u = \sin x$	$dv = x\cos x \, dx$
$du = \cos x \, dx$	$v = ??$

IBP

IBP Box?

(3)

$u = x\sin x$	$dv = \cos x \, dx$
$du = \text{PROD}$	$v = \sin x$

$$\int \sin[\ln(x)]dx.$$

Ex 5.

$$\int x\sin(x)\cos(x)\,dx. = \star$$

IBP Box??

$$\boxed{\begin{array}{ll} ① \; u = x & dv = \sin x \cos x \, dx \\ du = dx & v = \frac{1}{2}\sin^2 x \end{array}}$$

?? u-sub??

u-sub Box

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$$

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\sin^2 x + C$$

$$\star = x \cdot \frac{1}{2}\sin^2 x - \int \frac{1}{2}\sin^2 x \, dx$$

$$= \frac{x}{2}\sin^2 x - \int \frac{1}{2}\left(\frac{1-\cos 2x}{2}\right) dx$$

$$\underbrace{1 - 2\sin^2 x = \cos(2x)}$$

$$= \frac{x}{2}\sin^2 x - \frac{1}{4}\int 1 - \cos 2x \, dx$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{x}{2}\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C$$

The difficult ones

$$\int u\,dv = uv - \int v\,du$$

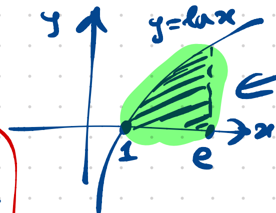
* $\int e^x \cos x\,dx.$ wrap around

* $\int x^2 e^x\,dx.$ do IBP twice

Ex 4

??

$$\int_1^e \underbrace{\ln x}_{u}\,\overbrace{dx}^{dv} = x\cdot \ln x\Big|_1^e - \boxed{\int_1^e x\cdot \cancelto{1}{\frac{1}{x}}\,dx}$$



← area is 1.

IBP Box

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}\,dx & v = x \end{array}}$$

same box

$$= \left(e\cdot \cancelto{1}{\ln e} - 1\cdot \cancelto{0}{\ln 1}\right) - \cancel{\frac{1}{2}x^2\cdot \ln x\Big|_1^e}$$

???

$$= e - \int_1^e 1\,dx \qquad F(e) - F(1)$$

$$= e - x\Big|_1^e = e - (e - 1) = \boxed{1}$$

F(b) − F(a)

Ex 4'

$$\int \ln x\,dx$$

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\ln x - \int 1\,dx = x\ln x - x + C$$

$$\int u\,dv = uv - \int v\,du$$

ILATE how to pick u.

(I, L, A: algebra, T, E: exponential)

Ex 7.

$$\int x^2 e^x\,dx.$$

IBP Box

$$\boxed{\begin{array}{ll} u = x^2 & dv = e^x\,dx \\ du = 2x\,dx & v = e^x \end{array}}$$

$$= x^2 e^x - \int e^x \cdot 2x\,dx$$

$$= x^2 e^x - \underbrace{\int 2x e^x\,dx}_{uv - \int v\,du}$$

$$\boxed{\begin{array}{ll} u = 2x & dv = e^x\,dx \\ du = 2\,dx & v = e^x \end{array}}$$

$$= x^2 e^x - \left(2x e^x - \int 2e^x\,dx\right)$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int xe^x\, dx$
8. $\int xe^{3x}\, dx$
9. $\int x^2e^{-x}\, dx$
10. $\int (x^2 - 2x + 1)e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3e^x\, dx$
16. $\int p^4e^{-p}\, dp$
17. $\int (x^2 - 5x)e^x\, dx$
18. $\int (r^2 + r + 1)e^r\, dr$
19. $\int x^5e^x\, dx$
20. $\int t^2e^{4t}\, dt$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Order in which to choose u

Choose u according to the *ILATE* rule:

- I – Inverse Functions
- L – Logarithmic Functions
- A – Algebraic Expressions (polynomials, rational functions, etc.)
- T – Trigonometric Functions
- E – Exponential Functions

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1 - x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln (x + x^2)\, dx$
29. $\int \sin (\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x \sec x^2\, dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$
33. $\int x (\ln x)^2\, dx$
34. $\int \frac{1}{x (\ln x)^2}\, dx$
35. $\int \frac{\ln x}{x^2}\, dx$
36. $\int \frac{(\ln x)^3}{x}\, dx$
37. $\int x^3 e^{x^4}\, dx$
38. $\int x^5 e^{x^3}\, dx$
39. $\int x^3\sqrt{x^2 + 1}\, dx$
40. $\int x^2 \sin x^3\, dx$
41. $\int \sin 3x \cos 2x\, dx$
42. $\int \sin 2x \cos 4x\, dx$
43. $\int \sqrt{x} \ln x\, dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
45. $\int \cos \sqrt{x}\, dx$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\, dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$
51. $\int x \tan^{-1} x\, dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\, dx$