MATH 1552

Chapter 8-10

Trig sub, partial fractions & L'Hop
Improper integrals, sequences/series
integral test

Math 1552

Section 8.4:
Trigonometric Substitution

https://strawpoll.com/polls/3RnYl3G2zye

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 **Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)**	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

Exam 1 vs. expectations

Trig sub.

$a^2 - x^2$
$a^2 + x^2$
$x^2 - a^2$

Three types of trig-sub.

Today's Learning Goals

- Identify which types of integrals can be solved with a trigonometric substitution
- Learn which substitution matches which general form
- Evaluate integrals using the method of trigonometric substitution

Trigonometric Substitutions

We use a trig substitution when no other integration method will work, and when the integral contains one of these terms:

$$a^2 - x^2$$
$$x^2 - a^2$$
$$a^2 + x^2$$

Ex 1

$$\int \sqrt{4 - x^2}\,dx$$

$2^2 - x^2$

$a = 2$

$x = 2\sin\theta$

Form 1:

When the integral contains a term of the form
$$a^2 - x^2,$$
use the substitution:
$$\boxed{x = a\sin\theta}$$

Ex 2

$$\int \frac{1}{(9 + x^2)^{3/2}}\,dx$$

$3^2 + x^2$

$a = 3$

$x = 3\tan\theta$

Form 2:

When the integral contains a term of the form
$$a^2 + x^2,$$
use the substitution:
$$\boxed{x = a\tan\theta}$$

Ex 3

$$\int \frac{1}{x^4\sqrt{x^2 - 1}}\,dx$$

$x^2 - 1^2$

$a = 1$

$x = \sec\theta$

Form 3:

When the integral contains a term of the form
$$x^2 - a^2,$$
use the substitution:
$$\boxed{x = a\sec\theta}$$

Ex 1.

$$\int \sqrt{4-x^2}\,dx$$

"U-sub Box"

$\boxed{x = 2\sin\theta}$

$dx = 2\cos\theta\, d\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$2^2 - x^2 \quad a = 2 \quad x = 2\sin\theta$

Form 1:

When the integral contains a term of the form

$a^2 - x^2,$

use the substitution:

$\boxed{x = a\sin\theta}$

$$= \int \sqrt{4-\underbrace{(2\sin\theta)}_{x}{}^2}\;\underbrace{2\cos\theta\, d\theta}_{dx}$$

$$= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int \sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\, d\theta$$

$* \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int 2\cdot \underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta}}\; 2\cdot\cos\theta\, d\theta$$

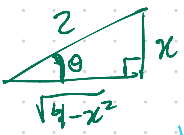
$* \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$* \sin 2\theta = 2\sin\theta\cos\theta$

$$= \int 4\cos\theta\cdot\cos\theta = \int 4\cos^2\theta\, d\theta$$

$\boxed{x = 2\sin\theta}$

$\frac{x}{2} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$



$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$$= 4\int \frac{1+\cos 2\theta}{2}\, d\theta = 2\int 1+\cos 2\theta\, d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = 2\theta + \sin 2\theta + C$$

$$= 2\cdot\sin^{-1}\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C}$$

$$\int \frac{1}{(9+x^2)^{3/2}}\,dx$$

Form 2:

When the integral contains a term of the form

$a^2 + x^2,$

use the substitution:

$\boxed{x = a\tan\theta}$

$3^2+x^2 \quad a=3 \quad x=3\tan\theta$

Form 2:

When the integral contains a term of the form a^2+x^2, use the substitution: $\boxed{x = a\tan\theta}$

$$\int \frac{1}{(9+x^2)^{3/2}}\,dx = \int \frac{1}{\left(9+(3\tan\theta)^2\right)^{3/2}}\, 3\sec^2\theta\, d\theta$$

"u-sub Box"

$\boxed{x=3\tan\theta}$

$dx = 3\sec^2\theta\, d\theta$

this is for

$$= \int \frac{1}{(9+9\tan^2\theta)^{3/2}}\, 3\sec^2\theta\, d\theta$$

$9+9x = 9(1+x)$ ✓

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \int \frac{1}{\left(9\cdot(1+\tan^2\theta)\right)^{3/2}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{3\sec^2\theta}{9^{3/2}(1+\tan^2\theta)^{3/2}}\, d\theta = \int \frac{3\sec^2\theta}{27(\sec^2\theta)^{3/2}}\, d\theta$$

$(a^2)^{3/2} = a^{2\cdot\frac{3}{2}}$

$\sec^2\theta = (\sec\theta)^2$

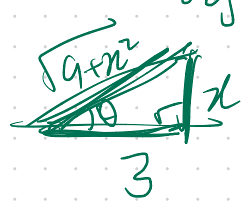
$$= \frac{1}{9}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \boxed{\frac{1}{9}\frac{x}{\sqrt{9+x^2}} + C}$$

$x = 3\tan\theta$

$\frac{x}{3} = \tan\theta = \frac{\text{opp}}{\text{adj}}$



① $\cos^2\theta = 1 - \sin^2\theta$
② $1 + \tan^2\theta = \sec^2\theta$
③ $\sec^2\theta - 1 = \tan^2\theta$

META for Trig sub

* identify which form to use

$a^2 - x^2$ $\quad$ $a^2 + x^2$ $\quad$ $x^2 - a^2$

① $x = a\sin\theta$ $\quad$ ② $x = a\tan\theta$ $\quad$ ③ $x = a\sec\theta$

* Fill in "u-sub Box" & make substitutions.

* simplify to get something you can integrate.

* draw reference triangle & get back to x's

$x^2-1 \quad a=1 \quad x=\sec\theta$

$$\int \frac{1}{x^4\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec^4\theta\,\underbrace{\sqrt{\sec^2\theta-1}}_{x^2-1}}\;\underbrace{\sec\theta\tan\theta\,d\theta}_{dx}$$

$\uparrow$ x^4

Form 3:

When the integral contains a term of the form x^2-a^2,

use the substitution: $\boxed{x = a\sec\theta}$

$x=\sec\theta$

$dx=\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sec\theta\tan\theta}{\sec^4\theta\sqrt{\tan^2\theta}}\,d\theta$$

$\sec^2\theta-1=\tan^2\theta$

$$= \int \frac{1}{\sec^3\theta}\cdot\frac{\cancel{\tan\theta}^{\,1}}{\cancel{\tan\theta}}\,d\theta$$

$\sqrt{a^2}=|a|$

$\frac{a}{a^4}=a^{-3}$

$$= \int \cos^3\theta\,d\theta = \int \cos^2\theta\cos\theta\,d\theta$$

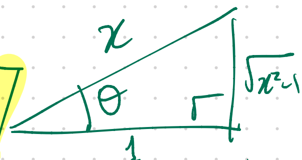
$$= \int (1-\sin^2\theta)\cos\theta\,d\theta = \int 1-u^2\,du = u-\tfrac{1}{3}u^3+C$$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

$$= \sin\theta - \tfrac{1}{3}\sin^3\theta + C$$

$$= \boxed{\frac{\sqrt{x^2-1}}{x} - \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C}$$



$\frac{x}{1}=\sec\theta=\frac{hyp}{adj}$

$\sin\theta=\frac{opp}{hyp}$

$=\frac{\sqrt{x^2-1}}{x}$

$$\int_0^2 \frac{dx}{8+2x^2}$$

$$\int_0^2 \frac{dx}{8+2x^2} = \int_0^2 \frac{1}{2(4+x^2)}\,dx = \boxed{\frac{1}{2}\int_0^2 \frac{1}{4+x^2}\,dx}$$

trig-sub Box.

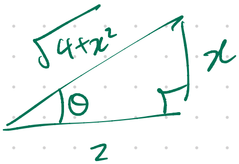
$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}}$$

$$= \frac{1}{2}\int_*^* \frac{1}{4+(2\tan\theta)^2}\, 2\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_*^* \frac{1}{4+4\tan^2\theta}\, 2\sec^2\theta\, d\theta$$

$\frac{x}{2} = \tan\theta$

$$= \frac{1}{2}\int_*^* \frac{1}{4}\cdot\frac{1}{1+\tan^2\theta}\, 2\sec^2\theta\, d\theta$$



$$= \frac{1}{\cancel{2}}\,\frac{1}{4}\int_*^* \frac{1}{\sec^2\theta}\cdot\cancel{2}\sec^2\theta\, d\theta$$

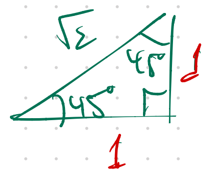
$$= \frac{1}{4}\int_*^* 1\, d\theta = \frac{1}{4}\theta\Big|_*^*$$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^2$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{2}{2}\right) - \frac{1}{4}\tan^{-1}\left(\frac{0}{2}\right)$$

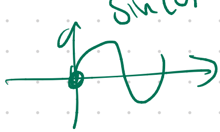
$$= \frac{1}{4}\tan^{-1}(1) - \frac{1}{4}\tan^{-1}(0)$$

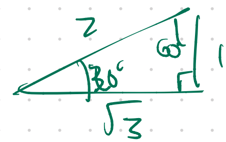


$$= \frac{1}{4}\cdot\frac{\pi}{4} - 0$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin(0) = 0$





$$= \pi/16$$

$$\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$$

$$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$$

$$\int \frac{x\,dx}{\sqrt{1+x^4}}$$

Section 8.4: 5, 11, 13, 17, 19, 29, 35 (extra practice: 39)

EXERCISES 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

16. $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

17. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

19. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

20. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

21. $\displaystyle\int \sqrt{\frac{x+1}{1-x}}\,dx$

22. $\displaystyle\int x\sqrt{x^2-4}\,dx$

23. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

24. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

25. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

26. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

27. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

28. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

29. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

30. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

31. $\displaystyle\int \frac{x^3\,dx}{x^2-1}$

32. $\displaystyle\int \frac{x\,dx}{25+4x^2}$

33. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

34. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

38. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

39. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

40. $\displaystyle\int \frac{dx}{1+x^2}$

41. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

42. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

43. $\displaystyle\int \frac{x\,dx}{\sqrt{1+x^4}}$

44. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

45. $\displaystyle\int \sqrt{\frac{4-x}{x}}\,dx$
(*Hint:* Let $x = u^2$.)

46. $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\,dx$
(*Hint:* Let $u = x^{3/2}$.)

47. $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\,dx$

48. $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Complete the Square Before Using Trigonometric Substitutions

For Exercises 49–52, complete the square before using an appropriate trigonometric substitution.

49. $\displaystyle\int \sqrt{8-2x-x^2}\,dx$

50. $\displaystyle\int \frac{1}{\sqrt{x^2-2x+5}}\,dx$

51. $\displaystyle\int \frac{\sqrt{x^2+4x+3}}{x+2}\,dx$

52. $\displaystyle\int \frac{\sqrt{x^2+2x+2}}{x^2+2x+1}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 53–56 for y as a function of x.

53. $x\dfrac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

54. $\sqrt{x^2-9}\,\dfrac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

55. $(x^2+4)\dfrac{dy}{dx} = 3, \quad y(2) = 0$

56. $(x^2+1)^2\dfrac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications and Examples

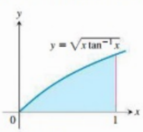
57. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

58. **Area** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

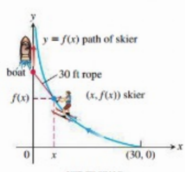
59. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.
 a. Find the area of the region.
 b. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \sqrt{x}\tan^{-1} x$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



61. Evaluate $\int x^3\sqrt{1-x^2}\,dx$ using
 a. integration by parts.
 b. a u-substitution.
 c. a trigonometric substitution.

62. **Path of a water skier** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.
 a. Show that $f'(x) = \dfrac{-\sqrt{900-x^2}}{x}$.
(*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)
 b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



63. Find the average value of $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$ on the interval $[1, 3]$.

64. Find the length of the curve $y = 1 - e^{-x}$, $0 \le x \le 1$.

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Section 8.5:

The Method of Partial Fractions

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution
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Today's Learning Goals

- Review partial fraction decomposition from algebra
- Learn to write partial fraction decompositions for functions with denominators that factor into products of linear and/or irreducible quadratic terms
- Evaluate integrals using the method of partial fractions

When to Use Partial Fractions:

both polynomials $\to \dfrac{p(x)}{q(x)}$ } rational function

- Use the method of partial fractions to evaluate the integral of a *rational function* when:
  - The degree of the numerator is *less than* that of the denominator.
  - The denominator can be *completely factored* into linear and/or irreducible quadratic terms.

Ex. $\boxed{\dfrac{x+4}{x^3-5x-6}}$

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a "1", factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.
3. Factor the denominator completely into linear and/or irreducible quadratic terms.
4. For each linear term of the form $(x-a)^k$, you will have k partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\dots+\frac{A_k}{(x-a)^k}}$$

(note: if k=1, there is only one fraction, etc.)

5. For each irreducible quadratic term of the form $(x^2+bx+c)^m$, you will have m partial fractions of the form:

$$\boxed{\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\frac{A_3x+B_3}{(x^2+bx+c)^3}+\dots+\frac{A_mx+B_m}{(x^2+bx+c)^m}}$$

(note: if m=1, there is only one fraction, etc.)

6. Solve for all the constants A_i and B_i. To solve:
   - Multiply everything by the common denominator.
   - Combine all like terms, then solve equations for all the A_i and B_i terms; OR plug in values to find equations for A_i and B_i terms.
7. Integrate using all the integration methods we have learned.

Ex 1.

$$\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{A}{(x+6)} + \frac{B}{(x-1)}\,dx$$

Step 1: Factor the denominator

$$x^2+5x-6 = (x+6)(x-1)$$

Put them in denominator of two new fractions w/ arbitrary constants as numerator. Set equal to function you want to integrate.

$$\frac{x+4}{x^2+5x-6} \overset{*}{=} \frac{A}{x+6} + \frac{B}{x-1}$$

Step 2: Solve for A & B

$$\frac{A}{x+6}+\frac{B}{x-1} = \frac{A(x-1)}{(x+6)(x-1)} + \frac{B(x+6)}{(x+6)(x-1)}$$

$$= \frac{A(x-1)+B(x+6)}{x^2+5x-6} \overset{*}{=} \frac{x+4}{x^2+5x-6}$$

Solve by setting numerator equal.

$$A(x-1)+B(x+6) = x+4$$

$$\Rightarrow Ax - A + Bx + 6B = x+4$$

collect like terms

$$\Rightarrow (A+B)x + (-A+6B) = x+4$$

$$\Rightarrow \begin{cases} A+B=1 \\ -A+6B=4 \end{cases} \rightarrow 7B=5$$

$$\boxed{B=5/7,\ A=2/7}$$

got A & B. ✓

Ex 2

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$$

Ex 1.

$$\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{A}{(x+6)} + \frac{B}{(x-1)}\,dx$$

$\Rightarrow$ $7B=5$

$\boxed{B=5/7,\ A=2/7}$ got A & B. ✓

$$= \int \frac{2/7}{x+6} + \frac{5/7}{x-1}\,dx$$

$$= \frac{2}{7}\int \frac{1}{x+6}\,dx + \frac{5}{7}\int \frac{1}{x-1}\,dx$$

① ②

① $\boxed{\begin{array}{l} u = x+6 \\ du = dx \end{array}}$

$\int \frac{1}{u}\,du = \ln|u| + C$

② $\boxed{\begin{array}{l} u = x-1 \\ du = dx \end{array}}$

$$= \boxed{\frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C}$$

Ex 2

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$$

Ex 2

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = \int_1^{\sqrt{3}} \frac{A}{t} + \frac{Bt+C}{t^2+1}\,dx$$

degree must be 1

numerator has degree ONE LESS than denominator.

degree of t^2+1 is $\underline{\underline{2}}$

Step 1:

$$t^3+t = t(t^2+1)$$

Solve for A, B, C

$$\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$

Step 2: mult. out & compare numerators

$$\frac{A}{t}\cdot\frac{t^2+1}{t^2+1} + \frac{Bt+C}{t^2+1}\cdot\frac{t}{t} = \frac{\boxed{At^2+A+Bt^2+Ct}}{t(t^2+1)} = \frac{\boxed{3t^2+t+4}}{t^3+t}$$

$$\Rightarrow (A+B)t^2 + Ct + A = 3t^2 + 1t + 4$$

$\Rightarrow$ $A+B=3$

$$\boxed{\begin{aligned} C&=1\\ A&=4\\ B&=-1 \end{aligned}}$$

got A, B, C

$$\frac{1+2}{3} = \frac{1}{3}+\frac{2}{3} \quad ✓$$

$$\frac{1}{1+2} \neq \frac{1}{1}+\frac{1}{2} \quad ✗$$

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = \int_1^{\sqrt{3}} \frac{A}{t} + \frac{Bt+C}{t^2+1}\,dx$$

$$\frac{-t}{t^2+1} + \frac{1}{t^2+1}$$

$$= \int_1^{\sqrt{3}} \frac{4}{t} + \frac{-t+1}{t^2+1}\,dt$$

② $u=t^2+1$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$= \int_1^{\sqrt{3}} \frac{4}{t}\,dt - \int_1^{\sqrt{3}} \frac{t}{t^2+1}\,dt + \int_1^{\sqrt{3}} \frac{1}{t^2+1}\,dt$$

$$= 4\ln|t| - \frac{1}{2}\ln|t^2+1| + \tan^{-1}(t)\Big|_1^{\sqrt{3}} \; \cancel{+C}$$

Ex 3 repeated factors.

$$\int \frac{x^2-1}{x(x^2+1)^2}dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\,dx$$

repeated factor

repeat twice

Step 2.

set numerators equal.

$$\frac{\boxed{x^2-1}}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

match this denom

$$= \frac{A}{x}\cdot\frac{(x^2+1)^2}{(x^2+1)^2} + \frac{Bx+C}{x^2+1}\cdot\frac{x(x^2+1)}{x(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}\cdot\frac{x}{x}$$

$$x^2-1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx+C)(x^3+x) + Dx^2+Ex$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

$$= \underbrace{(A+B)}_{0}x^4 + \underbrace{C}_{0}x^3 + \underbrace{(2A+B+D)}_{1}x^2 + \underbrace{(C+E)}_{0}x + \underbrace{A}_{-1}$$

$A+B=0 \rightarrow B=1$

$C=0$

$2A+B+D=1 \rightarrow (2)(-1)+(1)+D=1 \rightarrow D=2$

$C+E=0 \rightarrow E=0$

$A=-1$

$A=-1$
$B=1$
$C=0$
$D=2$
$E=0$

$$\int \frac{x^2-1}{x(x^2+1)^2}dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\,dx$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin\vartheta}{\cos^2\vartheta+\cos\vartheta-2}\,d\vartheta$$

$$= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx$$

u-sub u-sub

$$= \boxed{-\ln|x| + \frac{1}{2}\ln|x^2+1| - \frac{1}{x^2+1} + C}$$

$$\int \frac{x^3 + 4x^2}{2x^2 + 8x - 10} dx$$

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Section 4.5
L'Hopital's Rule

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 Test #1 (4.8, 5.1-5.6, 8.2-8.3)	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

Today's Learning Goals

- Understand which forms are indeterminate
- Apply L'Hopital's Rule to evaluate limits
- Rewrite limits in forms appropriate to applying L'Hopital's Rule

Indeterminate Forms

$$\boxed{\frac{0}{0}, \frac{\infty}{\infty} \quad 1^{\infty}, 0^{0}, \infty^{0}}$$

$$\cancel{0\cdot\infty}, \cancel{\infty-\infty}$$

Which of the following limits does NOT contain an indeterminate form?

A. $\lim_{x\to\infty}(x+1)^{3x}$

B. $\lim_{x\to 0^+} x^{6x}$

C. $\lim_{x\to\infty} x^2 e^{-x}$

D. $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}}$

L'Hopital's Rule

Let f and g be two functions. Then IF:

a) f and g are differentiable,

b) $\frac{f(x)}{g(x)}$ has the indeterminate form of $\frac{0}{0}$ OR $\frac{\infty}{\infty}$

c) $\lim_{x\to c}\frac{f'(x)}{g'(x)} = L$

THEN: $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)} = L$

$$\lim_{x\to\infty}\frac{e^x+x^2}{e^x+x} \overset{\text{"}\frac{\infty}{\infty}\text{" L'Hop}}{=} \lim_{x\to\infty}\frac{(e^x+x^2)'}{(e^x+x)'} = \lim_{x\to\infty}\frac{e^x+2x}{e^x+1}$$

"$\frac{\infty}{\infty}$" L'Hop

$$= \lim_{x\to\infty}\frac{(e^x+2x)'}{(e^x+1)} = \lim_{x\to\infty}\frac{e^x+2}{e^x}$$

$$= \lim_{x\to\infty}\frac{e^x}{e^x}+\frac{2}{e^x} = \lim_{x\to\infty} 1+\frac{2}{e^x}$$

$$= 1+0 = \boxed{1}$$

$\ln(e)=1$

$\ln(1)=0$

$\ln(0)$ DNE.

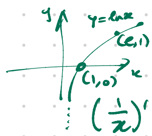
$$\lim_{x\to 0^+}(\sin(x)\cdot\ln(x))$$

Opt 1

$$= \lim_{x\to 0^+}\frac{\sin x}{1/\ln x} \quad ??$$

"$\frac{0}{0}$" L'Hop

$$= \lim_{x\to 0^+}\frac{\cos x}{\frac{-1}{(\ln x)^2}\cdot\frac{1}{x}}$$



$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

$$\left(\frac{1}{\ln x}\right)' = \frac{-1}{(\ln x)^2}\cdot\frac{1}{x}$$

$$(\ln x)' = \frac{1}{x}$$

$$\boxed{\lim_{x\to\infty}\frac{1}{x}\cdot x}$$

① $\lim_{x\to a} f(x)\cdot g(x)$

$= \lim_{x\to a}$ ② $f(x) * \lim_{x\to a}$ ③ $g(x)$

if both all 3 limits exist

Opt 2

"$\frac{-\infty}{\infty}$" L'Hop

$$= \lim_{x\to 0^+}\frac{\ln x}{1/\sin x} = \lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}$$

"$\frac{0}{0}$" L'Hop

$$= \lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0^+}\frac{2\sin x\cdot\cos x}{x(-\sin x)+1\cdot\cos x}$$

$$= \frac{2(0)\cdot 1}{(0)(0)+1(1)} = \frac{0}{1} = \boxed{0}$$

Evaluate the limit:

$(3^x-1)' =$

$$\lim_{x\to 0}\frac{3^x-1}{4^x-1}$$

$(3^x)' \stackrel{?}{=} (e^{x\cdot\ln 3})' = \ln 3\, e^{\ln 3\cdot x} = \boxed{\ln 3\cdot 3^x}$

$3^x = (e^{\ln 3})^x$

$= e^{\ln 3 x}$

A. 0

B. 1

C. ln(3/4)

D. (ln3)/(ln4)

"$\frac{0}{0}$" L'Hop

$= \lim_{x\to 0}\frac{\ln 3\cdot 3^x - \cancel{0}}{\ln 4\cdot 4^x - \cancel{0}}$

$= \frac{\ln 3\cdot 3^0}{\ln 4\cdot 4^0} = \frac{\ln 3\cdot 1}{\ln 4\cdot 1}$

Use L'Hopital's rule and logarithms to evaluate the following limits.

$$\text{Logarithm rule}: \lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))} = e^{\lim_{x\to a}\ln(f(x))}$$

$$\lim_{x\to 0^+} x^{\frac{1}{\ln(5x)}}$$

y

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$$

$\ln(a^b) = b\ln a$

Set $y = \left(1+\frac{a}{x}\right)^x$

Then $\ln y = \ln\left(\left(1+\frac{a}{x}\right)^x\right)$

$= \underbrace{x\ln\left(1+\frac{a}{x}\right)}_{\text{"}\ln y\text{"}} \leftarrow$ "$\infty * 0$"

opt 1

$= \frac{x}{1/\ln\left(1+\frac{a}{x}\right)}$ or

opt2

$= \frac{\ln\left(1+\frac{a}{x}\right)}{1/x}$ ✓ ← "$\frac{0}{0}$" L'Hop

$$\boxed{\lim_{x\to\infty}\ln y} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{1/x} \stackrel{\text{L'Hop "}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+a/x} * \frac{-a}{x^2}}{-1/x^2} = \lim_{x\to\infty}\frac{+a}{1+\frac{a}{x}}\cdot\frac{\cancel{x^2}\,1}{\cancel{x^2}} = \boxed{a}\ ??$$

$\lim_{x\to\infty}\ln y = a$ So $\lim_{x\to\infty} e^{\ln y} = \boxed{e^a}$

Evaluate the limit:

$$\lim_{x\to 0^+}(1+2x)^{\frac{1}{x}}$$

A. e^2

B. $e^{1/2}$

C. 1

D. Infinity

Some Common Limits

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ 5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ 7) $\lim_{n\to\infty} n^{1/n} = 1$

Section 4.5: 25, 42, 51, 60 (extra practice: 13, 15, 42, 57, 63)

EXERCISES 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x \to -2} \frac{x+2}{x^2-4}$
2. $\lim_{x \to 0} \frac{\sin 5x}{x}$
3. $\lim_{x \to \infty} \frac{5x^2-3x}{7x^2+1}$
4. $\lim_{x \to 1} \frac{x^3-1}{4x^3-x-3}$
5. $\lim_{x \to 0} \frac{1-\cos x}{x^2}$
6. $\lim_{x \to \infty} \frac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x \to 2} \frac{x-2}{x^2-4}$
8. $\lim_{x \to -5} \frac{x^2-25}{x+5}$
9. $\lim_{t \to -3} \frac{t^3-4t+15}{t^2-t-12}$
10. $\lim_{t \to -1} \frac{3t^3+3}{4t^3-t+3}$
11. $\lim_{x \to \infty} \frac{5x^3-2x}{7x^3+3}$
12. $\lim_{x \to \infty} \frac{x-8x^2}{12x^2+5x}$
13. $\lim_{t \to 0} \frac{\sin t^2}{t}$
14. $\lim_{t \to 0} \frac{\sin 5t}{2t}$
15. $\lim_{x \to 0} \frac{8x^2}{\cos x - 1}$
16. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$
17. $\lim_{\theta \to \pi/2} \frac{2\theta - \pi}{\cos(2\pi - \theta)}$
18. $\lim_{\theta \to -\pi/3} \frac{3\theta + \pi}{\sin(\theta + (\pi/3))}$
19. $\lim_{\theta \to \pi/2} \frac{1-\sin\theta}{1+\cos 2\theta}$
20. $\lim_{x \to 1} \frac{x-1}{\ln x - \sin \pi x}$
21. $\lim_{x \to 0} \frac{x^2}{\ln(\sec x)}$
22. $\lim_{x \to \pi/2} \frac{\ln(\csc x)}{(x-(\pi/2))^2}$
23. $\lim_{t \to 0} \frac{t(1-\cos t)}{t - \sin t}$
24. $\lim_{t \to 0} \frac{t \sin t}{1-\cos t}$
25. $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \sec x$
26. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x$
27. $\lim_{\theta \to 0} \frac{3^{\sin\theta}-1}{\theta}$
28. $\lim_{\theta \to 0} \frac{(1/2)^\theta - 1}{\theta}$
29. $\lim_{x \to 0} \frac{x2^x}{2^x-1}$
30. $\lim_{x \to 0} \frac{3^x-1}{2^x-1}$
31. $\lim_{x \to \infty} \frac{\ln(x+1)}{\log_2 x}$
32. $\lim_{x \to \infty} \frac{\log_2 x}{\log_3(x+3)}$
33. $\lim_{x \to 0^+} \frac{\ln(x^2+2x)}{\ln x}$
34. $\lim_{x \to 0^+} \frac{\ln(e^x-1)}{\ln x}$
35. $\lim_{y \to 0} \frac{\sqrt{5y+25}-5}{y}$
36. $\lim_{y \to 0} \frac{\sqrt{ay+a^2}-a}{y}, \quad a > 0$
37. $\lim_{x \to \infty} (\ln 2x - \ln(x+1))$
38. $\lim_{x \to 0^+} (\ln x - \ln \sin x)$
39. $\lim_{x \to 0^+} \frac{(\ln x)^2}{\ln(\sin x)}$
40. $\lim_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$
41. $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$
42. $\lim_{x \to 0^+} (\csc x - \cot x + \cos x)$
43. $\lim_{\theta \to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1}$
44. $\lim_{h \to 0} \frac{e^h - (1+h)}{h^2}$
45. $\lim_{t \to \infty} \frac{e^t + t^2}{e^t - t}$
46. $\lim_{x \to \infty} x^2 e^{-x}$
47. $\lim_{x \to 0} \frac{x - \sin x}{x \tan x}$
48. $\lim_{x \to 0} \frac{(e^x-1)^2}{x \sin x}$
49. $\lim_{\theta \to 0} \frac{\theta - \sin\theta\cos\theta}{\tan\theta - \theta}$
50. $\lim_{x \to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x}$

Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x \to 1^+} x^{1/(1-x)}$
52. $\lim_{x \to 1^+} x^{1/(x-1)}$
53. $\lim_{x \to \infty} (\ln x)^{1/x}$
54. $\lim_{x \to e^+} (\ln x)^{1/(x-e)}$
55. $\lim_{x \to 0^+} x^{-1/\ln x}$
56. $\lim_{x \to \infty} x^{1/\ln x}$
57. $\lim_{x \to \infty} (1+2x)^{1/(2\ln x)}$
58. $\lim_{x \to 0} (e^x + x)^{1/x}$
59. $\lim_{x \to 0^+} x^x$
60. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$
61. $\lim_{x \to \infty} \left(\frac{x+2}{x-1}\right)^x$
62. $\lim_{x \to \infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$
63. $\lim_{x \to 0^+} x^2 \ln x$
64. $\lim_{x \to 0^+} x(\ln x)^2$
65. $\lim_{x \to 0^+} x \tan\left(\frac{\pi}{2} - x\right)$
66. $\lim_{x \to 0^+} \sin x \cdot \ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x \to \infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$
68. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$
69. $\lim_{x \to (\pi/2)^-} \frac{\sec x}{\tan x}$
70. $\lim_{x \to 0^+} \frac{\cot x}{\csc x}$
71. $\lim_{x \to \infty} \frac{2^x - 3^x}{3^x + 4^x}$
72. $\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x}$
73. $\lim_{x \to \infty} \frac{e^{x^2}}{xe^x}$
74. $\lim_{x \to 0^+} \frac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $\lim_{x \to 3} \frac{x-3}{x^2-3} = \lim_{x \to 3} \frac{1}{2x} = \frac{1}{6}$ b. $\lim_{x \to 3} \frac{x-3}{x^2-3} = \frac{0}{6} = 0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $\lim_{x \to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x \to 0} \frac{2x-2}{2x-\cos x}$
 $= \lim_{x \to 0} \frac{2}{2+\sin x} = \frac{2}{2+0} = 1$

 b. $\lim_{x \to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x \to 0} \frac{2x-2}{2x-\cos x} = \frac{-2}{0-1} = 2$

Section 8.5: 11, 13, 23, 33, 39, 41 (extra practice: 9, 17, 35)

EXERCISES 8.5

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\frac{5x-13}{(x-3)(x-2)}$
2. $\frac{5x-7}{x^2-3x+2}$
3. $\frac{x+4}{(x+1)^2}$
4. $\frac{2x+2}{x^2-2x+1}$
5. $\frac{z+1}{z^2(z-1)}$
6. $\frac{z}{z^3-z^2-6z}$
7. $\frac{t^2+8}{t^2-5t+6}$
8. $\frac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\int \frac{dx}{1-x^2}$
10. $\int \frac{dx}{x^2+2x}$
11. $\int \frac{x+4}{x^2+5x-6}\,dx$
12. $\int \frac{2x+1}{x^2-7x+12}\,dx$
13. $\int_4^8 \frac{y\,dy}{y^2-2y-3}$
14. $\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$
15. $\int \frac{dt}{t^3+t^2-2t}$
16. $\int \frac{x+3}{2x^3-8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
18. $\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
19. $\int \frac{dx}{(x^2-1)^2}$
20. $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$
22. $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$
23. $\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$
24. $\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$
25. $\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$
26. $\int \frac{s^4+81}{s(s^2+9)^2}\,ds$
27. $\int \frac{x^2-x+2}{x^3-1}\,dx$
28. $\int \frac{1}{x^4+x}\,dx$
29. $\int \frac{x^2}{x^4-1}\,dx$
30. $\int \frac{x^2+x}{x^4-3x^2-4}\,dx$
31. $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$
32. $\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$
34. $\int \frac{x^4}{x^2-1}\,dx$
35. $\int \frac{9x^3-3x+1}{x^3-x^2}\,dx$
36. $\int \frac{16x^3}{4x^2-4x+1}\,dx$
37. $\int \frac{y^4+y^2-1}{y^3+y}\,dy$
38. $\int \frac{2y^4}{y^3-y^2+y-1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–54.

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$
40. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$
41. $\int \frac{\cos y\,dy}{\sin^2 y+\sin y-6}$
42. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$
43. $\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{(4x^2+1)(x-2)^2}\,dx$
44. $\int \frac{(x+1)^2\tan^{-1}(3x)+9x^3+x}{(9x^2+1)(x+1)^2}\,dx$
45. $\int \frac{1}{x^{3/2}-\sqrt{x}}\,dx$
46. $\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,dx$ (*Hint:* Let $x = u^6$.)
47. $\int \frac{\sqrt{x+1}}{x}\,dx$ (*Hint:* Let $x + 1 = u^2$.)
48. $\int \frac{1}{x\sqrt{x+9}}\,dx$
49. $\int \frac{1}{x(x^4+1)}\,dx$ (*Hint:* Multiply by $\frac{x^3}{x^3}$.)
50. $\int \frac{1}{x^6(x^5+4)}\,dx$
51. $\int \frac{1}{\cos 2\theta\sin\theta}\,d\theta$
52. $\int \frac{1}{\cos\theta+\sin 2\theta}\,d\theta$
53. $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$
54. $\int \frac{\sqrt{x}}{\sqrt{2-\sqrt{x}}+\sqrt{x}}\,dx$

Use any method to evaluate the integrals in Exercises 55–66.

55. $\int \frac{x^3-2x^2-3x}{x+2}\,dx$
56. $\int \frac{x+2}{x^3-2x^2-3x}\,dx$
57. $\int \frac{2^x-2^{-x}}{2^x+2^{-x}}\,dx$
58. $\int \frac{2^x}{2^{2x}+2^x-2}\,dx$
59. $\int \frac{1}{x^4-1}\,dx$
60. $\int \frac{x^4-1}{x^5-5x+1}\,dx$
61. $\int \frac{\ln x+2}{x(\ln x+1)(\ln x+3)}\,dx$
62. $\int \frac{2}{x(\ln x-2)^3}\,dx$
63. $\int \frac{1}{\sqrt{x^2-1}}\,dx$
64. $\int \frac{x}{x+\sqrt{x^2+2}}\,dx$
65. $\int x^5\sqrt{x^3+1}\,dx$
66. $\int x^2\sqrt{1-x^2}\,dx$

Initial Value Problems

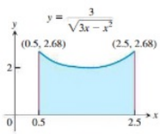
Solve the initial value problems in Exercises 67–70 for x as a function of t.

67. $(t^2-3t+2)\frac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$
68. $(3t^4+4t^2+1)\frac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$
69. $(t^2+2t)\frac{dx}{dt} = 2x+2 \quad (t, x > 0), \quad x(1) = 1$
70. $(t+1)\frac{dx}{dt} = x^2+1 \quad (t > -1), \quad x(0) = 0$

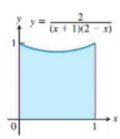
Applications and Examples

In Exercises 71 and 72, find the volume of the solid generated by revolving the shaded region about the indicated axis.

71. The x-axis



72. The y-axis



73. Find the length of the curve $y = \ln(1-x^2)$, $0 \le x \le \frac{1}{2}$.

74. Integrate $\int \sec\theta\,d\theta$ by

a. multiplying by $\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$ and then using a u-substitution.

b. writing the integral as $\int \frac{1}{\cos\theta}\,d\theta$. Then multiply by $\frac{\cos\theta}{\cos\theta}$, use a trigonometric identity and a u-substitution, and finally integrate using partial fractions.

Math 1552

Section 8.8
Improper Integrals

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 **Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)**	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

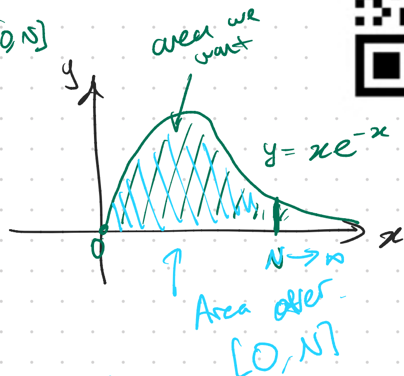
https://strawpoll.com/polls/2ayLkVVOVZ4

Ex. $\displaystyle\int_0^\infty x e^{-x}\,dx$

Step 1: Replace ∞ with N

Step 2: Integrate definite integral on interval $[0, N]$

Step 3: Take the limit



Step 1. replace ∞ w/ N.

$$\int_0^N x e^{-x}\,dx$$

$\int u\,dv = uv - \int v\,du$

Step 2: Integrate to get a formula for area over $[0,N]$.

IBP BOX

$u = x$	$dv = e^{-x}dx$
$du = dx$	$v = -e^{-x}$

$$= -xe^{-x}\Big|_0^N + \int_0^N +e^{-x}\,dx$$

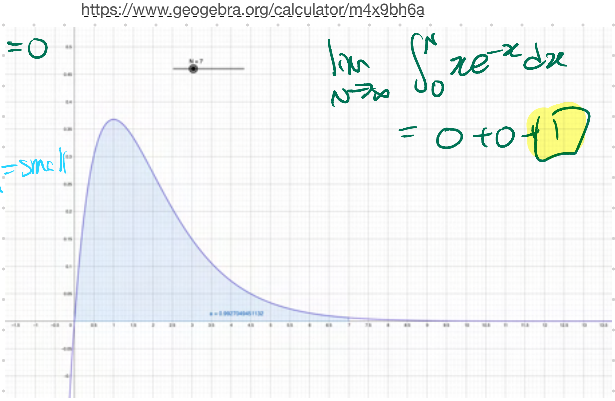
$$= -xe^{-x}\Big|_0^N - e^{-x}\Big|_0^N = -xe^{-x} - e^{-x}\Big|_0^N$$

$$= \left(-Ne^{-N} - e^{-N}\right) - \left(-0e^{-0} - e^{-0}\right) = \boxed{\underset{①}{-Ne^{-N}} \underset{②}{-e^{-N}} \underset{③}{+1}}$$

Geogebra w/ slider.

① $\displaystyle\lim_{N\to\infty} -Ne^{-N} = \lim_{N\to\infty} \frac{-N}{e^N} \overset{\text{"}\frac{\infty}{\infty}\text{" L'Hop}}{=} \lim_{N\to\infty} \frac{-1}{e^N} = 0$

https://www.geogebra.org/calculator/m4x9bh6a



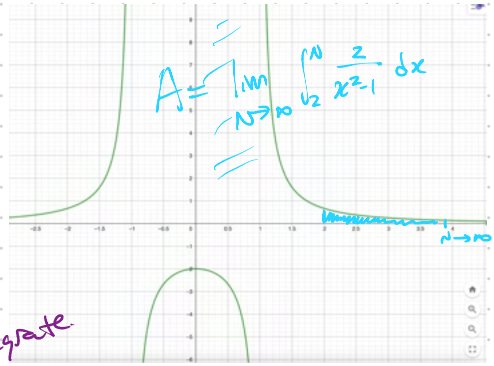
$$\lim_{N\to\infty} \int_0^N xe^{-x}\,dx = 0 + 0 + 1$$

② $\displaystyle\lim_{N\to\infty} e^{-N} = \lim_{N\to\infty} \frac{1}{e^N} = 0$ $\quad \frac{1}{BIG} = \text{small}$

③ $\displaystyle\lim_{N\to\infty} 1 = 1$

Example 2: Find the area

$$A = \int_2^\infty \frac{2}{x^2-1}\,dx$$



Step 1: $\int_2^\infty \frac{2}{x^2-1}\,dx = \lim_{N\to\infty} \int_2^N \frac{2}{x^2-1}$

horizontal asymptote

Step 2: Evaluate definite integral and write answer in terms of a function of N.

Step 3: take limit as $N\to\infty$

?? how to integrate

$$\boxed{\int_2^N \frac{2}{x^2-1}\,dx} = \int_2^N \frac{-1}{x+1} + \frac{1}{x-1}\,dx$$

partial fractions.

integrate

$$\frac{2}{x^2-1} = \frac{\boxed{2}}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{\boxed{A(x-1)+B(x+1)}}{(x+1)(x-1)}$$

$$\Rightarrow 2 = A(x-1) + B(x+1)$$

$$\Rightarrow 2 = Ax - A + Bx + B$$

$$\Rightarrow 2 = (A+B)x + (-A+B)$$

$$\begin{cases} \boxed{A+B=0} \\ -A+B=2 \end{cases}$$

$0A + 2B = 2$

$\boxed{B=1}$

$B=1$ $\boxed{A=-1}$

$$\boxed{\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)}$$

$$= -\ln(x+1) + \ln(x-1)\Big|_2^N = \left(-\ln(N+1) + \ln(N-1)\right) - \left(-\ln(3) + \ln(1)\right)$$

0

$$= -\ln(N+1) + \ln(N-1) + \ln(3)$$

Next Step 3: Take limit as $N\to\infty$

$$= \ln\left(\frac{N-1}{N+1}\right) + \ln(3) \xrightarrow[N\to\infty]{} \overset{0}{\ln(1)} + \boxed{\ln(3)}$$

SIDE NOTE: "$\infty - \infty$" is not always 0.

For example

Example

$$\lim_{x\to+\infty} \sqrt{x^2-2} - x = \lim_{x\to+\infty} \frac{(\sqrt{x^2-x}-x)\cdot(\sqrt{x^2-x}+x)}{\sqrt{x^2-x}+2} =$$

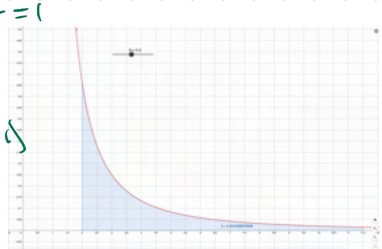
$$= \lim_{x\to+\infty} \frac{x^2-x-x^2}{\sqrt{x^2-x}+x} = \lim_{x\to+\infty} \frac{-x}{\sqrt{x^2-x}+x} =$$

$$= \lim_{x\to+\infty} \frac{-x}{x^{\frac{2}{2}}+x} = \lim_{x\to+\infty} \frac{-x}{2x} = \frac{-1}{2}$$

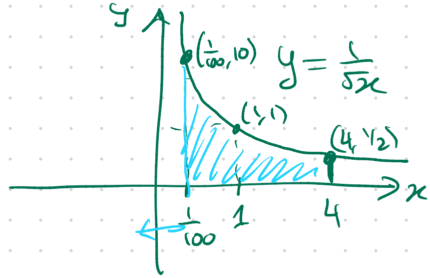
$$\lim_{N\to\infty} \frac{N-1}{N+1} \overset{\frac{\infty}{\infty}}{=} \lim_{N\to\infty} \frac{1}{1} = 1$$

So

$$\lim_{N\to\infty} \ln\left(\frac{N-1}{N+1}\right) = \ln(1)$$



Ex. Evaluate. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



Step 1: replace the x-value where the asymptote is with ε and take the limit.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^4 \frac{1}{\sqrt{x}}\,dx$$

Step 2: Evaluate the definite integral

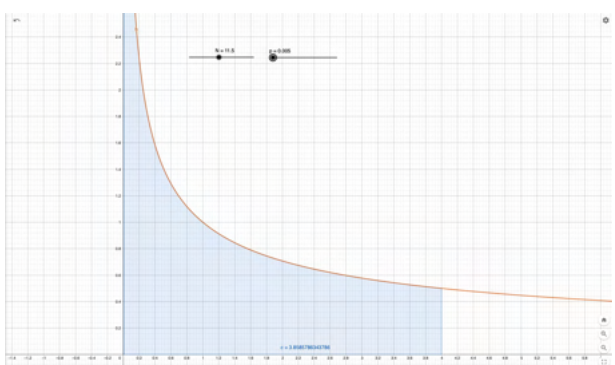
$$\int_\varepsilon^4 \frac{1}{\sqrt{x}}\,dx = \int_\varepsilon^4 x^{-1/2}\,dx = \left.\frac{x^{-1/2+1}}{-1/2+1}\right|_\varepsilon^4 = \left.\frac{x^{1/2}}{1/2}\right|_\varepsilon^4$$

$$= \left.2\sqrt{x}\,\right|_\varepsilon^4 = 2\sqrt{4} - 2\sqrt{\varepsilon}$$

$$= 4 - 2\sqrt{\varepsilon}$$

Step 3: take the limit as $\varepsilon \to 0^+$

$$\lim_{\varepsilon \to 0^+} 4 - 2\sqrt{\varepsilon} = 4 - 2\sqrt{0} = \boxed{4}$$



Ex.

$$\boxed{\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx} = \lim_{N\to-\infty} \int_N^0 \frac{1}{1+x^2}\,dx \quad \text{or} \quad \lim_{N\to\infty} \int_{-N}^{0} \frac{1}{1+x^2}\,dx$$

I am an improper integral

$\tan^{-1}(a) = \theta$

$\Rightarrow \tan(\theta) = a$

Improper integrals

A definite integral is improper if:

- The function has a vertical asymptote at $x=a$, $x=b$, or at some point c in the interval (a,b).
- One or both of the limits of integration are infinite (positive or negative infinity).

$$\int_N^0 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_N^0$$

$$= \tan^{-1}(0) - \tan^{-1}(N) = -\left(-\frac{\pi}{2}\right)$$

$$= \boxed{\pi/2}$$

$$\lim_{N\to-\infty} \tan^{-1}(N) = -\frac{\pi}{2} \quad \text{b/c}$$

vert. asymptote

Which integral(s) is (are) improper?

1) $\int_0^{\pi} \tan(2x)\,dx$ improper

2) $\int_0^{?} \frac{x-3}{x^2-2x-3}\,dx$ $\qquad \frac{x-3}{x^2-2x-3} = \frac{x-3}{(x-3)(x+1)}$

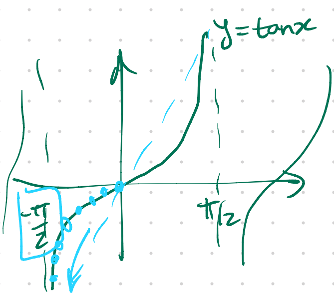
3) $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

4) $\int_0^{3} \frac{x-2}{x^2-6x+8}\,dx$ $\qquad \frac{x-2}{x^2-6x+8} = \frac{x-2}{(x-2)(x-4)}$

definite not improper.

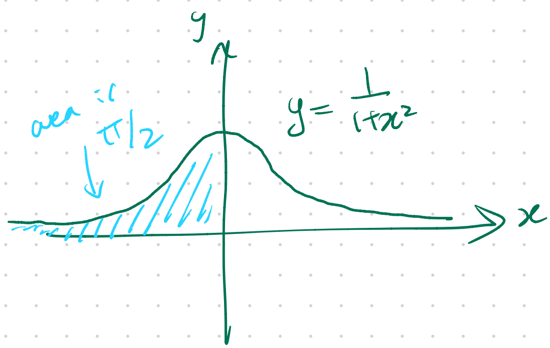
$$\boxed{\lim_{x\to-\frac{\pi}{2}^+} \tan(x) = -\infty \quad \text{DNE}}$$

FACT



Convergence of an Integral

- If an improper integral evaluates to a finite number, we say it *converges*.
- If the integral evaluates to $\pm\infty$ or to, $\infty - \infty$, we say the integral *diverges*.



$$\int_{\frac{\pi}{2}}^{\pi} \tan(x)\,dx$$

$$\int_{-1}^{32} x^{-1/5}\,dx$$

$$\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$$

idea

$x = \frac{1}{4}\sin x$?

$\frac{1}{2}\sin x$?

$\sin x$?

$1-\sin^2 x = \cos^2 x$

$$\int \frac{x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{1}{\sqrt{1+u^2}}\,du$$

u-sub Box

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

More Trig Sub practice

If $t = \ln 4$ then $x = e^{\ln 4} = 4$

If $t=0$ $x = e^0 = 1$

$$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_1^4 \frac{1}{\sqrt{x^2+9}}\,dx = \int_*^* \frac{1}{\sqrt{(3\tan\theta)^2+9}}\,3\sec^2\theta\,d\theta$$

u-sub Box

$x = e^t$

$dx = e^t\,dt$

trig sub Box

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$$= \int_*^* \frac{1}{\sqrt{9\tan^2\theta+9}}\,3\sec^2\theta\,d\theta$$

$$= 3\int_*^* \frac{\sec^2\theta}{\sqrt{9\sec^2\theta}}\,d\theta = \cancel{3}\int_*^* \frac{\sec^{\cancel{2}}\theta}{\cancel{3}\cancel{\sec\theta}}\,d\theta$$

x^2+a^2

$x = a\tan\theta$

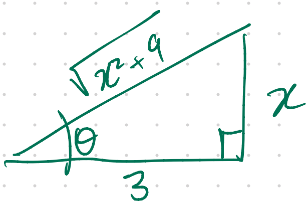
$x^2+a^2 = a^2\tan^2\theta + a^2 = a^2(\tan^2\theta+1) = a^2\sec^2\theta$

$$\sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta} = \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}$$

$$= \int_*^* \sec\theta\,d\theta$$

??

$$= \ln\left|\sec\theta+\tan\theta\right|\Big|_*^*$$



$$\boxed{\sec\theta = \frac{\sqrt{x^2+9}}{3}}$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\Big|_1^4$$

$x = 3\tan\theta$

$$\boxed{\frac{x}{3} = \tan\theta}$$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$$= \ln\left|\frac{\sqrt{4^2+9}}{3}+\frac{4}{3}\right| - \ln\left|\frac{\sqrt{10}}{3}+\frac{1}{3}\right|$$

$$= \ln(3) - \ln\left(\frac{\sqrt{10}+1}{3}\right)$$

$$= \ln\left(3\Big/\left(\frac{\sqrt{10}+1}{3}\right)\right) = \ln\left(\frac{9}{\sqrt{10}+1}\right)$$

$\ln(a) - \ln(b) = \ln(a/b)$

Section 8.4: 5, 11, 13, 17, 19, 29, 35 (extra practice: 39)

EXERCISES 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\int \frac{dx}{\sqrt{9+x^2}}$

2. $\int \frac{3\,dx}{\sqrt{1+9x^2}}$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\int_0^2 \frac{dx}{8+2x^2}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\int \sqrt{25-t^2}\,dt$

8. $\int \sqrt{1-9t^2}\,dt$

9. $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\int \frac{x}{\sqrt{9-x^2}}\,dx$

16. $\int \frac{x^2}{4+x^2}\,dx$

17. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

18. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

19. $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

20. $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

21. $\int \sqrt{\frac{x+1}{1-x}}\,dx$

22. $\int x\sqrt{x^2-4}\,dx$

23. $\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

24. $\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$

25. $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

26. $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

27. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

28. $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

29. $\int \frac{8\,dx}{(4x^2+1)^2}$

30. $\int \frac{6\,dt}{(9t^2+1)^2}$

31. $\int \frac{x^3\,dx}{x^2-1}$

32. $\int \frac{x\,dx}{25+4x^2}$

33. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

34. $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

36. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

37. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

38. $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

39. $\int \frac{dx}{x\sqrt{x^2-1}}$

40. $\int \frac{dx}{1+x^2}$

41. $\int \frac{x\,dx}{\sqrt{x^2-1}}$

42. $\int \frac{dx}{\sqrt{1-x^2}}$

43. $\int \frac{x\,dx}{\sqrt{1+x^4}}$

44. $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

45. $\int \sqrt{\frac{4-x}{x}}\,dx$
(*Hint:* Let $x = u^2$.)

46. $\int \sqrt{\frac{x}{1-x^3}}\,dx$
(*Hint:* Let $u = x^{3/2}$.)

47. $\int \sqrt{x}\sqrt{1-x}\,dx$

48. $\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Complete the Square Before Using Trigonometric Substitutions

For Exercises 49–52, complete the square before using an appropriate trigonometric substitution.

49. $\int \sqrt{8-2x-x^2}\,dx$

50. $\int \frac{1}{\sqrt{x^2-2x+5}}\,dx$

51. $\int \frac{\sqrt{x^2+4x+3}}{x+2}\,dx$

52. $\int \frac{\sqrt{x^2+2x+2}}{x^2+2x+1}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 53–56 for y as a function of x.

53. $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

54. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

55. $(x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

56. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

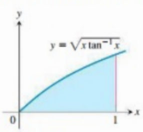
Applications and Examples

57. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

58. **Area** Find the area enclosed by the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

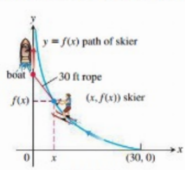
59. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.
   a. Find the area of the region.
   b. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \sqrt{x}\tan^{-1} x$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



61. Evaluate $\int x^3\sqrt{1-x^2}\,dx$ using
   a. integration by parts.
   b. a u-substitution.
   c. a trigonometric substitution.

62. **Path of a water skier** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.
   a. Show that $f'(x) = \frac{-\sqrt{900-x^2}}{x}$.
   (*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)
   b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



NOT TO SCALE

63. Find the average value of $f(x) = \frac{\sqrt{x+1}}{\sqrt{x}}$ on the interval [1, 3].

64. Find the length of the curve $y = 1 - e^{-x}$, $0 \le x \le 1$.

Section 8.8: 1, 4, 11, 21, 71 (extra practice: 7, 13, 15, 45)

EXERCISES 8.8

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^{\infty} \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^{1} \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^{1} \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^{1} \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^{\infty} \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^{\infty} \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\displaystyle\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$

22. $\displaystyle\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^{0} e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^{1} x\ln x\,dx$

26. $\displaystyle\int_0^{1} (-\ln x)\,dx$

27. $\displaystyle\int^{2} \frac{ds}{}$ 28. $\displaystyle\int^{1} \frac{4r\,dr}{}$

27. $\displaystyle\int_0^{2} \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^{1} \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^{2} \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^{4} \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^{2} \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–68, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_{1/2}^{2} \frac{dx}{x\ln x}$

36. $\displaystyle\int_{-1}^{1} \frac{d\theta}{\theta^2-2\theta}$

37. $\displaystyle\int_{1/2}^{\infty} \frac{dx}{x(\ln x)^3}$

38. $\displaystyle\int_0^{\infty} \frac{d\theta}{\theta^2-1}$

39. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

40. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

41. $\displaystyle\int_0^{1} \frac{\ln x}{x^2}\,dx$

42. $\displaystyle\int_1^{2} \frac{dx}{x\ln x}$

43. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

44. $\displaystyle\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

45. $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

46. $\displaystyle\int_0^{1} \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

47. $\displaystyle\int_0^{2} \frac{dx}{1-x^2}$

48. $\displaystyle\int_0^{2} \frac{dx}{1-x}$

49. $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

50. $\displaystyle\int_{-1}^{1} -x\ln|x|\,dx$

51. $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$

52. $\displaystyle\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$

53. $\displaystyle\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

54. $\displaystyle\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$

55. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$

56. $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

57. $\displaystyle\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

58. $\displaystyle\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$

59. $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

60. $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$

61. $\displaystyle\int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$

62. $\displaystyle\int_2^{\infty} \frac{1}{\ln x}\,dx$

$\displaystyle\int_4 \; t^{3/2}-1$ $\displaystyle\int_2 \; \ln x$

63. $\displaystyle\int_1^{\infty} \frac{e^x}{x}\,dx$

64. $\displaystyle\int_{e^e}^{\infty} \ln(\ln x)\,dx$

65. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{e^x-x}}\,dx$

66. $\displaystyle\int_1^{\infty} \frac{1}{e^x-2^x}\,dx$

67. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$

68. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

Theory and Examples

69. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^{2} \frac{dx}{x(\ln x)^p}$ b. $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

Math 1552

Sections 10.1:

Sequences

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 Test #1 (4.8, 5.1-5.6, 8.2-8.3)	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 NO CLASS Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, ...

Today's Learning Goals

- Use proper notation to denote a sequence.
- Understand how to find lower and upper bounds for sequences.
- Determine if a sequence is monotonic.
- Find limits of sequences when possible.

Ex. Find a formula for the sequence and determine the limit.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \ldots, \frac{101}{102}$

$\boxed{a_n = \frac{n}{n+1}, \quad n \geq 1}$

closed Formula for the sequence.

any are ok

$a_n = \frac{n}{n+1}, \quad n \geq 3 \rightarrow \frac{3}{4}, \frac{4}{4+1}, \frac{5}{6}, \ldots$

$n \geq 3$

$\frac{4}{5}$

$\boxed{a_n = \frac{n-1}{n}, \quad n \geq 2}$

$\boxed{a_n = \frac{n+1}{n+2}, \quad n \geq 0}$

Write the general term of the sequence below.

$-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots$

A) $a_n = \frac{(-1)^n n}{n+1}$

B) $a_n = \frac{(-1)^{n+1} n}{n+1}$

C) $a_n = \frac{(-1)^n (n+1)}{n+2}$

D) $a_n = \frac{(-1)^{n+1} (n+1)}{n+2}$

What is the limit of the sequence?

$\lim_{n\to\infty} a_n =$ L'Hop "$\frac{\infty}{\infty}$"

$= \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{1} = \boxed{1}$

(b) $\frac{\ln(2)}{3}, -\frac{\ln(3)}{5}, \frac{\ln(4)}{7}, -\frac{\ln(5)}{9}, \ldots$

Q_1: Find a_n a general formula (don't forget $n \geq n_0$)

"n_0" first n.

Q_2: Find $\lim_{n\to\infty} a_n = ?$

??

$\boxed{a_n = (-1)^{n+1} \frac{\ln(n+1)}{n+2}, \quad n \geq 1}$

$n=1$

$a_1 = \frac{\ln(1+1)}{1+2} = \frac{\ln(2)}{3}$

$n=2$

$a_2 = \frac{\ln(2+1)}{2+2} = \frac{\ln(3)}{4}$

(c) $1, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{4}}, \frac{4}{\sqrt{5}}, \frac{5}{\sqrt{6}}, \ldots$

4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 Test #1 (4.8, 5.1-5.6, 8.2-8.3)	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series

Fibonacci Sequence

$1, 1, 2, 3, 5, 8, 13, \ldots$

Today's Learning Goals

- Use proper notation to denote a sequence.
- Understand how to find lower and upper bounds for sequences.
- Determine if a sequence is monotonic.
- Find limits of sequences when possible.

Write the general term of the sequence below.

$-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots$ $\quad -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots$

X A) $a_n = \frac{(-1)^n n}{n+1}$, $n \geq 2$

✓ B) $a_n = \frac{(-1)^{n+1} n}{n+1}$ $n \geq 2$

✓ C) $a_n = \frac{(-1)^n (n+1)}{n+2}$, $n \geq 1$

X D) $a_n = \frac{(-1)^{n+1} (n+1)}{n+2}$ $n \geq 1$

(b) $\frac{\ln(2)}{3}, -\frac{\ln(3)}{5}, \frac{\ln(4)}{7}, -\frac{\ln(5)}{9}, \ldots, \ldots, \boxed{\frac{\ln(100)}{201}}$

Q1: Find an a general formula

(don't forget $n \geq n_0$) "n_0" first n.

Q2: Find $\lim_{n\to\infty} a_n = ?$

??

$\boxed{\cancel{a_n = (-1)^{n+1} \frac{\ln(n+1)}{n+2}}, \; n \geq 1}$

$n=1$

$a_1 = \frac{\ln(1+1)}{1+2} = \frac{\ln(2)}{3}$ ✓

$n=2$

$a_2 = \frac{\ln(2+1)}{2+2} = \cancel{\frac{\ln(3)}{4}}$

??

$\boxed{a_n = (-1)^{n+1} \frac{\ln(n+1)}{2n+1}, \; n \geq 1}$

$n=1$

$a_1 = (-1)^{1+1} \frac{\ln(1+1)}{2(1)+1} = (-1)^2 \frac{\ln(2)}{3} = \frac{\ln(2)}{3}$ ✓

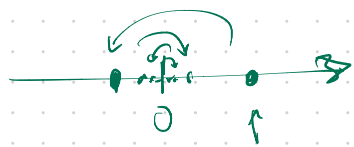
$n=2$

$a_2 = (-1)^{2+1} \frac{\ln(2+1)}{2(2)+1} = (-1)^3 \frac{\ln(3)}{5} = \frac{-\ln 3}{5}$ ✓

Next

Find $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \cancel{(-1)^{n+1}}^{1} \frac{\ln(n+1)}{2n+1}$ $\overset{\text{L'Hop } \frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{(1/n+1)(1)}{2}$

$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{1}{n+1} = 0.$



(c) $\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{4}}, \frac{4}{\sqrt{5}}, \frac{5}{\sqrt{6}}, \ldots$

(c) $\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{4}}, \frac{4}{\sqrt{5}}, \frac{5}{\sqrt{6}}, \ldots \frac{99}{\sqrt{100}}, \frac{100}{\sqrt{101}}$

$$\frac{1}{a/b} = \frac{b}{a}$$

$$\frac{c/d}{a/b} = \frac{c}{d} \cdot \frac{b}{a}$$

$$a_n = \frac{n}{\sqrt{n+1}}, \quad n \geq 1$$

L'Hop "$\frac{\infty}{\infty}$"

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{(n)'}{(\sqrt{n+1})'} = \lim_{n\to\infty} \frac{1}{\frac{1}{2\sqrt{n+1}} \cancelto{1}{(n+1)'}}$$

$$= \lim_{n\to\infty} 2\sqrt{n+1} = +\infty \text{ DNE}$$

Ex. Find the limit

$a_n = \frac{4-7n^2}{n^6+3}$, $n \geq 0$

a_0, a_1, a_2

$\frac{4}{3}, \frac{-3}{4}, \frac{-24}{67}, \ldots$

(a) $\lim_{n\to\infty} \frac{4-7n^2}{n^6+3} \overset{\text{L'Hop } \frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{-14n}{6n^{5}} = \lim_{n\to\infty} \frac{-14}{6n^4}$

$= \boxed{0}$ "$\frac{1}{\text{Big}}$ = small"

Example:

Determine whether or not the sequence converges. If so, find the limit.

$\left\{\frac{n^2}{n+1}\right\}$

$\{(-1)^n\}$

$\left\{(-1)^n \frac{1}{2^n}\right\}$

$\left\{\frac{2^n}{n!}\right\}$

Find the limit, if it exists.

$\left\{\frac{2n+1}{1-3n}\right\}$

A. 0
B. -2/3
C. 2/3
D. Diverges

Some Common Limits very likely to show up in your near future (once or twice)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ 5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ 7) $\lim_{n\to\infty} n^{1/n} = 1$

$\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = \frac{-1}{x^2}$

(b) $\lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n-1}\right)^n}_{a_n}$

$a_n = \left(\frac{n+1}{n-1}\right)^n$, $n \geq 2$

$a_2, a_3, a_4, \ldots, a_{100},$

$\left(\frac{2+1}{2-1}\right)^2, \left(\frac{3+1}{3-1}\right)^3, \left(\frac{4+1}{4-1}\right)^4, \ldots, \left(\frac{101}{99}\right)^{100}$

$\left(\frac{3}{1}\right)^2, \left(\frac{4}{2}\right)^3, \left(\frac{5}{3}\right)^4, \ldots, (1.02)^{100}$

7.38954

First analyze

$b_n = \ln(a_n)$ do this first.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln(a_n)$

$b_n = \ln\left(\left(\frac{n+1}{n-1}\right)^n\right) = n \ln\left(\frac{n+1}{n-1}\right)$

$\lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \overset{\text{L'Hop } \frac{0}{0}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1/n-1} * \left(\frac{n+1}{n-1}\right)}{-1/n^2}$

$f(x) = \left(\frac{x+1}{x-1}\right)^x$ $\qquad \left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = \frac{-1}{x^2}$

$a_n = \left(\frac{n+1}{n-1}\right)^n, \quad n \geq 2$

(b) $\lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n-1}\right)^n}_{a_n}$

$a_2,\ a_3,\ a_4, \ldots, a_{100},$

$\left(\frac{2+1}{2-1}\right)^2, \left(\frac{3+1}{3-1}\right)^3, \left(\frac{4+1}{4-1}\right)^4, \ldots, \left(\frac{101}{99}\right)^{100}$

First analyze

$b_n = \ln(a_n)$

do this first.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln(a_n)$

$\left(\frac{3}{1}\right)^2, \left(\frac{4}{2}\right)^3, \left(\frac{5}{3}\right)^4, \ldots, (1.02)^{100}$

7.38954

$e^2 \approx 7.3890560$

$b_n = \ln\left(\left(\frac{n+1}{n-1}\right)^n\right) = n \ln\left(\frac{n+1}{n-1}\right)$

L'Hop "$\frac{0}{0}$"

$$\lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{n+1/n-1} * \left(\frac{n+1}{n-1}\right)'}{-1/n^2}$$

$$= \lim_{n\to\infty} -n^2 \cdot \frac{\cancel{n-1}}{n+1}\left[\frac{\overset{\cancel{n}-1-\cancel{n}-1}{(n-1)(1) - (n+1)(1)}}{(n-1)^{\cancel{2}}}\right]$$

$$= \lim_{n\to\infty} \frac{-n^2 \cdot (-2)}{(n+1)(n-1)} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = \boxed{2}$$

$b_n = \ln(a_n)$ $\qquad \lim_{n\to\infty} b_n = 2$

$e^{b_n} = e^{\ln(a_n)}$ $\qquad \lim_{n\to\infty} \underbrace{e^{b_n}}_{a_n} = \boxed{e^2}$

Ex. Determine if the sequence is monotone.

(a) $a_n = \dfrac{3n+1}{n+1}$ $(, n \geq 1)$

$a_1 = \dfrac{4}{2} = 2$

$a_2 = \dfrac{7}{3} = 2.333$

$a_3 = \dfrac{10}{4} = 2.5$

$\vdots$

guess monotone increasing

$$a_n = \frac{3n+1}{n+1} \overset{?}{\leq} a_{n+1} = \frac{3(n+1)+1}{(n+1)+1} = \frac{3n+4}{n+2}$$

$$\boxed{\frac{3n+1}{n+1} \overset{?}{\leq} \frac{3n+4}{n+2}} \quad \checkmark$$

$$\Rightarrow (3n+1)(n+2) \overset{?}{\leq} (3n+4)(n+1)$$

$$\Rightarrow \cancel{3n^2} + n + 6n + 2 \overset{?}{\leq} \cancel{3n^2} + 4n + 3n + 4$$

$$\Rightarrow \cancel{7n} + \boxed{2} \leq \cancel{7n} + \boxed{4} \qquad 2 \leq 4 \ \checkmark$$

l.u.b is 2

g.l.b is 1

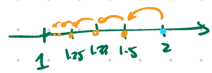
LUB and GLB

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (l.u.b.).
- A *lower bound* of a set S is a number *m* that is less than or equal to each element in S.
- The largest possible lower bound is called the *greatest lower bound* (g.l.b.).

$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{100}{99}, \frac{101}{100}, \ldots$

Find the l.u.b. and g.l.b. of the sequence: $\left\{\frac{n+1}{n}\right\}$, $n \geq 1$

A. l.u.b.=1, g.l.b.=0
B. l.u.b.=2, g.l.b.=0
C. l.u.b.=2, g.l.b.=1
D. No l.u.b., g.l.b.=0



not monotone functions.

$y = x^2$

$y = \sin x$

$a_n = n^3$ or $a_n = n$

$1, 2, 3, 4, 5, 6, \ldots$

monotone increasing.

$a_n = -n \quad a_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

monotone decreasing.

Monotone Sequences

A sequence is called *monotonic* if one of the following statements hold:

(i) $a_n < a_{n+1}$ for all n (strictly increasing)

(ii) $a_n \leq a_{n+1}$ for all n (monotonically increasing)

(iii) $a_n > a_{n+1}$ for all n (strictly decreasing)

(iv) $a_n \geq a_{n+1}$ for all n (monotonically decreasing)

Convergence Theorem

If a sequence $\{a_n\}$ is *monotonic* and *bounded*, then it converges.

If the sequence is increasing, then L=l.u.b.

If the sequence is decreasing, then L=g.l.b.

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot \boxed{3 \cdot 2 \cdot 1} = 24$

$5! = 120 \ \ldots$

$1, -1, 1, -1, +1, -1, \ldots$

down up.

non-monotone

(b) $a_n = \dfrac{2^n}{n!}$

$a_{n+1} = \dfrac{2^{n+1}}{(n+1)!}$

guess $a_n \geq a_{n+1}$

Check

$$\frac{2^n}{n!} \overset{?}{\geq} \frac{2^{n+1}}{(n+1)!} = \frac{2^n \cdot 2}{(n+1)\,n!} \Rightarrow \frac{n!}{n!} \geq \frac{\cancel{2^n} \cdot 2}{\cancel{2^n}} \cdot \frac{1}{n+1}$$

$$\Rightarrow 1 \geq \frac{2}{n+1} \quad ??$$

Section 10.1: 3, 17, 35, 41, 57, 129 (extra practice: 13, 19, 39, 93, 127, 133)

EXERCISES 10.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \dfrac{1-n}{n^2}$
2. $a_n = \dfrac{1}{n!}$
3. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$
4. $a_n = 2 + (-1)^n$
5. $a_n = \dfrac{2^n}{2^{n+1}}$
6. $a_n = \dfrac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$
8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$
9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$
10. $a_1 = -2, \quad a_{n+1} = na_n/(n+1)$
11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$
12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

Finding a Sequence's Formula

In Exercises 13–30, find a formula for the nth term of the sequence.

13. $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs
14. $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs
15. $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers, with alternating signs
16. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs
17. $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \ldots$ — Powers of 2 divided by multiples of 3
18. $-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \ldots$ — Integers differing by 2 divided by products of consecutive integers
19. $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1
20. $-3, -2, -1, 0, 1, \ldots$ — Integers, beginning with -3
21. $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer
22. $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer
23. $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \ldots$ — Integers differing by 3 divided by factorials
24. $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15{,}625}, \ldots$ — Cubes of positive integers divided by powers of 5
25. $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's
26. $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated
27. $\frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \ldots$
28. $\sqrt{5} - \sqrt{4}, \sqrt{6} - \sqrt{5}, \sqrt{7} - \sqrt{6}, \sqrt{8} - \sqrt{7}, \ldots$
29. $\sin\left(\frac{\sqrt{2}}{1+4}\right), \sin\left(\frac{\sqrt{3}}{1+9}\right), \sin\left(\frac{\sqrt{4}}{1+16}\right), \sin\left(\frac{\sqrt{5}}{1+25}\right), \ldots$
30. $\sqrt{\frac{5}{8}}, \sqrt{\frac{7}{11}}, \sqrt{\frac{9}{14}}, \sqrt{\frac{11}{17}}, \ldots$

41. $a_n = \left(\dfrac{n+1}{2n}\right)\left(1 - \dfrac{1}{n}\right)$
42. $a_n = \left(2 - \dfrac{1}{2^n}\right)\left(3 + \dfrac{1}{2^n}\right)$
43. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$
44. $a_n = \left(-\dfrac{1}{2}\right)^n$
45. $a_n = \sqrt{\dfrac{2n}{n+1}}$
46. $a_n = \dfrac{1}{(0.9)^n}$
47. $a_n = \sin\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$
48. $a_n = n\pi \cos(n\pi)$
49. $a_n = \dfrac{\sin n}{n}$
50. $a_n = \dfrac{\sin^2 n}{2^n}$
51. $a_n = \dfrac{n}{2^n}$
52. $a_n = \dfrac{3^n}{n^3}$
53. $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$
54. $a_n = \dfrac{\ln n}{\ln 2n}$
55. $a_n = 8^{1/n}$
56. $a_n = (0.03)^{1/n}$
57. $a_n = \left(1 + \dfrac{7}{n}\right)^n$
58. $a_n = \left(1 - \dfrac{1}{n}\right)^n$
59. $a_n = \sqrt[n]{10n}$
60. $a_n = \sqrt[n]{n^2}$
61. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$
62. $a_n = (n+4)^{1/(n+4)}$
63. $a_n = \dfrac{\ln n}{n^{1/n}}$
64. $a_n = \ln n - \ln(n+1)$
65. $a_n = \sqrt[n]{4^n n}$
66. $a_n = \sqrt[n]{3^{2n+1}}$
67. $a_n = \dfrac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)
68. $a_n = \dfrac{(-4)^n}{n!}$
69. $a_n = \dfrac{n!}{10^{6n}}$
70. $a_n = \dfrac{n!}{2^n \cdot 3^n}$
71. $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$
72. $a_n = \dfrac{(n+1)!}{(n+3)!}$
73. $a_n = \dfrac{(2n+2)!}{(2n-1)!}$
74. $a_n = \dfrac{3e^n + e^{-n}}{e^n + 3e^{-n}}$
75. $a_n = \dfrac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}}$
76. $a_n = \left(1 - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \cdots + \left(\dfrac{1}{n-2} - \dfrac{1}{n-1}\right) + \left(\dfrac{1}{n-1} - \dfrac{1}{n}\right)$

Section 10.1: 3, 17, 35, 41, 57, 129 (extra practice: 13, 19, 39, 93, 127, 133)

Convergence and Divergence

Which of the sequences $\{a_n\}$ in Exercises 31–100 converge, and which diverge? Find the limit of each convergent sequence.

31. $a_n = 2 + (0.1)^n$ **32.** $a_n = \dfrac{n + (-1)^n}{n}$

33. $a_n = \dfrac{1 - 2n}{1 + 2n}$ **34.** $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$

35. $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$ **36.** $a_n = \dfrac{n + 3}{n^2 + 5n + 6}$

37. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$ **38.** $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

39. $a_n = 1 + (-1)^n$ **40.** $a_n = (-1)^n\left(1 - \dfrac{1}{n}\right)$

77. $a_n = (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \cdots + (\ln(n - 1) - \ln(n - 2)) + (\ln n - \ln(n - 1))$

78. $a_n = \ln\left(1 + \dfrac{1}{n}\right)^n$ **79.** $a_n = \left(\dfrac{3n + 1}{3n - 1}\right)^n$

80. $a_n = \left(\dfrac{n}{n + 1}\right)^n$ **81.** $a_n = \left(\dfrac{x^n}{2n + 1}\right)^{1/n}, \quad x > 0$

82. $a_n = \left(1 - \dfrac{1}{n^2}\right)^n$ **83.** $a_n = \dfrac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

84. $a_n = \dfrac{(10/11)^n}{(9/10)^n + (11/12)^n}$ **85.** $a_n = \tanh n$

86. $a_n = \sinh(\ln n)$ **87.** $a_n = \dfrac{n^2}{2n - 1}\sin\dfrac{1}{n}$

88. $a_n = n\left(1 - \cos\dfrac{1}{n}\right)$ **89.** $a_n = \sqrt{n}\sin\dfrac{1}{\sqrt{n}}$

90. $a_n = (3^n + 5^n)^{1/n}$ **91.** $a_n = \tan^{-1} n$

92. $a_n = \dfrac{1}{\sqrt{n}}\tan^{-1} n$ **93.** $a_n = \left(\dfrac{1}{3}\right)^n + \dfrac{1}{\sqrt{2^n}}$

94. $a_n = \sqrt[n]{n^2 + n}$ **95.** $a_n = \dfrac{(\ln n)^{200}}{n}$

96. $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$ **97.** $a_n = n - \sqrt{n^2 - n}$

98. $a_n = \dfrac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$

99. $a_n = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{x}\,dx$ **100.** $a_n = \displaystyle\int_1^n \frac{1}{x^p}\,dx, \quad p > 1$

In Exercises 121–124, determine if the sequence is monotonic and if it is bounded.

121. $a_n = \dfrac{3n + 1}{n + 1}$ **122.** $a_n = \dfrac{(2n + 3)!}{(n + 1)!}$

123. $a_n = \dfrac{2^n 3^n}{n!}$ **124.** $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

Which of the sequences in Exercises 125–134 converge, and which diverge? Give reasons for your answers.

125. $a_n = 1 - \dfrac{1}{n}$ **126.** $a_n = n - \dfrac{1}{n}$

127. $a_n = \dfrac{2^n - 1}{2^n}$ **128.** $a_n = \dfrac{2^n - 1}{3^n}$

129. $a_n = ((-1)^n + 1)\left(\dfrac{n + 1}{n}\right)$

130. The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n + 1)\}.$$

131. $a_n = \dfrac{1 + \sqrt{2n}}{\sqrt{n}}$ **132.** $a_n = \dfrac{n + 1}{n}$

133. $a_n = \dfrac{4^{n+1} + 3^n}{4^n}$ **134.** $a_1 = 1, \quad a_{n+1} = 2a_n - 3$