Math 1553 **Exam 3** Spring 2016

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Name: KEY

GT username:

1. No books or notes are allowed.
2. All calculators and/or electronic devices are not allowed.
3. Show all work and fully justify your answer to receive full credit. *Hint: on most problems it is easy to check your answers!*
4. Please $\boxed{\text{BOX}}$ your answers.
5. Good luck!

Page	Max. Possible	Points
1	24	
2	30	
3	30	
4	16	
Total	100	

1. The vector $v = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 2 & 0 & -4 & 0 \\ -2 & 3 & -8 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}.$$

Find the eigenvalue λ associated to v. *Hint: write down the definition of eigenvalue.* (12 pts.)

$$Av = \begin{bmatrix} 2 & 0 & -4 & 0 \\ -2 & 3 & -8 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \\ -12 \\ -6 \end{bmatrix} = -6 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

So $\boxed{\lambda = -6}$ since $Av = \lambda v$, $\lambda = -6$.

2. If A is the 2×2 matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

then what is A? *Hint: Diagonalize!* (12 pts.)

$$P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{(1)(1) - (-2)(2)} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 2 & -6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix} = \boxed{\begin{bmatrix} 14/5 & -2/5 \\ -2/5 & 11/5 \end{bmatrix}}$$

3. Short answer section (two pages). Place one or more numbers, vectors, a matrix, or a variable in the box provided. (10 pts. each)

(a) Find all eigenvalues of $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$x_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \frac{3 \pm \sqrt{13}}{2}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 3 \\ 1 & 3-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$\boxed{3,\ \frac{3 \pm \sqrt{13}}{2}}$$

$$= 0 + (3-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} - 0$$

$$= (3-\lambda)\left[(2-\lambda)(1-\lambda) - 3\right] = (3-\lambda)\left[\lambda^2 - 3\lambda - 1\right]$$

quadratic formula needed

(b) Find an eigenvector of $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 2$.

Find null$(A - 2I)$

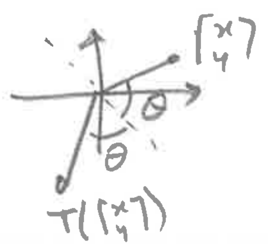
$$A - 2I = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

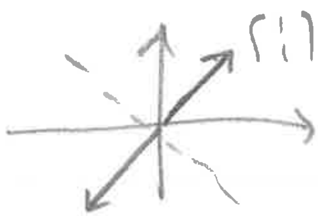
$$\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$X = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) List an eigenvector of the standard matrix of the linear transformation which reflects the standard basis vectors in $\mathbb{R}^2$ across the line $y = -x$.

T_A :





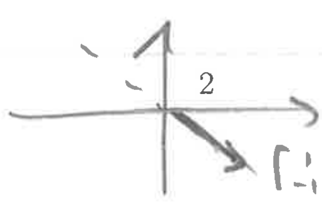
$$\boxed{\begin{array}{ccc} \lambda = -1 & & \lambda = 1 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{or} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

$$\lambda = -1$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \quad \lambda = 1$$

(d) Use the diagonalization $A = PDP^{-1}$ below to find A^5.

$$A = \begin{bmatrix} -1 & 0 & -3 \\ -4 & 1 & -6 \\ 0 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}^{-1}}_{P^{-1}}.$$

Find P^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -3 \\ -4 & 1 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 & -3 \\ -4 & 1 & -6 \\ 0 & 0 & 2 \end{pmatrix}}$$

(e) Give the eigenspace of the matrix $\begin{bmatrix} 5 & -3 & 3 \\ 6 & -4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 2$.

Find null$(A-2I)$

$$A - 2I = \begin{bmatrix} 3 & -3 & 3 \\ 6 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 x_3 both free

$$X = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

(f) If the eigenvalues of A are $1, 2$, and 3, then what are the eigenvalues of A^3?

$1^3, 2^3, 3^3$

Since

$A^n v = \lambda^n v$

$$\boxed{1, 8, 27}$$

4. In the questions below, assume that A is a square $n \times n$ matrix. Circle TRUE if the statement is **always** true, otherwise circle FALSE. (2 pts. each)

(a) If A has n linearly independent eigenvectors, then A has n distinct eigenvalues. TRUE FALSE

(b) The first standard basis vector e_1 is an eigenvector of the standard matrix of the linear transformation which rotates $\mathbb{R}^2$ by $180\,°$. TRUE FALSE

(c) If A is diagonalizable, then A is invertible. TRUE FALSE

(d) The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}$. TRUE FALSE

$$\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \checkmark$$

(e) If zero is an eigenvalue of A, then A is not invertible. TRUE FALSE

(f) If A is not invertible, then zero is an eigenvalue of A. TRUE FALSE

(g) If $A = PBP^{-1}$, then $\det(A) = \det(B)$. TRUE FALSE

(h) If λ is an eigenvalue of A, and B is the reduced row echelon form of A, then λ is an eigenvalue of B too. TRUE FALSE