EXAM 1 KEY. (A)

1. Determine whether $\begin{bmatrix}1\\6\\4\end{bmatrix}$ belongs to the span of the vectors $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}2\\4\\2\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\begin{bmatrix}-3\\-7\\-4\end{bmatrix}$. Justify your answer for full credit. (15 pts.)

Solve

$$c_1\begin{bmatrix}1\\2\\1\end{bmatrix}+c_2\begin{bmatrix}2\\4\\2\end{bmatrix}+c_3\begin{bmatrix}0\\1\\1\end{bmatrix}+c_4\begin{bmatrix}-3\\-4\\-4\end{bmatrix}=\begin{bmatrix}1\\6\\4\end{bmatrix}$$

$$[A|b]=\left[\begin{array}{cccc|c}1&2&0&-3&1\\2&4&1&-4&6\\1&2&1&-4&4\end{array}\right]\sim\begin{array}{r}\\-2R_1+R_2\\-R_1+R_2\end{array}\left[\begin{array}{cccc|c}1&2&0&-3&1\\0&0&1&2&4\\0&0&1&2&3\end{array}\right]$$

$$\sim\begin{array}{r}\\ \\-R_2+R_3\end{array}\left[\begin{array}{cccc|c}1&2&0&-3&1\\0&0&1&2&4\\0&0&0&0&-1\end{array}\right]$$

inconsistent system.

So $\boxed{\begin{bmatrix}1\\6\\4\end{bmatrix}\notin \text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\4\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-3\\-7\\-4\end{bmatrix}\right\}}$

(~~it's~~ not possible)

2. Solve the system of linear equations.

$$\begin{aligned} 2x + y \quad &+z = 3 \\ x \quad &+z = 0 \\ x + 2y \quad &+z = 4 \\ x + y \quad &+z = 2 \end{aligned}$$

$2+2-1=3$ ✓
$1-1=0$ ✓
$1+4-1=4$ ✓
$1+2-1=2$ ✓

(15 pts.)

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

$$\sim \begin{array}{r} \\ -2R_1+R_2 \\ -R_1+R_2 \\ -R_1+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{array}\right] \sim \begin{array}{r} \\ \\ \frac{1}{2}R_3 \\ -\frac{1}{2}R_3+R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{-R_3+R_4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1 \end{aligned}}$$

Check ✓

3. If $Ax = b$ has a unique solution, does that imply that the columns of A are linearly independent? Justify your answer for full credit. (3 pts. answer, 2 pts justification)

Yes. A has no free variables

$\Rightarrow$ $Ax=0$ has unique soln

$\Rightarrow$ cols of A are lin ind. ✓

4. Suppose A is row equivalent to

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 5 & 10 & -5 & 0 & 1 \end{bmatrix}$$

(a) Describe the solutions to $A\mathbf{x} = 0$ in parametric vector form. (10 pts.)

$$\begin{array}{ccccccc} & & x & y\checkmark r & z\checkmark s & u & v \\ A \sim & & \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 0 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right] \\ -5R_1+R_3 & & & & & & \end{array}$$

$$\begin{cases} x = -2r + s \\ y = r \\ z = s \\ u = 0 \\ v = 0 \end{cases}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

(b) Describe the set of solutions to $A\mathbf{x} = 0$ geometrically in a few words. (4 pts.)

$\boxed{\text{It's a plane in } \mathbb{R}^5.}$

(c) Is $\mathbf{x} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ a solution to $A\mathbf{x} = 0$? Justify your answer. (3 pts.)

$\boxed{\text{yes}}$

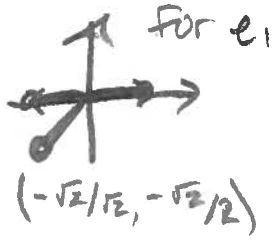
plug in.

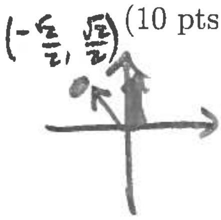
$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7-4-3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first reflects the vectors in $\mathbb{R}^2$ about the vertical y-axis, and then rotates the resulting vector by $45°$ counter-clockwise.

(a) Find the standard matrix T. (10 pts.)

$A = [T(e_1) \ T(e_2)]$





$$A = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

(b) Is T one-to-one? Explain for full credit. (3 pts. for ans., 2 pts. for just.)

yes. linearly independ columns.

(c) Is T onto? Explain for full credit. (2 pts. for answer, 1 pt. for justification)

yes. 2 pivots from part (a)

so #pivots = #rows

$\Rightarrow$ $Ax = b$ always consistent

$\Rightarrow$ $T(x)$ is onto.

6. Determine whether or not the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}$ are linearly independent. Fully justify your answer for full credit. (15 pts.)

free

$$A = \begin{bmatrix}1 & 0 & 3\\0 & 2 & 4\\1 & 1 & 5\end{bmatrix} \underset{-R_1+R_3}{\sim} \begin{bmatrix}1 & 0 & 3\\0 & 2 & 4\\0 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

No. B/c there are free variables so $Ax=0$ has a non trivial soln.

$\Longrightarrow$ cols of A are lin dep

7. For each 3×3 matrix below, determine if the matrix is in row reduced echelon form (RREF) or not. In each case, if the matrix is RREF circle the pivots, and if it is not then explicitly explain which property of RREF is being violated. (3 pts. each)

(a) $\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ not a staircase RREF/NOT RREF

(b) $\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ RREF/NOT RREF

(c) $\begin{bmatrix}1 & \cancel{2} & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ must clear above pivots RREF/NOT RREF

(d) $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ RREF/NOT RREF

(e) $\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}$ leading entries are not all 1. RREF/NOT RREF