EXAM 1 KEY (B)

1. Determine whether $\begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$ belongs to the span of the vectors $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Justify your answer for full credit. (15 pts.)

Solve

$$c_1 \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 2 \\ -3 & -9 & 1 & -1 & -2 \\ -2 & -6 & 1 & 0 & -3 \end{array}\right] \sim \begin{array}{c} \\ 3R_1+R_2 \\ 2R_1+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

inconsistent system
$\Rightarrow$ no soln.

So

$$\boxed{\begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} \notin \text{Span}\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}}$$

2. Solve the system of linear equations.

$$\begin{aligned} 2x + 2y \quad + z &= 2 \\ x + y \quad\quad &= 0 \\ 2x + y \quad + z &= 1 \\ 3x + 2y \quad + z &= 1 \end{aligned}$$

$-2 + 2 + 2 = 2$ ✓

$-1 + 1 = 0$ ✓

$-2 + 1 + 2 = 1$ ✓

$-3 + 2 + 2 = 1$ ✓

(15 pts.)

<u>Solve</u>

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} \\ -2R_1+R_2 \\ -2R_1+R_3 \\ -3R_1+R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \sim \begin{array}{l} \\ \\ \\ -R_3+R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Check ✓

$$\sim \begin{array}{l} \\ -R_3+R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{l} R_2+R_1 \\ -R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= -1 \\ y &= 1 \\ z &= 2 \end{aligned}}$$

3. If $Ax = b$ has a unique solution, does that imply that the columns of A are linearly independent? Justify your answer for full credit. (3 pts. answer, 2 pts justification)

Yes. A has no free vars. $\Rightarrow$ $Ax=0$ has unique soln

$\Rightarrow$ cols of A are <u>lin ind</u>.

4. Suppose A is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 8 & 1 \end{bmatrix}$$

(a) Describe the solutions to $A\mathbf{x} = 0$ in parametric vector form. (10 pts.)

$$\begin{array}{cc} & \begin{array}{ccccc} x & y \leftarrow r & z & u \leftarrow s & v \end{array} \\ A \sim & \begin{bmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ 4R_1+R_3 & \end{array}$$

$$\begin{cases} x = -r + 2s \\ y = r \\ z = 0 \\ u = s \\ v = 0 \end{cases}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

(b) Describe the set of solutions to $A\mathbf{x} = 0$ geometrically in a few words. (4 pts.)

It's a plane in $\mathbb{R}^5$

(c) Is $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ a solution to $A\mathbf{x} = 0$? Justify your answer. (3 pts.)

$\boxed{\text{No.}}$

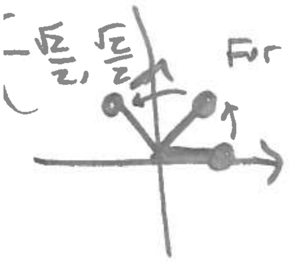
Plug in.

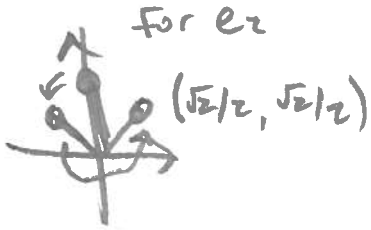
$$\begin{bmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+2-2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first rotates the vectors in $\mathbb{R}^2$ counter-clockwise by $45\,^\circ$, and then reflects the resulting vector across the vertical y-axis.

(a) Find the standard matrix T. (10 pts.)

$A = [T(e_1)\ T(e_2)]$





$$A = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

(b) Is T one-to-one? Explain for full credit. (3 pts. for ans., 2 pts. for just.)

yes. Cols. are lin. ind.

(c) Is T onto? Explain for full credit. (2 pts. for answer, 1 pt. for justification)

yes. 2 pivots (from (b))

$\Rightarrow$ pivot in each row (b/c 2x2)

$\Rightarrow$ $Ax = b$ never inconsistent

$\Rightarrow$ T is onto.

6. Determine whether or not the vectors $\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\\0\end{bmatrix}$, $\begin{bmatrix}-1\\-2\\6\end{bmatrix}$ are linearly independent. Fully justify your answer for full credit. (15 pts.)

$$A = \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\\2 & 0 & 6\end{bmatrix} \underset{-2R_1+R_3}{\sim} \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\\0 & -4 & 8\end{bmatrix}$$

free var x_3.

$$\underset{4R_2+R_3}{\sim} \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

$\boxed{\text{No}}$ They are lin dep b/c free variables imply $Ax=0$ has non-trivial sol'n.

7. For each 3×3 matrix below, determine if the matrix is in row reduced echelon form (RREF) or not. In each case, if the matrix is RREF circle the pivots, and if it is not then explicitly explain which property of RREF is being violated. (3 pts. each)

(a) $\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ RREF/NOT RREF

(b) $\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ not "staircase" RREF/NOT RREF

(c) $\begin{bmatrix}1 & \cancel{2} & \cancel{3}\\0 & \cancel{4} & \cancel{5}\\0 & 0 & \cancel{6}\end{bmatrix}$ leading entries are not 1 RREF/NOT RREF

(d) $\begin{bmatrix}1 & \cancel{2} & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ must clear above leading 1's RREF/NOT RREF

(e) $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ RREF/NOT RREF