B

1. Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 6 & 0 & 4 & 0 \\ 3 & 0 & 1 & -2 \\ 6 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find Nul(A) the null space of A. Be specific in your answer. (10 pts.)

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 6 & 0 & 4 & 0 \\ 3 & 0 & 1 & -2 \\ 6 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 3 & 0 & 1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = s \begin{bmatrix} 4/3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{nul}(A) = \text{span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

(b) Find a basis for Col(A) the column space of A. (6 pts.)

$$\left\{ \begin{bmatrix} 3 \\ 6 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

(c) Describe Nul(A) geometrically in a few words. (2 pts.)

a line in $\mathbb{R}^4$

(d) Describe Col(A) geometrically in a few words. (2 pts.)

a 3-plane in $\mathbb{R}^5$

2. Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}$.

(a) Find A^{-1}. (8 pts.)

$$A^{-1} = \frac{1}{3\cdot 2 - 2\cdot 4} \begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix}}$$

(b) Find the coordinate vector $\mathbf{x}$ of $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the basis $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$.

Hint: these are the columns of A. (8 pts.)

Soln to

$Ax = 0$

is

$x = A^{-1} b$

So

$$x = \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3+2 \\ 6-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

(c) Does $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ belong to the column space of A for any $u_1, u_2 \in \mathbb{R}$? Justify your answer for full credit. (4 pts.)

yes. Since A is invertible

$\text{col}(A) = \mathbb{R}^2$ so

$Ax = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is never

inconsistent.

3. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find A^{-1}. (12 pts.)

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

det * 2 * 2

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -\cancel{0} & -\cancel{0} & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] = [I|A^{-1}]$$

det * -1

So $\boxed{A^{-1} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ -1 & -1 & 2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}}$

(b) Find $\det(A)$. (10 pts.)

$\det(A) = -4$ using row operations.

(c) Check your answer to part *(a)* by appealing to the definition of inverse. (4 pts.)

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ -2 & -2 & 4 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$A^{-1} \cdot A = I$ ✓

4. Find a basis for (8 pts.)

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 6 \end{bmatrix}} \right\}.$$

$$2\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

basis is $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}}$

5. If a 5×7 matrix A has exactly 4 pivot positions, then the null space Nul(A) is a subspace of $\mathbb{R}^k$ and the column space Col(A) is a subspace of $\mathbb{R}^\ell$. In this problem you will specify the values of k, ℓ, and state the rank and nullity of A. (2 pts. each)

(a) Nul(A) is a subspace of $\mathbb{R}^k$. Specify the value of k.

$k = 7$

(b) Col(A) is a subspace of $\mathbb{R}^\ell$. Specify the value of ℓ.

$\ell = 5$

(c) What is the dimension of Nul(A) the null space of A?

$\dim \text{nul}(A) = 7 - 4 = 3$

(d) What is the rank of A?

$\text{rank } A = 4$

6. Suppose A is a 2×2 matrix and the null space $\text{Nul}(A)$ is the line in $\mathbb{R}^2$ given by the equation $y = 5x$.

(a) What is $\det(A)$? Justify your answer for full credit. (4 pts.)

$\boxed{\det(A) = 0}$ since $\text{nul}(A) \neq \{0\}$.

(b) What is the rank of A? Justify your answer for full credit. (4 pts.)

$\boxed{\text{rank } A = 1}$ since

$\dim \text{nul}(A) + \text{rank}(A) = \#\text{ cols}$

$1 + \text{rank}(A) = 2$.

7. Suppose A and B are square 2×2 matrices and you can assume that A, B, and $A + B$ are all invertible, and that $AB = BA$. Find the matrix equal to the following expression, that is, simplify the following expression. (5 pts.)

$$(A+B)^{-1}[(A^2 - B^2)B^{-1} + (A+B)]B$$

$(A+B)^{-1}\left[(A+B)(A-B)\,B^{-1}\cdot B + (A+B)\,B\right]$

$= (A+B)^{-1}(A+B)\cdot(A-B) + (A+B)^{-1}(A+B)\,B$

$= A - B + B = \boxed{A}$

8. Suppose A is any 3×3 matrix such that the ref of A is $A \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Is it true that a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$? Either give a counter-example or justify your answer in some way for full credit. (5 pts.)

yes. Since $\text{rank}(A) = 3$, $\text{col}(A) = \mathbb{R}^3$

and

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.