A

1. Let A be the matrix

$$A = \begin{bmatrix} 2 & -2 & 1 & -4 \\ 1 & 5 & -4 & 4 \\ 1 & 2 & 2 & 4 \\ 3 & 6 & -3 & 15 \end{bmatrix}$$

(a) Find a basis for the $\lambda = 3$ eigenspace of A. (12 pts.)

$$A - 3I = \begin{bmatrix} -1 & -2 & 1 & -4 \\ 1 & 2 & -4 & 4 \\ 1 & 2 & -1 & 4 \\ 3 & 6 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & r & & s \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$X = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 3$ eigenspace is $V_3 = \boxed{\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.}$

(b) Suppose v_1, v_2, v_3 are any $\lambda = 3$ eigenvectors of A that are all distinct, so $Av_i = 3v_i$, for $i = 1, 2, 3$, and $v_i \neq v_j$ if $i \neq j$. Suppose also that none of the v_i's are scalar multiples of each other. Is the set $\{v_1, v_2, v_3\}$ a basis for the $\lambda = 3$ eigenspace of A? Justify your answer for full credit. (6 pts.)

No, never. In particular $\dim V_3 = 2$, so its basis must have 2 elements.

Also, if $v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $v_3 = v_1 + v_2 = \begin{bmatrix} -6 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Then $Av_i = 3v_i$, $v_i \neq v_j$ but $\{v_1, v_2, v_3\}$ is not a basis for V_3, since it is a linearly dependent set.

2. Find all real or complex eigenvalues of each matrix and an associated eigenvector for each eigenvalue. (10 pts. each)

(a) $\begin{bmatrix} 7 & 3 \\ -2 & 2 \end{bmatrix}$ $p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 7-\lambda & 3 \\ -2 & 2-\lambda \end{bmatrix} = (7-\lambda)(2-\lambda) + 6$

$p(\lambda) = \lambda^2 - 9\lambda + 14 + 6 = \lambda^2 - 9\lambda + 20 = (\lambda-5)(\lambda-4) = 0$

$\Leftrightarrow \lambda = 4, 5$

$\underline{\lambda = 4}$

$A - 4I = \begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\underline{x = r\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ for } \lambda = 4}$

$\underline{\lambda = 5}$

$A - 5I = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$ $\underline{x = r\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \text{ for } \lambda = 5}$

$4 \cdot 29 = 4 \cdot 30 - 4$
$= 116$

(b) $\begin{bmatrix} 3 & 13 \\ -2 & 1 \end{bmatrix}$ $p(\lambda) = \det\begin{bmatrix} 3-\lambda & 13 \\ -2 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) + 26$

$= \lambda^2 - 4\lambda + 3 + 26$

$= \lambda^2 - 4\lambda + 29 = 0$

$\lambda = \frac{4 \pm \sqrt{16-116}}{2} = \frac{4 \pm 10i}{2}$

$\underline{\lambda = 2 \pm 5i}$

$\underline{\lambda = 2 - 5i}$

$A - \lambda I = \begin{bmatrix} 3-(2-5i) & 13 \\ -2 & 1-(2-5i) \end{bmatrix} = \begin{bmatrix} 1+5i & 3i \\ -2 & -1+5i \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 - \frac{5}{2}i \\ 0 & 0 \end{bmatrix}$

$\underline{x = r\begin{bmatrix} -1/2 + 5/2\, i \\ 1 \end{bmatrix} \text{ for } \lambda = 2 - 5i}$

$\underline{\lambda = 2 + 5i}$

$\underline{x = r\begin{bmatrix} -1/2 - 5/2\, i \\ 1 \end{bmatrix} \text{ for } \lambda = 2 + 5i}$

3. Find all eigenvalues of the matrix (10 pts.)

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

$$p(\lambda) = \det \begin{bmatrix} 3-\lambda & 0 & 3 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda) \begin{vmatrix} 3-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(3-\lambda)(2-\lambda) - 6\right]$$

$$= (1-\lambda)\left[\lambda^2 - 5\lambda + 6 - 6\right]$$

$$= (1-\lambda)(\lambda - 5)\cdot\lambda = 0$$

$$\boxed{\lambda = 0, 1, 5}$$

4. Suppose A is the non-invertible matrix below which has eigenvalue $\lambda = 6$. Is A diagonalizable? Justify your answer for full credit. (10 pts.)

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 & -1 \end{bmatrix}$$

Yes.

$\lambda = 0$ has a 4-dim'l eigenspace since

all free

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 6$ is another eigenvalue, with eigenvector

$$x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Since eigenvectors from different λ's are lin ind, the matrix is diagonalizable.

7. Find a basis for the $\lambda = a+b$ and the $\lambda = a-b$ eigenspaces of the matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, assuming neither a nor b is zero. (8 pts.)

$\underline{\underline{\lambda = a+b}}$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ by inspection.

$\underline{\underline{\lambda = a-b}}$ $\quad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8. Let A be the matrix below. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Assume none of a, b, c, d, e are zero. *Hint: think geometrically and consider the previous problem.* (8 pts.)

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & e & d \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} a+b & 0 & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & 0 & a-b \end{bmatrix}}_{D} P^{-1}$$

by inspection & previous problem.

9. Let A be a 2×2 matrix which satisfies $A\mathbf{v}_1 = 3\mathbf{v}_1$ and $A\mathbf{v}_2 = -\mathbf{v}_2$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. If $\mathbf{x}$ is the vector $\mathbf{x} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$, compute $A^3\mathbf{x}$. (10 pts.)

$$x = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad c_1 = 4, \quad c_2 = -2$$

$$= 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \checkmark$$

$$A^3 x = \lambda_1^3 c_1 v_1 + \lambda_2^3 c_2 v_2$$

$$\begin{array}{r} 27 \\ \times\ 4 \\ \hline \end{array}$$

$25 \cdot 4 + 2 \cdot 4 = 108$

$$= 27 \cdot 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)^3 \cdot (-2) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= 108 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 108 - 4 \\ 216 + 2 \end{bmatrix} = \boxed{\begin{bmatrix} 112 \\ 218 \end{bmatrix}}$$

Suppose A is a 2×2 matrix with $\lambda = 0$ and $\lambda = 1$ as eigenvalues. Choose one: (1) show that $A^2\mathrm{x} = A\mathrm{x}$ for any x in $\mathbb{R}^2$, or (2) show that $A^2 = A$. (8 pts.)

(1) $x = c_1 v_1 + c_2 v_2$ for some v_1, v_2 with $Av_1 = v_1$, $Av_2 = 0$

hence

$Ax = c_1 v_1 + 0$

$A^2 x = A(c_1 v_1) = c_1 v_1 = A^2 x$ ✓

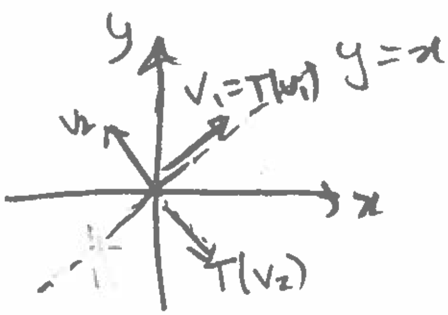
(2)

$A = P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}P^{-1}$ for some invertible P, so

$A^2 = \left(P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}P^{-1}\right)\left(P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}P^{-1}\right) = P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}^2 P^{-1} = P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}P^{-1}$

$= A$ ✓

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors in $\mathbb{R}^2$ about the line $y = x$. Find eigenvectors and state the associated eigenvalues for the standard matrix A of this linear transformation. *Hint: think geometrically.* (8 pts.)



$v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ satisfies

$T(v_1) = v_1$ so $\lambda = 1$ for $\begin{pmatrix}1\\1\end{pmatrix}$

$v_2 = \begin{pmatrix}-1\\1\end{pmatrix}$ satisfies

$T(v_2) = \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}$

so $\lambda = 2$ for $\begin{bmatrix}-1\\1\end{bmatrix}$