1. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & -2 & -6 \\ 1 & 2 & 2 & 1 \\ 1 & -1 & 5 & 6 \\ 3 & -3 & 6 & 21 \end{bmatrix}$$

(a) Find a basis for the $\lambda = 3$ eigenspace of A. (12 pts.)

$$A-3I = \begin{bmatrix} -1 & 1 & -2 & -6 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 6 \\ 3 & -3 & 6 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & r & s & \end{array} \\ \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$X = r\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda=3$ eigenspace $\boxed{V_3 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.}$

(b) Suppose v_1, v_2, v_3 are any $\lambda = 3$ eigenvectors of A that are all distinct, so $Av_i = 3v_i$, for $i = 1, 2, 3$, and $v_i \neq v_j$ if $i \neq j$. Suppose also that none of the v_i's are scalar multiples of each other. Is the set $\{v_1, v_2, v_3\}$ a basis for the $\lambda = 3$ eigenspace of A? Justify your answer for full credit. (6 pts.)

No In particular $\dim V_3 = 2$ so any basis of the $\lambda=3$ eigenspace must have 2 elements.

In particular, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

all satisfy $Av_i = 3v_i$, none are scalars of each other, but $\{v_1, v_2, v_3\}$ is a linearly dependent set, so not a basis.

2. Find all real or complex eigenvalues of each matrix and an associated eigenvector for each eigenvalue. (10 pts. each)

(a) $\begin{bmatrix} 8 & 2 \\ -4 & 2 \end{bmatrix}$ $p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 8-\lambda & 2 \\ -4 & 2-\lambda \end{bmatrix} = (8-\lambda)(2-\lambda)+8$

$p(\lambda) = \lambda^2 - 10\lambda + 16 + 8 = \lambda^2 - 10\lambda + 24 = (\lambda-6)(\lambda-4) = 0$

$\Leftrightarrow \lambda = 4, 6$

$\lambda = 4$

$A - 4I = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ $x = r\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ for $\lambda = 4$

$\lambda = 6$

$A - 6I = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x = r\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for $\lambda = 6$

$\begin{array}{r} 29 \\ \times 4 \\ \hline 30 \cdot 4 - 4 = 116 \end{array}$

(b) $\begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$ $p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} = (6-\lambda)(4-\lambda)+5$

$p(\lambda) = \lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29 = 0$

$\Leftrightarrow \lambda = \dfrac{10 \pm \sqrt{100 - 4(29)}}{2} = \dfrac{10 \pm \sqrt{100-116}}{2} = \dfrac{10 \pm \sqrt{16}\, i}{2}$

$\lambda = 5 \pm 2i$

$\lambda = 5 - 2i$

$A - (5-2i)I = \begin{bmatrix} 6-(5-2i) & -1 \\ 5 & 4-(5-2i) \end{bmatrix} = \begin{bmatrix} 1-2i & 1 \\ 5 & -1+2i \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{5}+\frac{2}{5}i \\ 0 & 0 \end{bmatrix}$

$x = r\begin{bmatrix} 1/5 - \frac{2}{5}i \\ 1 \end{bmatrix}$ for $\lambda = 5 - 2i$

$\lambda = 5 + 2i$

$x = r\begin{bmatrix} 1/5 + \frac{2}{5}i \\ 1 \end{bmatrix}$ for $\lambda = 5 + 2i$

3. Find all eigenvalues of the matrix (10 pts.)

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{bmatrix}$$

$$p(\lambda) = \det \begin{bmatrix} 4-\lambda & 0 & 4 \\ 0 & 1-\lambda & 1 \\ 3 & 0 & 3-\lambda \end{bmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(4-\lambda)(3-\lambda) - 12\right]$$

$$= (1-\lambda)\left[\lambda^2 - 7\lambda + 12 - 12\right]$$

$$= (1-\lambda)(\lambda)(\lambda - 7)$$

$$\underline{\underline{\lambda = 1, 0, 7}}$$

4. Suppose A is the non-invertible matrix below which has eigenvalue $\lambda = 5$. Is A diagonalizable? Justify your answer for full credit. (10 pts.)

$$A = \begin{bmatrix} 1 & 4 & 3 & -2 & -1 \\ 1 & 4 & 3 & -2 & -1 \\ 1 & 4 & 3 & -2 & -1 \\ 1 & 4 & 3 & -2 & -1 \\ 1 & 4 & 3 & -2 & -1 \end{bmatrix}$$

<u>yes</u>.

$\lambda = 0$ is an eigenvalue and

dim nul$(A) = 4$ since

$$\begin{array}{cccccc} & & \multicolumn{4}{c}{\text{all free}} \\ & & \swarrow & \swarrow & \swarrow & \swarrow \\ A \sim & \multicolumn{5}{c}{\begin{bmatrix} 1 & 4 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \end{array}$$

also $\lambda = 5$ is an eigenvalue, with eigenvector

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$ (by the way).

So There is a basis of $\mathbb{R}^5$ consisting entirely of eigenvectors, So A is diagonalizable.

5. Suppose A is a 2×2 matrix with $\lambda = 0$ and $\lambda = 1$ as eigenvalues. Choose one: (1) show that $A^2\mathrm{x} = A\mathrm{x}$ for any x in $\mathbb{R}^2$, or (2) show that $A^2 = A$. (8 pts.)

(1) $x = c_1 v_1 + c_2 v_2$ for some $c_1, c_2 \in \mathbb{R}$ and $Av_1 = v_1$, $Av_2 = 0$

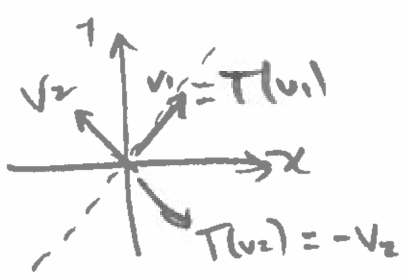
$Ax = c_1 v_1 + 0$

So $A^2 x = A(c_1 v_1) = c_1 v_1 = Ax$. ✓

(2) $A = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$ for some invertible P.

$A^2 = \left(P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}\right)\left(P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}\right) = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = A$ ✓

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors in $\mathbb{R}^2$ about the line $y = x$. Find eigenvectors and state the associated eigenvalues for the standard matrix A of this linear transformation. *Hint: think geometrically.* (8 pts.)



$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\lambda = 1$

$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $\lambda = -1$.

$$\boxed{\begin{array}{l} v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } \lambda = 1 \\ V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for } \lambda = -1 \end{array}}$$

7. Find a basis for the $\lambda = a+b$ and the $\lambda = a-b$ eigenspaces of the matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, assuming neither a nor b is zero. (8 pts.)

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda = a+b$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda = a-b$

by inspection

8. Let A be the matrix below. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Assume none of a, b, c, d, e are zero. *Hint: think geometrically and consider the previous problem.* (8 pts.)

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 \\ 0 & c & b & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & e & d \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b+c & 0 & 0 & 0 \\ 0 & 0 & b-c & 0 & 0 \\ 0 & 0 & 0 & d+e & 0 \\ 0 & 0 & 0 & 0 & d-e \end{bmatrix}}_{D} P^{-1}$$

by inspection & prev. problem

9. Let A be a 2×2 matrix which satisfies $A\mathbf{v}_1 = 2\mathbf{v}_1$ and $A\mathbf{v}_2 = -\mathbf{v}_2$, where $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. If $\mathbf{x}$ is the vector $\mathbf{x} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$, compute $A^3\mathbf{x}$. (10 pts.)

$$x = \begin{bmatrix} -5 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad c_1 = -1,\ c_2 = -3$$

$$= -1 \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{v_1} + (-3) \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{v_2} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \checkmark$$

$$A^3 x = \lambda_1^3 c_1 v_1 + \lambda_2^3 c_2 v_2$$

$$= 8 \cdot (-1) \begin{bmatrix} -1 \\ 2 \end{bmatrix} + (-1)^3 (-3) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= -8 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 16 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 14 \\ 19 \end{pmatrix}}$$

$$\left[\begin{array}{cc|c} -1 & 2 & -5 \\ 2 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 5 & -15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -3 \end{array}\right]$$