MATH 1553, SUMMER 2022
MIDTERM 1: THROUGH SECTION 2.5

Name	Key	GTID	

Please **read all instructions** carefully before beginning.

- Write your name on the top of each page (not just the cover page!).
- You have 55 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form."
- Please show your work unless specified otherwise. A correct answer without appropriate work may be given little or no credit.
- Please $\boxed{\text{box}}$ your answer for each question. (if needed)
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[Parts (a) through (e) are worth 2 points each]

a) Compute: $\begin{pmatrix} 2 & 1 \\ -3 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -1 \end{bmatrix}$

The remaining problems are True or false. Circle **T** if the statement is **always** true, and circle **F** otherwise. You do not need to justify your answer.

free vars

b) **T** **F** The matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$ is in reduced row echelon form.

c) **T** **F** The vector equation $x_1\begin{pmatrix} 2 \\ -3 \end{pmatrix} + x_2\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ is consistent.

$$-4\begin{bmatrix} 2 \\ -3 \end{bmatrix} - 4\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \checkmark$$

d) **T** **F** If A is an $m \times n$ matrix with $m > n$ and the system $Ax = 0$ has a unique solution, then $Ax = b$ is consistent for every b in $\mathbf{R}^m$.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

e) **T** **F** Suppose A is an 4×3 matrix whose first column is the sum of its second and third columns. Then the equation $Ax = 0$ has infinitely many solutions.

$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ & $v_1 = v_2 + v_3$

$\Rightarrow v_1 - v_2 - v_3 = 0$

$\Rightarrow x = s\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ are all solns to $Ax = 0$

Problem 2.

[Parts (a),(b) are 2pt each. Parts (c),(d) are 3pt each.]

a) Are there three nonzero vectors $v_1,\ v_2,\ v_3$ in $\mathbf{R}^3$ so that Span$\{v_1,\ v_2,\ v_3\}$ is a plane but v_3 is not in Span$\{v_1, v_2\}$? If your answer is yes, write such vectors $v_1,\ v_2,\ v_3$ and label each vector clearly.

yes $\quad v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}2\\2\\2\end{bmatrix} \quad v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

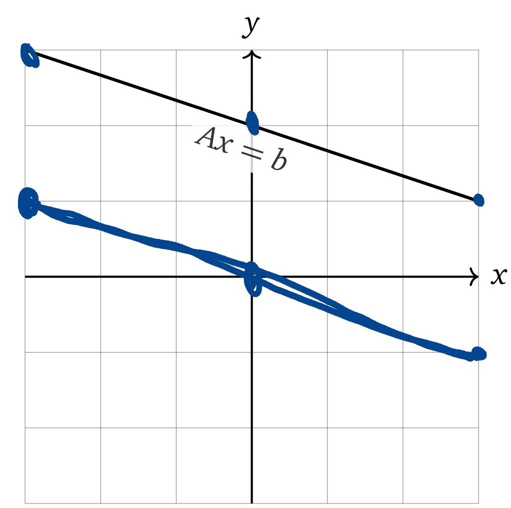
b) Write a matrix A with the property that the equation $Ax = \begin{pmatrix}0\\0\\1\end{pmatrix}$ is consistent.

$$A = \begin{bmatrix}0 & 0\\0 & 0\\1 & 2\end{bmatrix}$$

c) Write a vector equation which represents an inconsistent system of two linear equations in the variables x_1, x_2, x_3.

$$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}2\\2\end{bmatrix} + x_3\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

d) For some 2×2 matrix A and vector b in $\mathbf{R}^2$, the solution set of $Ax = b$ is drawn below. Draw the solution set of $Ax = 0$.



Problem 3.

[Parts (a) and (b) are 3pt each.]

Short answer. You do not need to show your work or justify your answers.

a) Suppose we are given a consistent linear system of 3 equations in 4 variables, and suppose that the augmented matrix corresponding to the system has 3 pivots. Then the solution set to the system is a:

(circle one answer) point line plane 3-space

in:

(circle one answer) $\mathbf{R}^2$ $\mathbf{R}^3$ $\mathbf{R}^4$ $\mathbf{R}^5$.

$$\begin{array}{cccc} & & \text{free} & \\ & & \downarrow & \end{array}$$

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right], \quad 4 \text{ vars}, \quad \text{if } Ax=b \to x \text{ in } \underline{\mathbb{R}^4}$$

b) Consider the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ -4 \\ 2 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \qquad -2v_1 = v_2$$

Describe Span$\{v_1, v_2\}$ geometrically: "it is a $\boxed{\text{line}}$ in $\mathbf{R}^{\boxed{4}}$."

Problem 4.

[10 points]

Please organize your work below and put a $\boxed{\text{box}}$ around your answer for each part.

a) Find the parametric form of the general solution of the following system of equations. Clearly indicate which variables (if any) are free variables.

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= -2 \\ -x_1 + 3x_2 + 2x_3 - 4x_4 &= 6 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 3 \end{aligned}$$

b) Write the set of solutions to

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= 0 \\ -x_1 + 3x_2 + 2x_3 - 4x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

in parametric vector form.

c) Write *one* specific non-zero vector that solves each system of equations (one vector for the system (a) and another vector for (b)). *Clearly show your work.*

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -2 \\ -1 & 3 & 2 & -4 & 6 \\ -1 & 3 & -1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 4 & -8 & 4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cccc} & s & & t \end{array} \\ \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -4 + 3s$

$x_2 = s$

$x_3 = 1 + 2t$

$x_4 = t$

(a) $$\boxed{X = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}$$

(b) $$\boxed{X = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}$$

(c) $$\boxed{\begin{aligned} X &= \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ for (a)} \\ X &= \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \text{ for (b)} \end{aligned}}$$

Problem 5.

[Parts (a) and (b) are 4pt each.]

Parts (a) and (b) are unrelated.

a) Caroll Spinney cannot stop thinking about the system of equations

$$x - 2y = h$$
$$3x + ky = 2,$$

where h and k are real numbers.

For what values of h and k (if any) is the system inconsistent?

$$\left[\begin{array}{cc|c} 1 & -2 & h \\ 3 & k & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & k+6 & 2-3h \end{array}\right]$$

$$\boxed{\begin{array}{l} k=-6 \\ h \neq 2/3 \end{array}}$$

b) Let $v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 4 \end{pmatrix}$.

Is $\{v_1, v_2, v_3\}$ linearly independent? If your answer is yes, justify why. If your answer is no, give a linear dependence relation for v_1, v_2, and v_3.

$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 4 \\ 2 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 3 & 12 \\ 0 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

s

$$\boxed{-2\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \checkmark$$

$$x = s\begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

Problem 6.

[Parts (a) and (b) are 3pt each.]

Parts (a) and (b) are unrelated.

a) Write an augmented matrix in RREF representing a system of three equations in two unknowns, whose solution set is the line $y = 3x$ in $\mathbf{R}^2$.

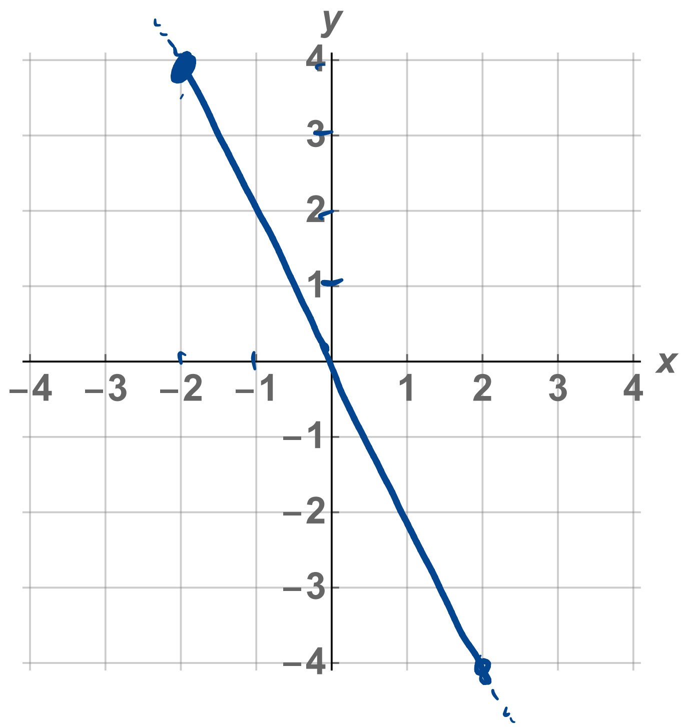
a vector in the line $y=3x$ is $\begin{bmatrix}1\\3\end{bmatrix}$.

Need $A * \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ← (b in $\mathbb{R}^3$ b/c 3 eqns)

$A = \begin{bmatrix}-3 & 1\\-3 & 1\\-3 & 1\end{bmatrix}$ works

So $[A|b] = \boxed{\left[\begin{array}{cc|c}1 & -1/3 & 0\\0 & 0 & 0\\0 & 0 & 0\end{array}\right]}$

b) Let $A = \begin{pmatrix}3 & -2\\-6 & 4\end{pmatrix}$. Draw the span of the columns of A below.



[Scratch work]

私はサーレです。
数学の教師です。
私は少し日本語を分かります。
毎週日本語を書きます。