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MATH 1553, SUMMER 2022
MIDTERM 2: THROUGH SECTION 3.6

Name	Key	**GTID**	

Please **read all instructions** carefully before beginning.

- Write your name on the top of each page (not just the cover page!).
- You have 55 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Please show your work unless specified otherwise. A correct answer without appropriate work may be given little or no credit.
- Please $\boxed{\text{box}}$ your answer for each question when needed.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[Parts (a) through (e) are worth 2 points each]

Parts (a)-(e) are True or false. Circle **T** if the statement is **always** true, and circle **F** otherwise. You do not need to justify your answer.

a) **T** **F** The transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ y-x \\ 0 \end{pmatrix}$ is one-to-one.

$$A = [T(e_1)\ T(e_2)] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

not pivot in every column

b) **T** **F** $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 \mid x + y = z \right\}$ is a subspace of $\mathbb{R}^3$.

$$V = \text{Nul}\left(\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\right)$$

c) **T** **F** If A is a 3×6 matrix and B is a 6×4 matrix, then the transformation $T(x) = ABx$ has domain $\mathbb{R}^4$ and codomain $\mathbb{R}^3$.

$\underset{3\times 6\ \ 6\times 4}{AB}$ is 3×4 $\quad \underbrace{ABx}$ is in $\mathbb{R}^3$; x is in $\mathbb{R}^4$

d) **T** **F** Suppose A is an 4×4 matrix with dim(Nul A)=2. Then the matrix A is not invertible.

to be invertible rank(A) must be 4.

e) **T** **F** Given that the RREF of $A = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$ is the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, a basis for Col($A$) is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

two linearly independent vectors in Col A.

Problem 2.

[Parts (a),(b) are 2pt each. Part (c) is 6pt.]

a) Suppose A is a 6×4 matrix, and the dimension of Nul(A) is equal to 3. Then the range of the transformation $T(x) = Ax$ is a:

(circle one answer) point line plane 3-space

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^4$ $\mathbf{R}^6$.

$A = 6\left[\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}\right]$ 4 3 free

rank $A = 1$

subspace of $\mathbb{R}^6$

range of T = Col A

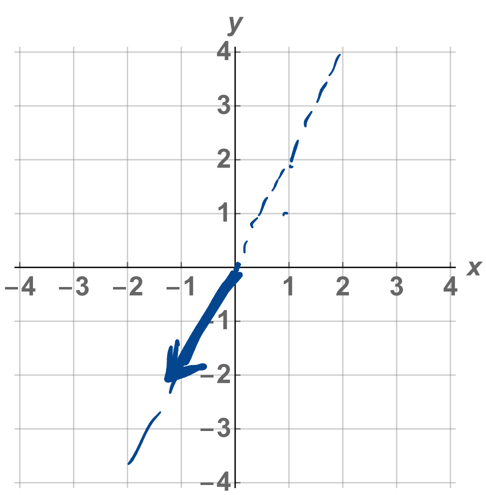
b) Consider the subspace $W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \text{ in } \mathbb{R}^5 \mid a = b = 0 \right\}$. The dimension of W

is equal to $\boxed{3}$.

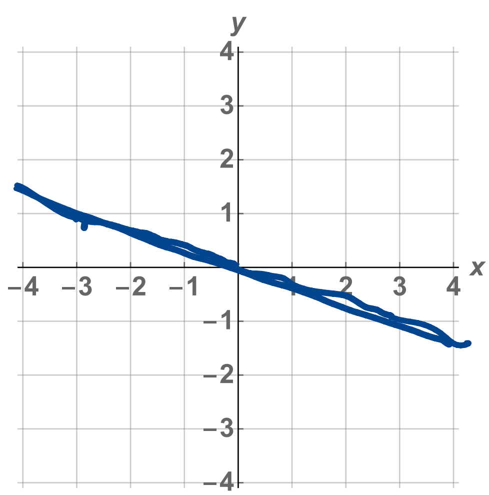
$$W = \text{Nul} \begin{array}{ccccc} & & c & d & e \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

c) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. Clearly draw a non-zero vector b which is in Col(A) but is **not a column** of A, and draw Nul(A). Briefly show work.

Draw b in Col(A) here.



Draw Nul(A) here.



$$-\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Problem 3. [10 points]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y-z \end{pmatrix}$, and let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix which reflects vectors in $\mathbb{R}^2$ about the line $y = x$.

a) Write the standard matrix A for T.

$$A = [T(e_1)\ T(e_2)\ T(e_3)] = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}}$$

b) Write the standard matrix B for U.

$$B = [T(e_1)\ T(e_2)] = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

c) Is T one-to-one? YES NO

d) Is U onto? YES NO

$$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{U} \mathbb{R}^2$$

e) Circle the composition that makes sense: $T \circ U$ $U \circ T$

T goes first

f) Write the standard matrix for the composition you chose in part (e).

$$B \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}}$$

Problem 4.

[10 points]

Frank Oz has put the matrix A below into its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ -3 & 9 & 1 & -1 \\ 2 & -6 & 0 & 4 \end{pmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis $\mathcal{B}$ for Nul(A). *Please* $\boxed{\textit{box}}$ *your answer.*

b) Is $x = \begin{pmatrix} 8 \\ 2 \\ 5 \\ -1 \end{pmatrix}$ in Nul(A)? YES NO

c) If you answered yes to part (b), write x as a linear combination of the vectors you found in part (a), otherwise justify why x is not in Nul(A).

(a) $x_1 - 3x_2 + 2x_4 = 0$
$x_2 = \text{free}$
$x_3 + 5x_4 = 0$
$x_4 = \text{free}$

$x_1 = 3s - 2t$
$x_2 = s$ free
$x_3 = -5t$
$x_4 = t$ free

$$\boxed{X = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -5 \\ 1 \end{bmatrix}}$$

(b) $$\begin{bmatrix} 1 & -3 & 0 & 2 \\ -3 & 9 & 1 & -1 \\ 2 & -6 & 0 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ 2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 8-6-2 \\ -24+18+5+1 \\ 16-12-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

(c) $$\left[\begin{array}{cc|c} 3 & -2 & 8 \\ 1 & 0 & 2 \\ 0 & -5 & 5 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{bmatrix} 8 \\ 2 \\ 5 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ -5 \\ 1 \end{bmatrix}}$$

Problem 5. [Parts (a) and (b) are 5pt each.]

Parts (a) and (b) are unrelated.

a) Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ and $T\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$. Find $T\begin{pmatrix}5\\2\end{pmatrix}$.

$$\begin{bmatrix}5\\2\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix} \quad \text{So} \quad \left[\begin{array}{cc|c}1 & -1 & 5\\ 1 & 2 & 2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -1 & 5\\ 0 & 3 & -3\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}1 & 0 & 4\\ 0 & 1 & -1\end{array}\right]$$

$c_1 = 4,\ c_2 = -1$

$$T\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = T\left(4\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix}\right)$$

$$= 4\,T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = 4\begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix} = \boxed{\begin{bmatrix}-6\\1\end{bmatrix}}$$

b) Find the inverse of the matrix A below. For full credit, *check your answer* using the definition of inverse.

$$A = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 3 & -2 & 1\end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 3 & -2 & 1 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & -5 & 1 & -3 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -3 & 5 & 1\end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ -3 & 5 & 1\end{bmatrix}}$$

$$A * A^{-1} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 3 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ -3 & 5 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \checkmark$$

[Scratch work]