Final Exam Review

In-Class Final Exam Review Set A, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
○	○	If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.
○	○	A $n \times n$ matrix A and its echelon form E will always have the same eigenvalues.
○	○	$x^2 - 2xy + 4y^2 \geq 0$ for all real values of x and y.
○	○	If matrix A has linearly dependent columns, then $\dim((\text{Row}A)^{\perp}) > 0$.
○	○	If λ is an eigenvalue of A, then $\dim(\text{Null}(A - \lambda I)) > 0$.
○	○	If A has QR decomposition $A = QR$, then $\text{Col}A = \text{Col}Q$.
○	○	If A has LU decomposition $A = LU$, then $\text{rank}(A) = \text{rank}(U)$.
○	○	If A has LU decomposition $A = LU$, then $\dim(\text{Null}\,A) = \dim(\text{Null}\,U)$.

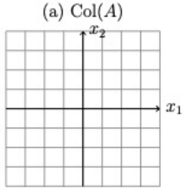
2. Give an example of the following.

i) A 4×3 lower triangular matrix, A. such that $\text{Col}(A)^{\perp}$ is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$. $A = \begin{pmatrix} \quad \end{pmatrix}$

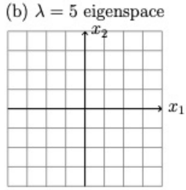
ii) A 3×4 matrix A, that is in RREF, and satisfies $\dim\left((\text{Row}\,A)^{\perp}\right) = 2$ and $\dim\left((\text{Col}\,A)^{\perp}\right) = 2$. $A = \begin{pmatrix} \quad \end{pmatrix}$

3. (3 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. On the grid below, sketch a) Col(A), and b) the eigenspace corresponding to eigenvalue $\lambda = 5$.

(a) Col(A)



(b) $\lambda = 5$ eigenspace



4. Fill in the blanks.

(a) If $A \in \mathbb{R}^{M \times N}$, $M < N$, and $A\vec{x} = 0$ does not have a non-trivial solution, how many pivot columns does A have? $\boxed{}$

(b) Consider the following linear transformation.

$$T(x_1, x_2) = (2x_1 - x_2,\ 4x_1 - 2x_2,\ x_2 - 2x_1)\,.$$

The domain of T is $\boxed{}$. The image of $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $T(\vec{x})$ is $\begin{pmatrix} \end{pmatrix}$. The co-domain of T is $\boxed{}$. The range of T is:

5. Four points in $\mathbb{R}^2$ with coordinates (t, y) are $(0, 1)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{4}, -\frac{1}{2})$. Determine the values of c_1 and c_2 for the curve $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$ that best fits the points. Write the values you obtain for c_1 and c_2 in the boxes below.

$$c_1 = \boxed{} \qquad c_2 = \boxed{}$$

In-Class Final Exam Review Set B, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
○	○	For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \text{proj}_W \vec{y}$ is orthogonal to W.
○	○	If A is $m \times n$ and has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.
○	○	If a matrix is invertible it is also diagonalizable.
○	○	If E is an echelon form of A, then $\text{Null}\, A = \text{Null}\, E$.
○	○	If the SVD of $n \times n$ singular matrix A is $A = U\Sigma V^T$, then $\text{Col}A = \text{Col}U$.
○	○	If the SVD of $n \times n$ matrix A is $A = U\Sigma V^T$, $r = \text{rank}A$, then the first r columns of V give a basis for $\text{Null}A$.

2. Give an example of:

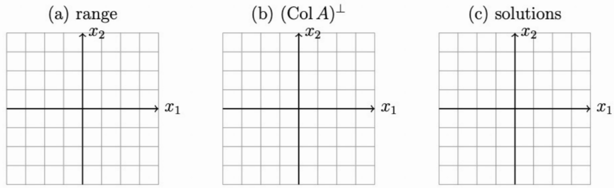
a) a vector $\vec{u} \in \mathbb{R}^3$ such that $\text{proj}_{\vec{p}} \vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$: $\quad \vec{u} = \begin{pmatrix} \quad \end{pmatrix}$

b) an upper triangular 4×4 matrix A that is in RREF, 0 is its only eigenvalue, and its corresponding eigenspace is 1-dimensional. $\quad A = \begin{pmatrix} \quad \end{pmatrix}$

c) A 3×4 matrix, A, and $\text{Col}(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$.

d) A 2×2 matrix in RREF that is diagonalizable and not invertible.

3. Suppose $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. On the grid below, sketch a) the range of $x \to Ax$, b) $(\text{Col}\, A)^{\perp}$, (c) set of solutions to $A\vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.



4. Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

   1. $A(\vec{v}_1 + 4\vec{v}_2)$
   2. A^{10}
   3. $\lim_{k \to \infty} A^k(\vec{v}_1 + 4\vec{v}_2)$

In-Class Final Exam Review Set C, Math 1554, Fall 2019

1. Indicate whether the statements are possible or impossible.

possible	impossible	
○	○	$Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a positive definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A.
○	○	The maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	○	The location of the maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	○	A is 2×2, the algebraic multiplicity of eigenvalue $\lambda = 0$ is 1, and $\dim(\text{Col}(A)^\perp)$ is equal to 0.
○	○	Stochastic matrix P has zero entries and is regular.
○	○	A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.
○	○	The map $T_A(\vec{x}) = A\vec{x}$ is one-to-one but not onto, A is $m \times n$, and $m < n$.

2. Transform $T_A = A\vec{x}$ reflects points in $\mathbb{R}^2$ through the line $y = 2 + x$. Construct a standard matrix for the transform using homogeneous coordinates. Leave your answer as a product of three matrices.

3. Fill in the blanks.

(a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? ______

(b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? ______

(c) A is a 6×4 matrix in RREF, and $\operatorname{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? ______

(d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? ______

(e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? ______

(f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is ______.

4. A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Sketch the nullspace of $Y = AC$.

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.

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Final Exam Review Worksheet, Spring 2020

1. (12 points) Indicate whether the statements are true or false.

		true	false
i)	If $A\vec{x} = \vec{b}$ has infinitely many solutions, then the RREF of A must have a row of zeros.	○	○
ii)	If A is $n \times n$ and $A\vec{x} = \vec{b}$ is inconsistent, then the columns of A are linearly dependent.	○	○
iii)	If A is a 3×3 matrix and $\det(A) = 2$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for Col(A).	○	○
iv)	A basis for a subspace must include the zero vector.	○	○
v)	If the columns of an $n \times n$ matrix span $\mathbb{R}^n$, then the matrix must be invertible.	○	○
vi)	A matrix, A, and any echelon form of A will have the same column space.	○	○
xii)	An $n \times n$ diagonalizable matrix must have n distinct eigenvalues.	○	○
xiii)	The geometric multiplicity of an eigenvalue is greater than or equal to the algebraic multiplicity of the same eigenvalue.	○	○
ix)	If S is a subspace of $\mathbb{R}^8$ and $\dim(S) = 6$, then $S^\perp$ is a two-dimensional subspace.	○	○
x)	If two vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then they are linearly independent.	○	○
xi)	If A is symmetric, and $v_1 \neq v_2$ are two eigenvectors of A, then v_1 and v_2 are orthogonal.	○	○
xii)	For a symmetric matrix A, the largest value of $\|Ax\|$ subject to the constraint that $\|x\| = 1$ is the largest singular value of A.	○	○

2. (10 points) Fill in the blanks.

(a) List all values of $k \in \mathbb{R}$ such that the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \\ -1 \end{pmatrix}$ are linearly dependent. $\boxed{}$.

(b) Suppose $\det(A^2B) = 4$, $\det(B) = \frac{1}{3}$, and A and B are $n \times n$ real matrices. List all possible values of $\det(A)$. $\boxed{}$.

(c) List all values of k such that $A\vec{x} = \vec{b}$ is inconsistent where $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2k \\ 0 & 0 & k \end{pmatrix}$. $\quad k = \boxed{}$

(d) Consider the row operation that reduces matrix A to RREF.

$$A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1 A} = E_1 A$$

By inspection, E_1 is the elementary matrix $E_1 = \begin{pmatrix} \end{pmatrix}$.

(e) If $S = \{\vec{x} \in \mathbb{R}^4 \,|\, x_1 = x_2\}$ then $\dim S = \boxed{}$.

(f) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{pmatrix}$, then a non-zero vector in Null$A$ is $\begin{pmatrix} \end{pmatrix}$.

(g) If the basis for the column space of an 11×15 matrix consists of exactly three vectors, how many pivot columns does the matrix have? $\boxed{}$

(h) If A is a 3×3 matrix with eigenvalues 5 and $1 - i$, then the third eigenvalue is $\boxed{}$.

(i) If $\vec{v}$ is the steady-state vector for a regular stochastic matrix, then $\vec{v}$ is an eigenvector of that matrix corresponding to the eigenvalue $\lambda = \boxed{}$.

(j) List all values of k such that $A = \begin{pmatrix} 4 & k \\ 0 & 4 \end{pmatrix}$ is diagonalizable. $\boxed{}$

3. (6 points) Fill in the blanks.

(a) The distance between the vector $\vec{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and the line spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\boxed{}$.

(b) If W is the plane spanned by the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, a basis of $W^{\perp}$ is given by $\vec{w} = \begin{pmatrix} \end{pmatrix}$.

(c) If A is a 3×3 matrix and $\dim(\text{Row}(A)) = 2$, then $\dim(\text{Null}(A^T)) = \boxed{}$.

(d) If $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^2$ satisfying $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$ and $\vec{u} \cdot \vec{v} = \frac{3}{2}$, then the length of the sum of the two vectors is $\|\vec{u} + \vec{v}\| = \boxed{}$.

(e) Let U be an $n \times n$ matrix with orthonormal columns. Then $U^t U =$ ______

(f) The maximum value of $Q(\vec{x}) = 10x_1^2 - 7x_2^2 - 4x_3^2$ subject to the constraints $\vec{x} \cdot \vec{x} = 1$ and $\vec{x} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ is equal to $\boxed{}$.

4. (8 points) Indicate whether the statements are possible or impossible.

		possible	impossible
i)	The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. $T = Ax$, and A has linearly independent columns.	○	○
ii)	The columns of a matrix with N rows are linearly dependent and span $\mathbb{R}^N$.	○	○
iii)	Matrix A is $n \times n$, $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, and $\dim(\text{Null}A) = 0$.	○	○
iv)	P is a stochastic matrix which has zero in the first entry of the first row, but is regular.	○	○
v)	There is a 2×2 real matrix A and a vector $\vec{u} \neq \vec{0}$, such that $\vec{u} \in \text{Null}(A)$ and $\vec{u} \in \text{Row}(A)$.	○	○
vi)	A is a non-singular matrix which is not diagonalizable.	○	○
vi)	$\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of matrix A that correspond to distinct eigenvalues, $A = A^T$, and $\vec{v}_1 \cdot \vec{v}_2 = 1$.	○	○
viii)	$\vec{y}$ is a non-zero vector in $\mathbb{R}^5$. The projection of $\vec{y}$ onto a subspace of $\mathbb{R}^5$ is the zero vector.	○	○

5. (2 points) Suppose A and B are $n \times n$ matrices and A is symmetric. Fill in the circles next to the expressions (if any) that are equal to

$$(B^T AB)^T$$

Leave the other circles empty.

- ○ $BA^T B^T$
- ○ $B^T AB$

6. (2 points) List the singular values of the matrix below. (No need to justify your answer.)

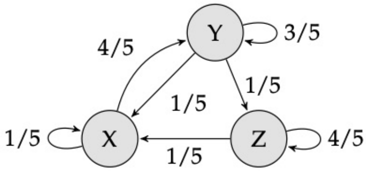
$$\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \_\_\_\_\_, \qquad \sigma_2 = \_\_\_\_\_,$$

7. (6 points) Let $A = \begin{pmatrix} -2 & -4 & 0 & 0 & 2 \\ -2 & -4 & 1 & 0 & 0 \\ -2 & -4 & 0 & 2 & 4 \\ -2 & -4 & 0 & 3 & 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 7 \\ 8 \end{pmatrix}$.

(a) Solve the system $A\vec{x} = \vec{b}$ where A and $\vec{b}$ are as above. Write your answer in parametric vector form for full credit.

(b) Write down a basis for $\text{Col}(A)$.

8. (4 points) Consider the following Markov chain.



(a) What is the transition matrix, P?

$$P = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

(b) Use your transition matrix from part (a) to calculate the steady-state probability vector, $\vec{q}$. Show your work.

9. (3 points) Apply the Gram-Schmidt process to construct an orthogonal basis for Col(A). Show your work.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

10. (3 points) Construct the LU factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 0 \end{pmatrix}$. Clearly indicate matrices L and U.

11. (5 points) Compute Σ and V in the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^T$$

$$\Sigma = \begin{bmatrix} \_\_\_ & 0 \\ 0 & \_\_\_ \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

12. (5 points) Find matrices D and P to construct the orthogonal diagonalization of A. Show your work.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} = PDP^T$$

$$D = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}, \qquad P = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$