Math 1554 Linear Algebra Fall 2022

Midterm 1

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

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Section Number (e.g. A3, G2, etc.) ________ TA Name ____________

Circle your instructor:

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Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

Midterm 1. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	If A has a pivot in every column then the system $A\vec{x} = \vec{b}$ has a unique solution. could be inconsistent
●	○	Suppose A is a 6×4 matrix with 4 pivots, then there is b such that $A\vec{x} = \vec{b}$ has no solution. rows w/ no pivots
○	●	The sets $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1 + \vec{v}_2, -\vec{v}_1 - \vec{v}_2\}$ have the same span. spans a line
○	●	If A and B are square $n \times n$ matrices, then $A^2 - B^2 = (A - B)(A + B)$. $AB \neq BA$
●	○	The matrix equation $A\vec{x} = \vec{0}$ is always consistent. $\vec{x} = \vec{0}$
○	●	Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are nonzero vectors in $\mathbb{R}^n$ and the sets $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$ are all linearly independent. Then, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. e.g. $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$
●	○	If $A\vec{v} = 0$, $A\vec{u} = 0$ and $\vec{w} = 3\vec{v} - 2\vec{u}$, then $A\vec{w} = 0$.
○	●	Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\vec{x}) = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$. Then T is ~~one-to-one~~ onto. (FYI true if $m=n$)

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
●	○	A 7×5 matrix A with linearly independent columns. tall matrix
○	●	A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is not onto and its standard matrix has linearly independent columns. $\Rightarrow$ range of $T = \mathbb{R}^3$
●	○	$T : \mathbb{R}^3 \to \mathbb{R}^2$ that is onto and its standard matrix has exactly one non-pivotal column. e.g. $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$
●	○	Two non-zero matrices A, B of size 2×2 with $AB = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$.

e.g. $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

You do not need to justify your reasoning for questions on this page.

(c) (2 points) Let

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 3h & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

be an augmented matrix of a system of linear equations. For which values of h does the system have a free variable? *Choose the best option.*

- ● 0 only
- ○ $\frac{1}{3}$ only
- ○ 1 only
- ○ for all values of h
- ○ for no values of h

(d) (2 points) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^1$ maps each of the standard unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ to 1. Which of the following statements is TRUE? *Select only one.*

- ○ T is one-to-one.
- ○ T is not onto.
- ● The solution set of $T(\vec{x}) = \vec{0}$ spans a plane in $\mathbb{R}^3$.
- ○ The range of T is $\{1\}$.

Midterm 1. Your initials: ______________

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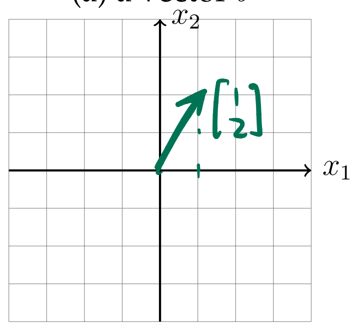
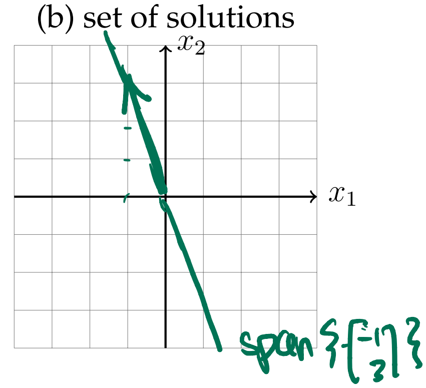
2. (4 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and sketch (a) a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent, and (b) the set of solutions to $A\vec{x} = \vec{0}$.

(a) a vector $\vec{b}$



(b) set of solutions



$A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$Ax=0$ $\quad \begin{bmatrix}3&1\\6&2\end{bmatrix}\sim\begin{bmatrix}3&1\\0&0\end{bmatrix}$

$x=s\begin{bmatrix}-1/3\\1\end{bmatrix}\sim\begin{bmatrix}-1\\3\end{bmatrix} \qquad \sim\begin{bmatrix}1&1/3\\0&0\end{bmatrix}$

3. (2 points) Consider the linear system in variables x_1, x_2, x_3 with unknown constants below.

$$a_1x_1 + a_2x_2 + a_3x_3 = b_1$$
$$c_1x_1 + c_2x_2 + c_3x_3 = b_2$$

Which of the following statements about the solution set of this system are possible? *Select all that apply.*

- ● The solution set is empty. ✓ inconsistent
- ○ The solution set is a single point. ✗
- ● The solution set is a line ✓ 1 free var
- ● The solution set is a plane ✓ 2 free var

$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ c_1 & c_2 & c_3 & b_2 \end{array}\right]$

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You do not need to justify your reasoning for questions on this page.

4. Fill in the blanks.

   (a) (3 points) Let A be a coefficient matrix of size 2×2 and B be a coefficient matrix of size 3×2. Construct an example of two augmented matrices $\left[A|\vec{b}\right]$ and $\left[B|\vec{d}\right]$ which are both in RREF and such that the systems $A\vec{x} = \vec{b}$ and $B\vec{x} = \vec{d}$ each have the exact same unique solution $x_1 = 3$ and $x_2 = 6$. If this is not possible write NP in each box.

$$\left[A|\vec{b}\right] = \boxed{\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 6 \end{array}\right]} \qquad \left[B|\vec{d}\right] = \boxed{\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array}\right]}$$

   (b) (2 points) Let $\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find c_1, c_2 such that $\vec{b} = c_1\vec{u}_1 + c_2\vec{u}_2$.

$$c_1 = \boxed{1} \qquad c_2 = \boxed{3}$$

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Check.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

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You do not need to justify your reasoning for questions on this page.

5. (8 points) Let T be a linear transformation that maps $\vec{v}_1$ to $T(\vec{v}_1)$ and $\vec{v}_2$ to $T(\vec{v}_2)$, where

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad T(\vec{v}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \quad T(\vec{v}_2) = \begin{pmatrix} 3 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

(i) What is domain and codomain of T?

domain is $\boxed{\mathbb{R}^2}$

codomain is $\boxed{\mathbb{R}^4}$

(ii) Is it true that $\mathbb{R}^2 = \text{span}\{\vec{v}_1, \vec{v}_2\}$? ● yes ○ no

(iii) Write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$\vec{e}_1 = \boxed{V_1 + V_2}$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \checkmark$$

$\vec{e}_2 = \boxed{V_1 + 2V_2}$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 2 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

(iv) What is the standard matrix of T?

$$\boxed{\begin{bmatrix} 4 & 7 \\ 2 & 1 \\ -2 & -4 \\ 2 & 3 \end{bmatrix}}$$

$$T(e_1) = T(v_1) + T(v_2) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

$$T(e_2) = T(v_2) + 2T(v_2) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ -4 \\ 3 \end{bmatrix}$$

(v) Is T one-to-one? ● yes ○ no

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6. **Show all work for problems on this page.**

Matrix A have 2 pivots.

(a) (3 points) For what value of k will the columns of A span a plane in $\mathbb{R}^3$?

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & k \end{pmatrix}$$

$$k = \boxed{3}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & k \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & k \\ 0 & 0 & 3-k \end{bmatrix}$$

(b) (4 points) Find b and c such that $AB = BA$.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$$

$$b = \boxed{4} \qquad c = \boxed{1}$$

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3+4c & 3b \\ 1+2c & b \end{bmatrix} = \begin{bmatrix} 3+b & 4+2b \\ 3c & 4c \end{bmatrix}$$

$$1+2c = 3c \Rightarrow \boxed{c=1}$$

$$b = 4c \overset{(c=1)}{\Longrightarrow} \boxed{b=4}$$

$$X = \begin{bmatrix} 2s \\ -3t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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7. (4 points) **Show your work for problems on this page.**
Write down the parametric vector form for solutions to the homogeneous equation $A\vec{x} = \vec{0}$.

$$A = \begin{bmatrix} 1 & -1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ -1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

$$\boxed{X = s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ -1 & 1 & 2 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF $\sim$

$$\begin{array}{ccccc} & & s & t & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 - 2x_3 = 0$
$x_2 + 3x_4 = 0$
$x_3 =$ free
$x_4 =$ free
$x_5 = 0$

$x_1 = 2s$
$x_2 = -3t$
$x_3 = s$ (free)
$x_4 = t$ (free)
$x_5 = 0$

8. (4 points) Determine whether the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.
Justify your answer in the space below.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ -9 \end{bmatrix}$$

● linearly independent ○ linearly dependent

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 5 & 8 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot in every column of the matrix
$A = [v_1\ v_2\ v_3] \Rightarrow$ cols lin ind.

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