Math 1554 Linear Algebra Fall 2022

# Midterm 2

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

Name: Key GTID Number: ____________________

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Section Number (e.g. A3, G2, etc.) ________ TA Name __________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler

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**Student Instructions**

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, $X$'s, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

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*You do not need to justify your reasoning for questions on this page.*

1. (a) (8 points) Suppose $A$ is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of $A$ and $\vec{b}$. Otherwise, select **false**.

| true | false | |
|---|---|---|
| ● | ○ | If $A$, $B$ and $C$ are $n \times n$ matrices, $A$ is invertible and $AB = AC$, then $B = C$. $A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B = C$ |
| ● | ○ | If $A$, $B$ and $C$ are $n \times n$ matrices and $ABC = I_n$, then $C$ is invertible. $(AB) = C^{-1}$ |
| ● | ○ | If $A = LU$ is an LU-factorization of a square matrix $A$, then $A$ is invertible if and only if $U$ is invertible. $U$ is REF of $A$ |
| ● | ○ | If $\vec{x}$ is a vector in $\mathbb{R}^3$ and $B$ is a basis for $\mathbb{R}^3$, then $[\vec{x}]_B$ has 3 entries. $B = \{v_1, v_2, v_3\}$ |
| ○ | ● | If $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, then the set of solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = 2b \neq b.$ |
| ○ | ● | The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. $\vec{x} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \Rightarrow 2\vec{x}$ not prob vec. |
| ● | ○ | If two matrices $A, B$ share an eigenvector $\vec{v}$, with eigenvalue $\lambda$ for matrix $A$ and eigenvalue $\mu$ for the matrix $B$, then $\vec{v}$ is an eigenvector of the matrix $(A + 2B)$ with eigenvalue $\lambda + 2\mu$. $(A+2B)v = Av + 2Bv = \lambda v + 2\mu v = (\lambda + 2\mu)v$ |
| ○ | ● | For any $2 \times 2$ real matrix $A$, we have $\det(-A) = -\det(A)$. |

(b) (4 points) Indicate whether the following situations are possible or impossible.

| possible | impossible | |
|---|---|---|
| ○ | ● | A matrix $A \in \mathbb{R}^{n \times n}$ such that $A$ is invertible and $A^T$ is singular. $\det A = \det(A^T)$ |
| ○ | ● | A $3 \times 3$ matrix $A$ with $\dim(\mathrm{Null}(A)) = 0$ such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solution. $A$ is invertible $\Rightarrow Ax = b$ always has unique soln. |
| ● | ○ | $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is onto and its standard matrix has determinant equal to $-1$. e.g. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| ○ | ● | Two square matrices $A, B$ with $\det(A)$ and $\det(B)$ both non-zero, and the matrix $AB$ is singular. |

$\det(AB) = \det A * \det B = 0$

only if $\det A = 0$

or $\det B = 0$.

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*You do not need to justify your reasoning for questions on this page.*

(c) (3 points) If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible write NP in the space.

(i) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & \text{NP} & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix}$ $v_1 + 2v_2 = v_3$ ✓

(iii) $\begin{bmatrix} 2 & 3 & h \\ 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & h \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & h-10 \\ 0 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & h-10 \\ 0 & 0 & h-8 \end{bmatrix}$ $h - 8 = 0$

$\Rightarrow h = 8$

(d) (2 points) Let $A$ be a $3 \times 3$ upper triangular matrix and assume that the volume of the parallelpiped determined by the columns of $A$ is equal to 1. Which of the following statements is FALSE?

- ○ $A$ is invertible.
- ● The diagonal entries of $A$ are either 1 or $-1$.
- ○ For every $3 \times 3$ matrix $B$ we have $|\det(AB)| = |\det(B)|$.
- ○ If $B$ is a matrix obtained by interchanging two rows of $A$, then the volume of the parallelepiped determined by the columns of $B$ is equal to 1.

e.g.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1/2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

and $\det A = 1$.

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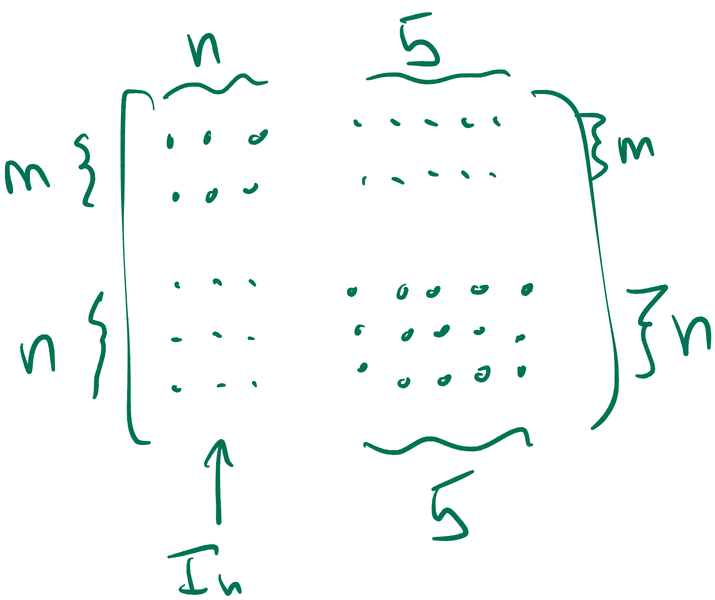
2. (2 points) Suppose $A$ and $B$ are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix
$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} AZ = I \\ AW = 0 \\ BX = 0 \\ BY = I \end{matrix}\right\} \Rightarrow \begin{matrix} Z = A^{-1} \\ W = 0 \\ X = 0 \\ Y = B^{-1} \end{matrix}$$

3. (2 points) Suppose $A$ is a $m \times n$ matrix and $B$ is $m \times 5$ matrix. Find the dimensions of the matrix $C$ in the block matrix
$$\begin{pmatrix} A & B \\ I_n & C \end{pmatrix}.$$

$C$ has $\boxed{n}$ rows and $\boxed{5}$ columns.



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4. Fill in the blanks.

(a) (3 points) Give a matrix $A$ whose column space is spanned by the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix}1\\1\\1\end{pmatrix}$. If this is not possible, write NP in the box.

1 $\text{Nul}\,A = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

$\Rightarrow$ A has one free variable

2 $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in Nul(A)

$\Rightarrow A\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{0}$

($\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a soln.)

$$A = \boxed{\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}}$$

3 So,

$$\begin{bmatrix}1 & 0 & c\\ 0 & 1 & b\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

multiply to get

$$\left.\begin{matrix}1+c=0\\1+b=0\end{matrix}\right\} \Rightarrow A = \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}$$

<u>Check</u>. $\text{Col}(A) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$? yes b/c $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ are pivot cols of A.

(b) (3 points) Use the determinant to find all values of $\lambda \in \mathbb{R}$ such that the following matrix is singular.

$\det A = 0$

$$\begin{pmatrix}1 & 1 & 2\\ 2 & 4 & 5\\ \lambda & 2 & 3\end{pmatrix}.$$

$$\lambda = \boxed{4/3}$$

$$\det A = 1\begin{vmatrix}4 & 5\\ 2 & 3\end{vmatrix} - 2\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} + \lambda\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}$$

$$= (12-10) - 2(3-4) + \lambda(5-8)$$

$$= 2+2-3\lambda = 0 \Rightarrow 3\lambda = 4 \Rightarrow \boxed{\lambda = 4/3}$$

det A=0

alt soln: compute $\det(A-\lambda I)$ and complete the square.

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5. (3 points) Find the value of $h$ such that the matrix

$$A = \begin{pmatrix} 5 & h \\ 1 & 3 \end{pmatrix}$$

has an eigenvalue with algebraic multiplicity 2.

$$h = \boxed{-1}$$

$\det A = 15 - h$

$\det A = 16$

$\Rightarrow h = -1$

$$\det A = \lambda^2 - \operatorname{trace}(A)\cdot\lambda + \det A$$
$$= \lambda^2 - 8\lambda + \det A$$
$$= (\lambda - c)^2 \text{ only if } c = 4 \text{ and } \det A = 16$$

6. (3 points) Let $\mathcal{P}_B$ be a parallelogram that is determined by the columns of the matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, and $\mathcal{P}_C$ be a parallelogram that is determined by the columns of the matrix $C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose $A$ is the standard matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathcal{P}_B$ to $\mathcal{P}_C$. What is the value of $|\det(A)|$?

$$|\det(A)| = \boxed{1/2}$$

$\det B = \det\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = 3-(-1) = 4 \Rightarrow \text{area } \mathcal{P}_B = 4$

and

$\det C = \det\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = -1-1 = -2 \Rightarrow \text{area } \mathcal{P}_C = 2$

$|\det(A)| * \text{area}(\mathcal{P}_B) = \text{area}(T(\mathcal{P}_B)) = \text{area}(\mathcal{P}_C)$

$\Rightarrow |\det A| * 4 = 2 \Rightarrow |\det A| = 1/2$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 = \text{free} \\ x_3 = 0 \end{cases} \quad \begin{cases} x_1 = s \\ x_2 = s \\ x_3 = 0 \end{cases}$$

$$X = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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7. (5 points) **Show all work for problems on this page.** Given that $4$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix},$$

find an eigenvector $\vec{v}$ of $A$ such that $A\vec{v} = 4\vec{v}$.

$$A - 4I = \begin{bmatrix} 6-4 & -2 & 2 \\ 2 & 2-4 & -2 \\ 1 & -1 & 4-4 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$s$

$\vec{v} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{pmatrix}.$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{bmatrix} \sim \begin{matrix} \\ -1R_1+R_2 \to \\ -2R_1+R_3 \to \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -3 & 3 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ \tfrac{4}{3}R_2+R_3 \to \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -4/3 & 1 \end{bmatrix}$$

$L = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{4}{3} & 1 \end{bmatrix}}$

$U = \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}}$

$$\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}0\\-1\\1\end{bmatrix}+\frac{1}{4}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}1/4\\1/2\\1/4\end{bmatrix}$$

✓

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9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k,\ k = 0, 1, 2, \ldots$.
Suppose $P$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = 0$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to $\lambda_1$, $\lambda_2$, and $\lambda_3$, respectively:

$$\vec{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

*Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.*

(i) If $\vec{x}_0 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$, then what is $\vec{x}_3$?

$$\vec{X}_3 = P^3 X_0 = P^3\left(\frac{1}{2}V_1 + \frac{1}{2}V_2\right)$$
$$= \frac{1}{2}P^3 V_1 + \frac{1}{2}P^3 V_2$$
$$= \frac{1}{2}(1)^3 V_1 + \frac{1}{2}\left(\frac{1}{2}\right)^3 V_2 = \frac{1}{2}V_1 + \frac{1}{16}V_2$$

$\vec{x}_3 = \boxed{\frac{1}{2}V_1 + \frac{1}{16}V_2}$

(ii) If $\vec{x}_0 = \begin{pmatrix}1/4\\1/2\\1/4\end{pmatrix}$, then what is $\vec{x}_1$?
*Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.*

$$\left[\begin{array}{ccc|c}1&0&-1&1/4\\1&-1&1&1/2\\0&1&0&1/4\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&-1&1/4\\0&-1&2&1/4\\0&1&0&1/4\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&0&-1&1/4\\0&-1&2&1/4\\0&0&2&1/2\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&1/2\\0&-1&0&-1/4\\0&0&1&1/4\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&1/2\\0&1&0&1/4\\0&0&1&1/4\end{array}\right]$$

$\vec{x}_1 = \boxed{\frac{1}{2}V_1 + \frac{1}{8}V_2}$

check ✓

$$\vec{X}_0 = \begin{bmatrix}1/4\\1/2\\1/4\end{bmatrix} = \frac{1}{2}V_1 + \frac{1}{4}V_2 + \frac{1}{4}V_3 \qquad \vec{X}_1 = PX_0 = \frac{1}{2}*1V_1 + \frac{1}{4}*\frac{1}{2}V_2 + \frac{1}{4}\cdot 0V_3$$
$$= \frac{1}{2}V_1 + \frac{1}{8}V_2$$

(iii) If $\vec{x}_0 = \begin{pmatrix}1/4\\1/2\\1/4\end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$$\vec{X}_k = P^k\vec{X}_0 = \frac{1}{2}(1)^k V_1 + \frac{1}{4}\left(\frac{1}{2}\right)^k V_2 + \frac{1}{4}*(0)^k V_3$$

$\lim_{k\to\infty} \vec{x}_k = \boxed{\frac{1}{2}V_1}$

$$= \frac{1}{2}V_1 + \cancel{\frac{1}{4}\left(\frac{1}{2}\right)^k V_2} \to 0 \text{ as } k\to\infty$$