Math 1554 Linear Algebra Fall 2022

Midterm 3

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

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Section Number (e.g. A3, G2, etc.) ________ TA Name ______________________

Circle your instructor:

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Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

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You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	A matrix $A \in \mathbb{R}^{n \times n}$ and its transpose A^{T} have the same eigenvectors.
●	○	An invertible matrix A is diagonalizable if and only if its inverse A^{-1} is diagonalizable.
●	○	If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{u}$ is orthogonal to $(\vec{w} - \vec{v})$.
○	●	If the vectors $\vec{u}$ and $\vec{v}$ are orthogonal then $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.
○	●	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\|\operatorname{proj}_W(\vec{y})\|$ is the shortest distance between W and $\vec{y}$.
●	○	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\vec{y} - \operatorname{proj}_W(\vec{y})$ is in $W^{\perp}$.
○	●	If W is a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ such that $\vec{y} \cdot \vec{w} = 0$ for some vector $\vec{w} \in W$, then $\vec{y} \in W^{\perp}$.
○	●	The line of best fit $y = \beta_0 + \beta_1 x$ for the points $(1, 2), (1, 3)$, and $(1, 4)$ is unique.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	●	A 5×5 real matrix A such that A has no real eigenvalues.
○	●	An $m \times n$ matrix U where $U^T U = I_n$ and $n > m$.
●	○	A 2-dimensional subspace W of $\mathbb{R}^3$ and a vector $\vec{y} \in W$ such that $\|\vec{v}_1 - \vec{y}\| = \|\vec{v}_2 - \vec{y}\|$ where $\vec{v}_1, \vec{v}_2 \in W^{\perp}$ and $\vec{v}_1 \neq \vec{v}_2$.
○	●	A matrix $A \in \mathbb{R}^{3 \times 4}$ such that the linear system $A\vec{x} = \vec{b}$ has a unique least-squares solution.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) An $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ has non-zero orthogonal columns and $A^T A = 2I_n$. Which of the following statements is FALSE?

- [x] $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ for every $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.
- [ ] $n \leq m$.
- [ ] If we apply the Gram-Schmidt process to $\{\vec{a}_1, \ldots, \vec{a}_n\}$ we obtain the same set $\{\vec{a}_1, \ldots, \vec{a}_n\}$.
- [ ] If $A = QR$ is the QR factorization of A, then R is a diagonal matrix.

2. (3 points) Find a, b, c so that the set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} -2 \\ 0 \\ a \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} -1 \\ b \\ c \end{pmatrix}$$

$U_1 \cdot U_2 = 0$

$\Rightarrow -2 + a = 0 \Rightarrow a = 2$

$a = \boxed{2}$

$b = \boxed{2}$

$c = \boxed{-1}$

$$\left.\begin{matrix} U_1 \cdot U_3 = 0 \\ U_2 \cdot U_3 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} -1 + b + c = 0 \\ 2 + 2c = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} c = -1 \\ b = 2 \end{matrix}$$

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You do not need to justify your reasoning for questions on this page.

3. (8 points) Fill in the blanks.

(a) Suppose $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ and that $\vec{v}$ is a unit vector. If $(2\vec{u}+\vec{v})\cdot(\vec{u}+5\vec{v}) = 13$, determine the length of $\vec{u}$. $\|\vec{u}\| = \boxed{2}$

$$(2u+v)\cdot(u+5v) = 2u\cdot u + 10u\cdot v + u\cdot v + 5v\cdot v$$

$$\Rightarrow 2\|u\|^2 + 5 = 13 \qquad \|u\|^2 = \frac{13-5}{2} = 4$$

(b) The normal equations for the least-squares solution to $A\vec{x} = \vec{b}$ are given by:

$$\boxed{A^TA\hat{x} = A^Tb}$$

(c) Compute the length (magnitude) of the vector $\vec{y}$.

$$\vec{y} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\|\vec{y}\| = \boxed{\sqrt{14}}$

$\|Ux\| = \|x\|$ if U has orthonormal cols.

So $\|y\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

(d) Let $A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$. The vector $\vec{v} = \begin{pmatrix} -1-i \\ -1+i \end{pmatrix}$ is an eigenvector of A. Find the associated eigenvalue λ for the eigenvector $\vec{v}$ of A. $\lambda = \boxed{2+2i}$

$$\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1-i \\ -1+i \end{bmatrix} = \begin{bmatrix} 2(-1-i)-2(-1+i) \\ 2(-1-i)+2(-1+i) \end{bmatrix} = \begin{bmatrix} -4i \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} -4i \\ -4 \end{bmatrix} = \lambda \begin{bmatrix} -1-i \\ -1+i \end{bmatrix} \Rightarrow \lambda * (-1-i) = -4i$$

$$\Rightarrow \lambda = \frac{-4i}{-1-i} = \frac{4i}{1+i} * \frac{1-i}{1-i} = \frac{4i-4i^2}{1-(-1)^2}$$

$$= \frac{4+4i}{2} = 2+2i$$

check

$$\lambda = \frac{-4}{(-1+i)} = \frac{-4}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{4+4i}{2} = 2+2i$$

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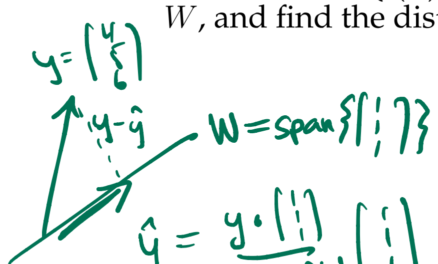
You do not need to justify your reasoning for questions on this page.

4. (4 points) Fill in the blanks.

(a) Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ and let $\vec{y} = \begin{pmatrix}4\\5\\6\end{pmatrix}$. Calculate the projection of $\vec{y}$ onto the subspace W, and find the distance from $\vec{y}$ to W.



$$\hat{y} = \frac{y\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{15}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\5\\5\end{bmatrix}$$

$\text{proj}_W(\vec{y}) = \boxed{\begin{pmatrix}5\\5\\5\end{pmatrix}}$

$\text{dist}(\vec{y}, W) = \boxed{\sqrt{2}}$

$$y - \hat{y} = \begin{bmatrix}4\\5\\6\end{bmatrix} - \begin{bmatrix}5\\5\\5\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

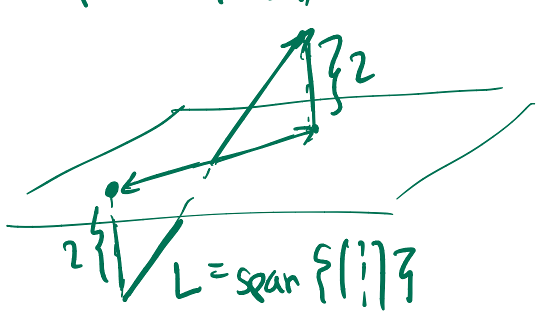
(b) Let $W = \text{span}\left\{\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\vec{x}_1}, \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\vec{x}_2}\right\}$, $L = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Find **all** vectors $u \in L$ such that the distance from u to W is equal to 2.

$$V_1 = X_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$V_2 = X_2 - \frac{X_2\cdot X_1}{X_1\cdot X_1}X_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\boxed{\begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}-2\\-2\\-2\end{bmatrix}}$$

$$W = \text{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$



Let $c\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = u$ then

$$\text{proj}_W(u) = \frac{u\cdot V_1}{V_1\cdot V_1}V_1 + \frac{u\cdot V_2}{V_2\cdot V_2}V_2$$

$$= \frac{c}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{c}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}c\\c\\0\end{bmatrix}$$

need $\left\|\begin{bmatrix}c\\c\\c\end{bmatrix} - \begin{bmatrix}c\\c\\0\end{bmatrix}\right\| = 2$ so $c = \pm 2$

$(\lambda-1)(\lambda+1)$

Midterm 3. Your initials: _______________

5. (6 points) **Show all work for problems on this page.**

   One of the eigenvalues of the matrix A is $\lambda = 1$. Diagonalize the matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$p(\lambda) = \det(A - \lambda I)$

$= \det\begin{bmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ -1-\lambda & -1 \end{vmatrix}$

$= (1-\lambda)[(-1-\lambda)(1-\lambda)+1] + (1-\lambda-1) + (-1-(-1-\lambda))$

$= -(\lambda-1)[\lambda^2 - 1 + 1] + (-\lambda) + \lambda$ als mult 2

$= -\lambda^3 + \lambda^2 = -\lambda^2(1-\lambda) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$

$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underline{\lambda = 0}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (s, t) $\quad x = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = 1}$

$A - I = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x = s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

6. (3 points) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 4$, and $\vec{v}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = -3$.

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$A = PDP^{-1}$ where $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\frac{1}{3-2}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} 18 & -21 \\ 14 & -17 \end{bmatrix}$

$A = \begin{bmatrix} 18 & -21 \\ 14 & -17 \end{bmatrix}$

Midterm 3. Your initials: ______________

7. (4 points) **Show all work for problems on this page.**
Let $\mathcal{B} = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ be a basis for a subspace W of $\mathbb{R}^4$, where

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}.$$

(a) Apply the Gram-Schmidt process to the set of vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ to find an orthogonal basis $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for W. *Clearly show all steps of the Gram-Schmidt process.*

$$V_1 = X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathcal{H} = \boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right\}}$$

$$V_2 = X_2 - \frac{X_1 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = V_2$$

$$V_3 = X_3 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{-3}{6} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3/2 \\ 1/2 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = V_3$$

(b) In the space below, **check** that the vectors in the basis $\mathcal{H}$ form an orthogonal set.

$$V_1 \cdot V_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = -1 + 0 + 2 - 1 = 0 \checkmark$$

$$V_1 \cdot V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = 0 - 3 + 1 + 2 = 0 \checkmark$$

$$V_2 \cdot V_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = 0 + 0 + 2 - 2 = 0 \checkmark$$

8. (4 points) **Show all work for problems on this page.**
If A has the following QR factorization

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 0 & 3 \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix},$$

compute the least-square solution to the equation $A\vec{x} = \vec{b}$.

$A = QR$

$\&$

$A^T A \hat{x} = A^T b$

$\Rightarrow (QR)^T QR\hat{x} = (QR)^T b$

$\Rightarrow \cancel{R^T} \cancel{Q^T Q} R\hat{x} = \cancel{R^T} Q^T b$

$\Rightarrow R\hat{x} = Q^T b$

$$Q^T b = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$[R \mid Q^T b] = \left[\begin{array}{cc|c} 7 & 2 & 0 \\ 0 & 3 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 7 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 7 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -2/7 \\ 0 & 1 & 1 \end{array}\right]$$

$$\hat{x} = \boxed{\begin{bmatrix} -2/7 \\ 1 \end{bmatrix}}$$

9. (4 points) Compute the least squares line $y = c_1 + c_2 x$ that best fits the data

x	-1	0	1
y	4	2	5

$$A = \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

$A^T A\hat{x} = A^T b$

$$\left[\begin{array}{cc|c} 3 & 0 & 11 \\ 0 & 2 & 1 \end{array}\right] \sim \begin{array}{cc} \left[\begin{array}{cc|c} 1 & 0 & 11/3 \\ 0 & 1 & 1/2 \end{array}\right] & \begin{array}{c} \leftarrow c_1 \\ \leftarrow c_2 \end{array} \end{array}$$

$$\boxed{y = \frac{11}{3} + \frac{1}{2}x}$$