LINEAR ALGEBRA

Week 13

Theorem 1

If G is a regular $m \times m$ transition matrix with $m \geq 2$, then the following statements are all true.

1. There is a stochastic matrix Π such that
$$\lim_{n\to\infty} G^n = \Pi = [\vec{q}\ \vec{q} \cdots \vec{q}]$$
2. Each column of Π is the same probability vector $\vec{q}$.
3. For any initial probability vector $\vec{x}_0$,
$$\lim_{n\to\infty} G^n \vec{x}_0 = \vec{q}$$
4. P has a unique eigenvector, $\vec{q}$, which has eigenvalue $\lambda = 1$.
5. The eigenvalues of G satisfy $|\lambda| \leq 1$.

We will apply this theorem when solving PageRank problems.

$G^n \vec{e}_i \approx \vec{q}$ for any i & $n >> 0$.

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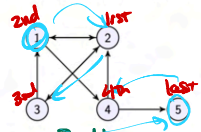
Example 1

Suppose we have 4 web pages that link to each other according to this diagram.



$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Page 1 has links to pages 2, 4.

Page 2 has links to pages 1, 3.

If a user on a page in this web is **equally likely** to go to any of the pages that their page links to, construct a Markov chain that represents how users navigate this web.

```
clc
format bank
%% rental car - long term analysi
A=[.8 .1 .2 ; .2 .6 .3 ; 0 .3 .5]
k=10
A^k


%% google PageRank
P0=[0   1/2 1/2 0   1/5 ;
    1/2 0   1/2 1/2 1/5 ;
    0   1/2 0   0   1/5 ;
    1/2 0   0   0   1/5 ;
    0   0   0   1/2 1/5 ];
K0=(1/5)*[1 1 1 1 1 ;
    1 1 1 1 1 ;
    1 1 1 1 1;
    1 1 1 1 1 ;
    1 1 1 1 1];
G0=.85*P0+.15*K0;
P1=[0 0 1 0 1/5
    1/3 0 0 1/2 1/5 ;
    1/3 0 0 1/2 1/5 ;
    1/3 1/2 0 0 1/5 ;
    0 1/2 0 0 1/5];
G1=.85*P1+.15*K0;
P2=[0 1/2 0   1/7 0   0   1/7 ;
    0 0   1/3 1/7 1/2 0   1/7 ;
    1 0   0   1/7 0   1/3 1/7 ;
    0 0   1/3 1/7 0   0   1/7 ;
    0 1/2 0   1/7 0   0   1/7 ;
    0 0   1/3 1/7 1/2   1/3 1/7;
    0 0   0   1/7 0   1/3 1/7];
K2=1/7*[1 1 1 1 1 1 1;
    1 1 1 1 1 1 1;
    1 1 1 1 1 1 1;
    1 1 1 1 1 1 1;
    1 1 1 1 1 1 1;
    1 1 1 1 1 1 1;
    1 1 1 1 1 1 1];
G2=.85*P2+.15*K2;
k=20;
test=[1 1 1 1 1 1 1]*G2;
format short
for i=1:k
    i;
    G1^i;
end
```

G^{15} in spot 3,1

17% chance that you end up @ site #3 after 15 steps starting in website #1

Transition Matrix, Importance, and PageRank

- The square matrix we constructed in the previous example is a **transition matrix**. It describes how users transition between pages in the web.
- The steady-state vector, $\vec{q}$, for the Markov-chain, can characterize the long-term behavior of users in a given web.
- If $\vec{q}$ is unique, the **importance** of a page in a web is given by its corresponding entry in $\vec{q}$.
- The **PageRank** is the ranking assigned to each page based on its importance. The highest ranked page has PageRank 1, the second PageRank 2, and so on.

Is the transition matrix in Example 1 a regular matrix?

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Adjustment 1

Adjustment 1

If a user reaches a page that doesn't link to other pages, then the user will choose any page in the web, with equal probability, and move to that page.

Let's denote this modified transition matrix as P_*. Our transition matrix in Example 1 becomes:

$$P^* = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 1/5 \\ 1/2 & 0 & 1/2 & 1/2 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/5 \\ 1/2 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 1/2 & 1/5 \end{bmatrix}$$

The first adjustment

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Adjustment 2

Adjustment 2

A user at any page will navigate any page among those that their page links to with equal probability p, and to any page in the web with equal probability $1-p$. The transition matrix becomes
$$G = pP_* + (1-p)K$$
All the elements of the $n \times n$ matrix K are equal to $1/n$.

p is referred to as the **damping factor**. Google is said to use $p = 0.85$.

With adjustments 1 and 2, our the Google matrix is:

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Computing Page Rank

- Because G is stochastic, for any initial probability vector $\vec{x}_0$,
$$\boxed{\lim_{n\to\infty} G^n \vec{x}_0 = \vec{q}}$$
$x_0 = \vec{e}_1$ ✓
- In practice we can compute the page rank for each page in the web by evaluating:
$$G^n \vec{x}_0$$
for large n. The elements of the resulting vector give the page ranks of each page in the web.

$G^n \vec{e}_1 =$ first col of G^n

On a MATH 1554 exam,

- problems that require a calculator will not be on your exam
- you may construct your G matrix using factions instead of decimal expansions

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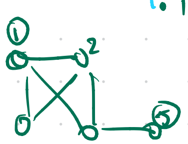
$$G = (.85) P^* + (.15) \begin{bmatrix} 1/5 & \cdots & 1/5 \\ \vdots & & \vdots \\ 1/5 & \cdots & 1/5 \end{bmatrix}$$

$$G = (.85) \underbrace{\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 1/5 \\ 1/2 & 0 & 1/2 & 1/2 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/5 \\ 1/2 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 1/2 & 1/5 \end{bmatrix}}_{P_0} + (.15) \underbrace{\begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}}_{K_0}$$

$$G^{15} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .2517 \\ .2985 \\ .1766 \\ .1568 \\ .1164 \end{bmatrix}$$

column $.85\, V + .15\, W$ = also be a prob vector

prob vector



$$G^{15} \approx \begin{bmatrix} .2517 & .2517 & .2517 & .2517 & .2517 \\ .2985 & .2985 & .2985 & .2985 & .2985 \\ .1766 & .1766 & .1766 & .1766 & .1766 \\ .1568 & .1568 & .1568 & .1568 & .1568 \\ .1164 & .1164 & .1164 & .1164 & .1164 \end{bmatrix}$$

row 3

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

If $A^T = A$ symmetric.

What can you do w/ A?

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices ✓
2. Orthogonal diagonalization
3. Spectral decomposition

$A = PDP^T$

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.
2. Construct a spectral decomposition of a matrix.

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course mat

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Symmetric Matrices

Definition

Matrix A is **symmetric** if $\boxed{A^T = A.}$

only works if A is square n×n.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

✓ $A = [*]$ ✓ $B = \begin{bmatrix} * & \star \\ \star & * \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ ✓ $C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = C^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

✗ $D = \begin{bmatrix} * & \star \\ 0 & 0 \end{bmatrix}$ ✗ $E = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix}$

a_{12}, a_{21}

$a_{ij} = a_{ji}$ (row, col)

$D = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

$\neq D^T = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

$E^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A^TA is Symmetric

A very common example: For **any** matrix A with columns $a_1, \dots, a_n$,

$$A^TA = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \cdots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \cdots & a_2^Ta_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \cdots & a_n^Ta_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

e.g. $A = [a_1 \; a_2]$

$A^TA = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 \\ a_2 \cdot a_1 & a_2 \cdot a_2 \end{bmatrix}$ always symmetric

$\boxed{A^T = A}$ Spectrum of A = set of eigenvalues.

Spectral Theorem.

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

ORTHOGONAL DIAGONALIZATION

$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = PDP^T$

want P to have orthonormal cols so that $P^T = P^{-1}$

$= \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^T$

(otherwise GS)

lucky! these were already orthogonal

Example 1

diag'ble

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

all eigenvectors $\lambda = 1$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \dots$

in the $\lambda_1 = 1$ eigenspace

geo 2

$\lambda_1 = 1$ $A - I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{x} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2 free vars in $A - \lambda I$ ✓

geo 1

$\lambda_2 = -1$. $A + I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{x} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Check $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0$ ✓

$\lambda_1 = 1$ $\lambda_2 = -1$

P n×n

$P^TP = I$

$\Rightarrow$ P has orthonormal cols & $P^{-1} = P^T$

Spectral Theorem

$A \stackrel{?}{=} A^T$

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real
2. The dimenison of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity.
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is orthogonal.

Proof (if time permits):

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = [\vec{u}_1 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank ______.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \underbrace{\begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \end{array} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P} \underbrace{\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} P^T$$

$$\begin{array}{ccccccccccc} A & & \lambda_1 & \vec{u}_1 & \vec{u}_1^T & + & \lambda_2 & \vec{u}_2 & \vec{u}_2^T & + & \lambda_3 \ \vec{u}_3 \ \vec{u}_3^T \end{array}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + (1) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} + (-1) \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} - 1 \begin{bmatrix} +1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

cf. $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

$$\begin{array}{c} 100 \\ 100 \begin{bmatrix} \cdots \\ \vdots \end{bmatrix} \end{array} \approx \sum_{i=1}^{\cancel{100}\ r} \lambda_i u_i u_i^T = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T \cdots + \lambda_{100} u_{100} u_{100}^T$$

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \overset{u_1}{\boxed{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}}\overset{u_2}{\boxed{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}} \begin{pmatrix} \overset{\lambda_1}{4} & 0 \\ 0 & \overset{\lambda_2}{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \underbrace{4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{} + 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

best possible rank 1 approx of $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

7.1 Exercises

Determine which of the matrices in Exercises 1–6 are symmetric.

1. $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

7. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

10. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

11. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$

12. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix P and a diagonal matrix D. To save you time, the eigenvalues in Exercises 17–22 are the following: (17) $-4, 4, 7$; (18) $-3, -6, 9$; (19) $-2, 7$; (20) $-3, 15$; (21) $1, 5, 9$; (22) $3, 5$.

13. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Then orthogonally diagonalize A.

In Exercises 25–32, mark each statement True or False (T/F). Justify each answer.

25. **(T/F)** An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

26. **(T/F)** There are symmetric matrices that are not orthogonally diagonalizable.

27. **(T/F)** An orthogonal matrix is orthogonally diagonalizable.

28. **(T/F)** If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

29. **(T/F)** For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

30. **(T/F)** If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

31. **(T/F)** An $n \times n$ symmetric matrix has n distinct real eigenvalues.

32. **(T/F)** The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

33. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

34. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB, and BB^T are symmetric matrices.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

39. Construct a spectral decomposition of A from Example 2.

40. Construct a spectral decomposition of A from Example 3.

41. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.
    a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.
    b. Show that B is a symmetric matrix and $B^2 = B$.
    c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

42. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.
    a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.
    b. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^\perp$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

Orthogonally diagonalize the matrices in Exercises 43–46. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue λ, find an orthonormal basis for $\mathrm{Nul}(A - \lambda I)$, as in Examples 2 and 3.

T 43. $\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$

T 44. $\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$

T 45. $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

T 46. $\begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$

$z = x^2 + y^2 = f(x,y)$

$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

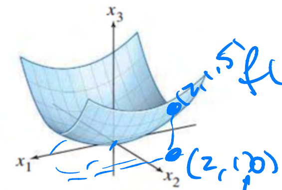
Section 7.2 : Quadratic Forms

$f(2,1) = 5$

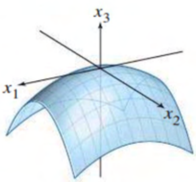
Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

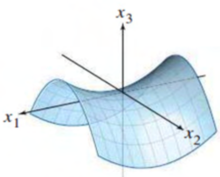
$z = 3x^2 + 4y^2$



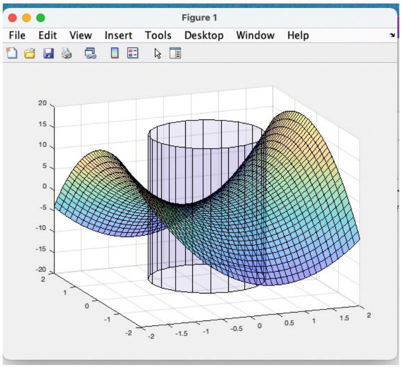
Positive definite

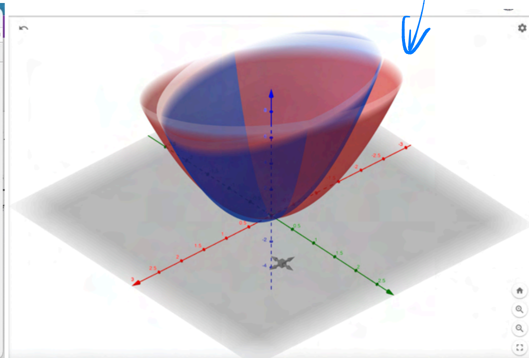


Negative definite



Indefinite





Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A\vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question ~~Does this inequality hold for all x, y?~~

$$\boxed{x^2 - 6xy + 9y^2 \geq 0} \leftarrow \text{quadratic homogeneous form}$$

Quadratic Forms

Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.

$$\underset{1\times2}{[x \;\; y]}\underset{2\times2}{\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}}\underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = [x \;\; y]\begin{pmatrix} x-3y \\ -3x+9y \end{pmatrix}$$

$$= x(x-3y) + y(-3x+9y)$$

$$= x^2 - 3xy - 3xy + 9y^2$$

$$x^T A x = \boxed{x^2 - 6xy + 9y^2}$$

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course m

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Example 1

Compute the quadratic form $\vec{x}^T A\vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\vec{x}^T A \vec{x} = [x \;\; y]\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [x \;\; y]\begin{bmatrix} 4x \\ 3y \end{bmatrix} = \boxed{\underbrace{4x^2+3y^2}_{Q_A(x,y)} \geq 0}$$

any $x, y \in \mathbb{R}$

$4x^2 + 3y^2 = 0 \Rightarrow x = y = 0$

$$\vec{x}^T B \vec{x} = [x \;\; y]\begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x \;\; y]\begin{bmatrix} 4x+y \\ x-3y \end{bmatrix}$$

$$= x(4x+y) + y(x-3y)$$

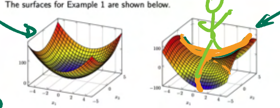
$$= 4x^2 + xy + xy - 3y^2$$

$$= \boxed{4x^2 + 2xy - 3y^2 = Q_B(x,y)}$$

Example 1 - Surface Plots

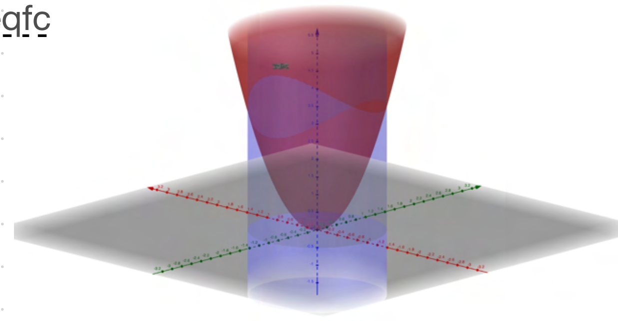
The surfaces for Example 1 are shown below.

Q_A Q_B



Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

https://www.geogebra.org/m/pbzpeqfc



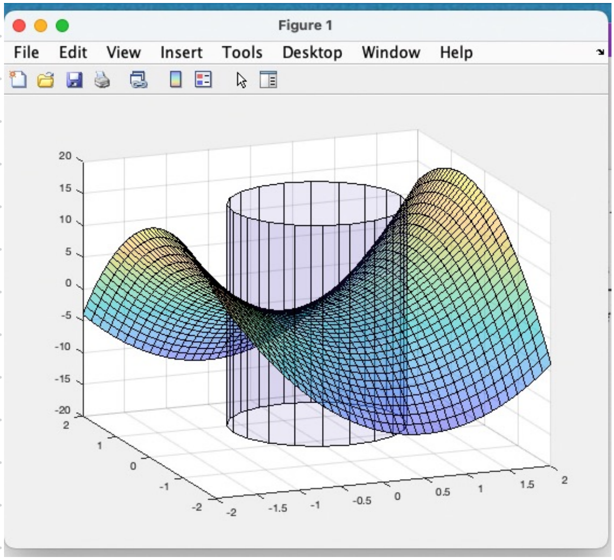
```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



Example 2

Write Q in the form $\vec{x}^T A \vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$Q(x) = \underbrace{5x_1^2 - 1x_2^2 + 3x_3^2}_{\text{square terms have coeff's on diagonal.}} + 6x_1x_3 - 12x_2x_3 + 0x_1x_2$$

$$A = \cancel{\begin{bmatrix} 5 & 0 & 6 \\ 0 & -1 & -12 \\ 0 & 0 & 3 \end{bmatrix}}$$

$$\vec{x}^T A x = 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3$$

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}$$

we want A matrix to be symmetric so that the shape of the graph can be known/controlled by the eigenvalues/eigenvectors.

Change of Variable

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A \vec{x}$ becomes:

once we know

$$A = PDP^T \qquad \boxed{P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$$

then

$$\underbrace{\boxed{\vec{x}^T A \vec{x}}}_{Q_A(x_1,x_2)} = \vec{x}^T P D P^T \vec{x}$$

$$= \underbrace{(P^T x)^T}_{\vec{y}^T} D \underbrace{P^T x}_{\vec{y}}$$

$$= \underbrace{\boxed{\vec{y}^T D \vec{y}}}_{Q_D(y_1,y_2)}$$

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A \vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$A = PDP^T$$

orthonormal cols.

$$P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 9\lambda + 14 = (\lambda-7)(\lambda-2) = 0 \qquad D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 7 \qquad \lambda_2 = 2$$

$$\underline{\lambda_1 = 7} \quad A - 7I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad s=2 \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

check?

$$\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$$

$$\underline{\lambda_2 = 2} \quad A - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\boxed{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix}3 & 2\\ 2 & 6\end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$$

$$D = \begin{pmatrix}7 & 0\\ 0 & 2\end{pmatrix}$$

orthonormal col.

$$P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$$

$$A = PDP^T$$

$$\underbrace{[x_1, x_2]\,P}_{y^T}\,D\,\underbrace{P^T \begin{bmatrix}x_1\\x_2\end{bmatrix}}_{y}$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 2\end{bmatrix}$$

$$Q_A(x_1,x_2) = [x_1\ x_2]\begin{bmatrix}3 & 2\\ 2 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2^2$$

Change of vars.

$$x^TAx = x^TPDP^Tx = (P^Tx)^T D P^Tx = \vec{y}^T D\vec{y}$$

$$Q_D(y_1,y_2) = [y_1\ y_2]\begin{bmatrix}7 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = 7y_1^2 + 2y_2^2$$

$$\boxed{P^Tx = \vec{y}}$$

$$Q_A(1,1) = 3(1)^2 + 4(1)(1) + 6(1)^2 = 13$$

$$\vec{x} = \begin{array}{c|l} \begin{bmatrix}1\\1\end{bmatrix} & \leftarrow x_1 \\ & \leftarrow x_2 \end{array} \qquad P^Tx = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}3/\sqrt{5}\\ -1/\sqrt{5}\end{bmatrix} \begin{array}{l}\leftarrow y_1\\ \leftarrow y_2\end{array}$$

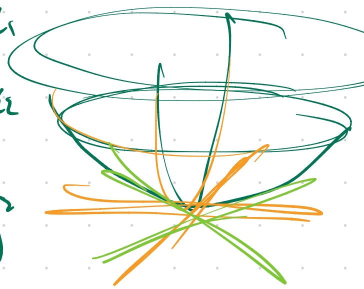
$$P(P^T\vec{x}) = Py$$

$$\boxed{\vec{x} = Py}$$

$$Q_D\left(\frac{3}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) = 7\left(\frac{3}{\sqrt{5}}\right)^2 + 2\left(\frac{-1}{\sqrt{5}}\right)^2 = \frac{63}{5} + \frac{2}{5} = \frac{65}{5} = 13 \checkmark$$

$$Q_D(1,0) = 7(1)^2 + 2(0)^2 = 7$$

$$\vec{y} = \begin{bmatrix}1\\0\end{bmatrix},\ P\vec{y} = \vec{x} \qquad \begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}\begin{array}{l}\leftarrow x_1\\ \leftarrow x_2\end{array}$$



$$Q_A\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 3\left(\frac{1}{\sqrt{5}}\right)^2 + 4\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) + 6\left(\frac{2}{\sqrt{5}}\right)^2$$

$$= \frac{3}{5} + \frac{8}{5} + \frac{24}{5} = \frac{35}{5} = 7$$

Principle Axes Theorem

Theorem

If A is a Symmetric matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{x}^T D\vec{x}$ with no cross-product terms.

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

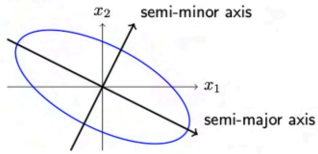
$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

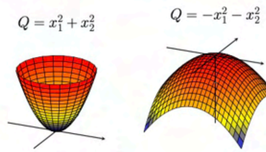
$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

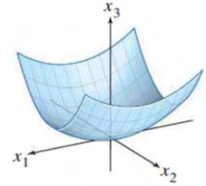
Example 5

Compute the quadratic form $Q = \vec{x}^T A\vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.

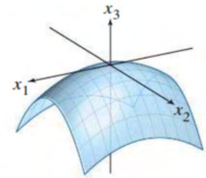


Classifying Quadratic Forms

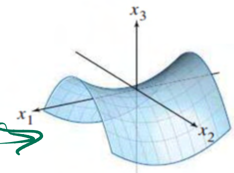




Positive definite



Negative definite



Indefinite

Definition

A quadratic form Q is

1. **positive definite** if $Q > 0$ for all $\vec{x} \neq \vec{0}$.
2. **negative definite** if $Q < 0$ for all $\vec{x} \neq \vec{0}$.
3. **positive semidefinite** if $Q \geq 0$ for all $\vec{x}$.
4. **negative semidefinite** if $Q \leq 0$ for all $\vec{x}$.
5. **indefinite** if sometimes $Q > 0$ & sometimes $Q < 0$.

$Q(x_1, x_2) = 2x_1^2 - 3x_2^2$

$Q(1,0) = 2$

$Q(0,1) = -3$

$D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ $\lambda_1 = 2$, $\lambda_2 = -3$

Quadratic Forms and Eigenvalues

Theorem

If A is a symmetric matrix with eigenvalues λ_i, then $Q = \vec{x}^T A \vec{x}$ is

1. **positive definite** iff $\lambda_i > 0$ (all λ_i's)
2. **negative definite** iff $\lambda_i < 0$
3. **indefinite** iff λ_i positive and negative.

positive semi-definite iff $\lambda_i \geq 0$ (for all i)

$\cancel{\lambda_3} = Q(x_1, x_2) = 2x_1^2 + 0x_2^2$

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y?

$$x^2 - 6xy + 9y^2 \overset{?}{\geq} 0$$

Is Q_A positive semi-definite. Yes b/c $\lambda_i \geq 0$

$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$

$Q_A(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$x^T A x = x^T P D P^T x$

$= (P^T x)^T D \, P^T x$ $\uparrow \begin{bmatrix} a \\ b \end{bmatrix}$

$p(\lambda) = \lambda^2 - 10\lambda + \overbrace{(1)(9) - (-3)(-3)}^{\det A = 0}$

$= \lambda^2 - 10\lambda = (\lambda - 10)(\lambda - 0) = 0$

$\boxed{\lambda_1 = 10, \ \lambda_2 = 0}$

$Q_D(a,b) = 10a^2 + 0b^2$

$\begin{bmatrix} a \\ b \end{bmatrix} = P^T \begin{bmatrix} x \\ y \end{bmatrix}$

$\boxed{D = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}}$

$A - 10\lambda I = \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$

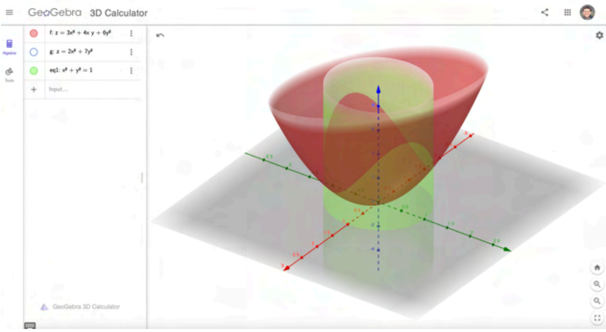
$v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

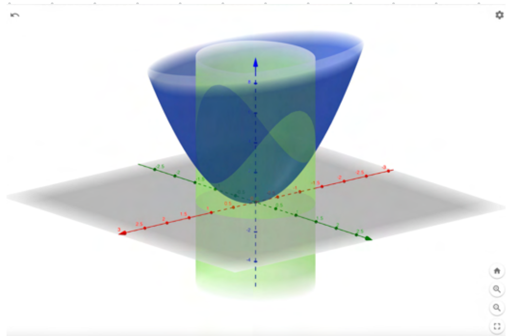
$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \boxed{\begin{bmatrix} -1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}}$

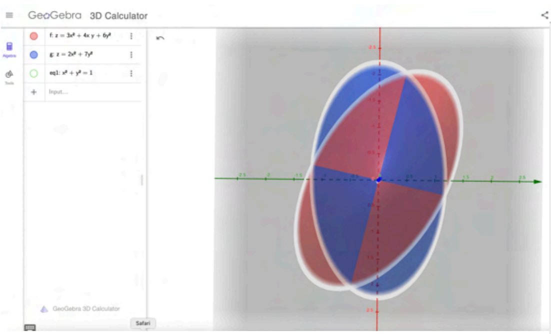
$\lambda_1 = 10$ $\lambda_2 = 0$

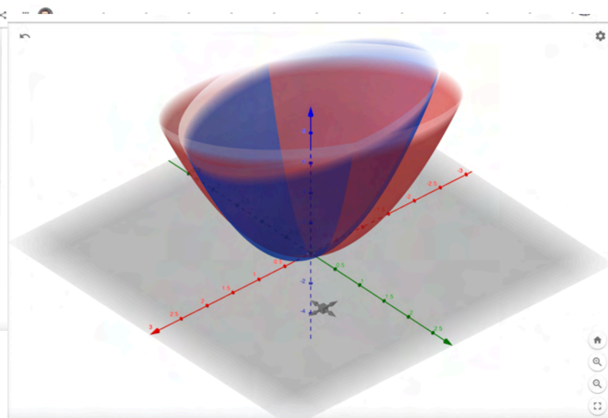
$x = s \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

$v_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

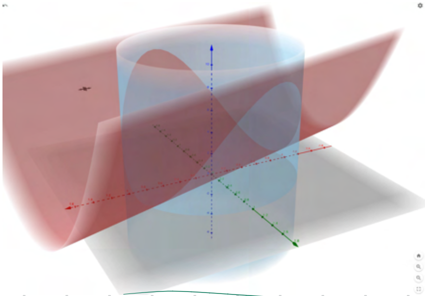


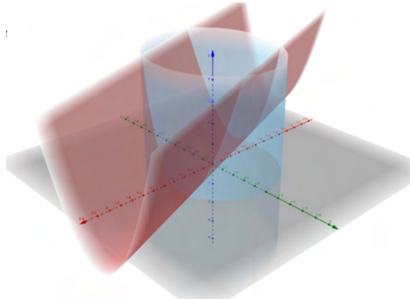






https://www.geogebra.org/m/c6yg2agh





f: $z = x^2 - 6xy + 9y^2$

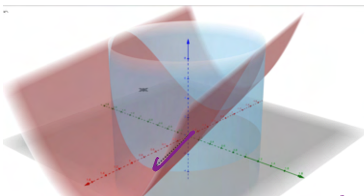
eq1: $x^2 + y^2 = 1$

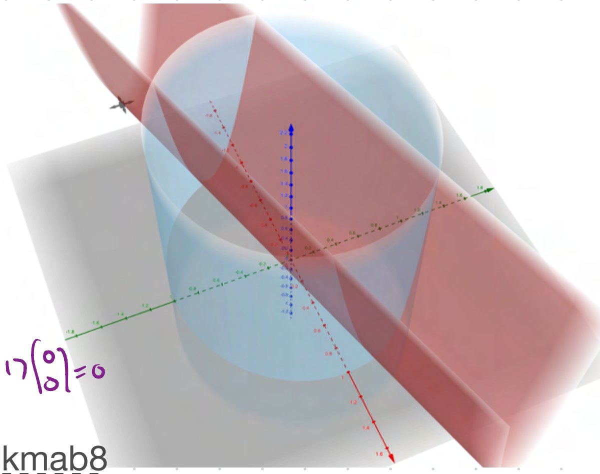
$u = \text{Vector}\left(\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right)$

$= \begin{pmatrix} 0.95 \\ 0.32 \end{pmatrix}$

$$Q_A(x_1,x_2) = x_1^2 - 6x_1x_2 + 9x_2^2 \geq 0$$

$$Q_D(y_1,y_2) = 10y_1^2 + 0y_2^2$$





$$\underbrace{[3\ \ 1]}_{x^T}\underbrace{\begin{pmatrix}1 & -3\\ -3 & 9\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}3\\ 1\end{pmatrix}}_{x} = [3\ \ 1]\begin{bmatrix}3-3\\ -9+9\end{bmatrix} = [3\ \ 1]\begin{pmatrix}0\\ 0\end{pmatrix} = 0$$

https://www.geogebra.org/m/akbkmab8

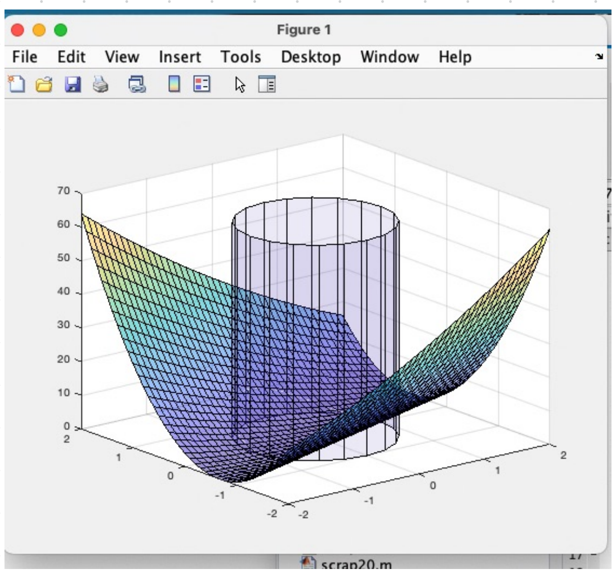
```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $3x_1^2 - 4x_1x_2 + 5x_2^2$ b. $3x_1^2 + 2x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $5x_1^2 + 16x_1x_2 - 5x_2^2$ b. $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$

b. $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 - 2x_2^2 + 5x_3^2 + 4x_1x_2 - 6x_1x_3$

b. $4x_3^2 - 2x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

8. Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct P using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$ **10.** $2x_1^2 + 6x_1x_2 - 6x_2^2$

11. $2x_1^2 - 4x_1x_2 - x_2^2$ **12.** $-x_1^2 - 2x_1x_2 - x_2^2$

13. $x_1^2 - 6x_1x_2 + 9x_2^2$ **14.** $3x_1^2 + 4x_1x_2$

15. [M] $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$

16. [M] $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$

17. [M] $11x_1^2 + 11x_2^2 + 11x_3^2 + 11x_4^2 + 16x_1x_2 - 12x_1x_4 + 12x_2x_3 + 16x_3x_4$

18. [M] $2x_1^2 + 2x_2^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.

b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.

c. The principal axes of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are eigenvectors of A.

d. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

e. If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.

f. A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is not a quadratic form.

b. If A is symmetric and P is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term.

c. If A is a 2×2 symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola.

d. An indefinite quadratic form is neither positive semidefinite nor negative semidefinite.

e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all positive.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of A.

23. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.

a. Q is positive definite if $\det A > 0$ and $a > 0$.

b. Q is negative definite if $\det A > 0$ and $a < 0$.

c. Q is indefinite if $\det A < 0$.

25. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

26. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.]

B works.]

27. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let A be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

SG Mastering: Diagonalization and Quadratic Forms 7-7

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

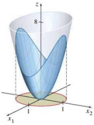


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

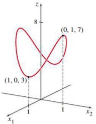
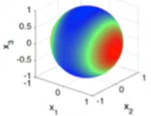


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.



13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

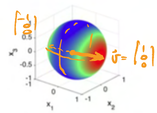
Example 1

The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = \|\vec{x}\|^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$



Find the largest and smallest values of Q on the surface of the sphere.

$Q\left(\begin{bmatrix}:\\:\end{bmatrix}\right) = ?$ plug in only unit vectors?

$$Q\left(\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}\right) = 9\cdot\left(\tfrac{1}{\sqrt{2}}\right)^2 + 4\,0^2 + 3\left(\tfrac{1}{\sqrt{2}}\right)^2 = \frac{9+3}{2} = \frac{12}{2} = 6$$

$$Q\left(\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}\right) = 9\left(\tfrac{1}{3}\right)^2 + 4\left(\tfrac{2}{3}\right)^2 + 3\left(\tfrac{2}{3}\right)^2 = \frac{1\cdot 9 + 4\cdot 4 + 3\cdot 4}{9} = \frac{37}{9} = 4.111\ldots$$

$$Q\left(\begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}\right) = 9\left(-1/\sqrt{2}\right)^2 + 4\,0^2 + 3\left(-1/\sqrt{2}\right)^2 = \frac{9+3}{2} = 6.$$

$Q\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$$3 \leq Q\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) \leq 9 \leftarrow Q\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$\left\|\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right\| = 1$$

Ex. Find the largest output z-value with restricted input ||x||=1 where z is given by:

$$z = 3x_1^2 + 7x_2^2.$$

<u>MAX</u> is 7 occurs at

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \quad \& \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}.$$

MIN is 3 occurs at

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \& \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

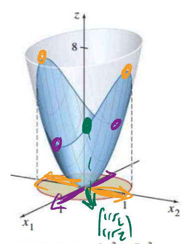


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

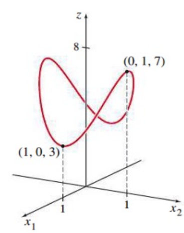


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.

$x = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

Plug in something else get output between 3 & 7.

$$z\left(1/\sqrt{2}, 1/\sqrt{2}\right) = 3\left(\tfrac{1}{\sqrt{2}}\right)^2 + 7\left(\tfrac{1}{\sqrt{2}}\right)^2 = \frac{3}{2} + \frac{7}{2} = 10/2 = 5$$

EXAMPLE 3 Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of the quadratic form $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, and find a unit vector at which this maximum value is attained.

SOLUTION By Theorem 6, the desired maximum value is the greatest eigenvalue of A. The characteristic equation turns out to be

$$0 = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 6)(\lambda - 3)(\lambda - 1)$$

The greatest eigenvalue is 6.

$\lambda_1 = 6$
$\lambda_2 = 3$
$\lambda_3 = 1$.

$P = (v_1 \; v_2 \; v_3)$

$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$Q_A(x_1, x_2, x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

$Q_A(1,0,0) = 3$
$Q_A(0,1,0) = 3$
$Q_A(0,0,1) = 4$

$P^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ or $\boxed{P\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$

plug into Q_A to get 6 output

$Q_A(x_1, x_2, x_3) = Q_D(y_1, y_2, y_3) = 6y_1^2 + 3y_2^2 + 1y_3^2$

$Q_D(1,0,0) = 6$ ✓

$\underbrace{P\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ $= \begin{pmatrix} * \\ * \\ * \end{pmatrix}$

$v_1 =$ "normalized $\lambda = 6$ eigenvector"

$\underline{\underline{\lambda_1 = 6}}$

$P - 6I = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{X} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{V}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$Q_A(x_1, x_2, x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

$Q_A\left(\underline{1/\sqrt{3}}, \underline{1/\sqrt{3}}, \underline{1/\sqrt{3}}\right) = 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)$

$= \frac{1}{3}(3+3+4+4+2+2) = \frac{18}{3} = \boxed{6}$

EXAMPLE 5 Let A be the matrix in Example 3 and let $\mathbf{u}_1$ be a unit eigenvector corresponding to the greatest eigenvalue of A. Find the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the conditions

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T\mathbf{u}_1 = 0 \tag{4}$$

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

subject to

$$||\vec{x}|| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$
$$M = \max\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $||\vec{x}|| = 1$,

- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm \vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm \vec{u}_n$.

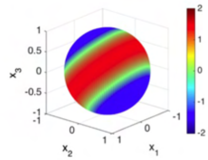
Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$-(a-b) = (b-a)$$

$$= (1-\lambda)\big((-\lambda)(-\lambda) - (1)(1)\big)$$

$$= (1-\lambda)(\lambda^2 - 1)$$

$$= -(\lambda-1)(\lambda^2-1)$$

$$p(\lambda) \checkmark \qquad \checkmark$$

$$= \boxed{-\lambda^3 + \lambda^2 + \lambda - 1}$$

$$p(\lambda).$$

$$= \boxed{-(\lambda-1)(\lambda-1)(\lambda+1)}$$

$$\lambda_1 = 1 \quad (\text{alg} = \text{geo} = 2)$$

$$\lambda_2 = -1 \quad (\text{alg} = \text{geo} = 1)$$

$$\boxed{Q_D(y_1, y_2, y_3) = y_1^2 + y_2^2 - y_3^2}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_*$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$ and to $\vec{x} \cdot \vec{u}_1 = 0$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$P^T x = y$

$\boxed{Py = x}$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$Q_A(x_1, x_2) = \boxed{3x_1^2 + 4x_1x_2 + 6x_2^2}$$

$$P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \Rightarrow \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2 \\ \frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{Q_D(y_1, y_2) = 7y_1^2 + 2y_2^2}$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Q(x_1, x_2) = 3\left(\frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2\right)^2 + 4\left(\frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2\right)\left(\frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2\right) + 6\left(\frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2\right)^2$$

$$= 3\left(\frac{1}{5}y_1^2 - \frac{4}{5}y_1y_2 + \frac{4}{5}y_2^2\right) + 4\left(\frac{2}{5}y_1^2 + \overbrace{\frac{1}{5}y_1y_2 - \frac{4}{5}y_1y_2}^{-3} - \frac{2}{5}y_2^2\right)$$

$$+ 6\left(\frac{4}{5}y_1^2 + \frac{4}{5}y_1y_2 + \frac{1}{5}y_2^2\right)$$

35

$$= \frac{1}{5}\left(3y_1^2 - 12y_1y_2 + 12y_2^2 + 8y_1^2 - 12y_1y_2 - 8y_2^2 + 24y_1^2 + 24y_1y_2 + 6y_2^2\right)$$

$$= \frac{1}{5}35y_1^2 + 0y_1y_2 + \frac{1}{5}10y_2^2 = \boxed{7y_1^2 + 2y_2^2}$$

$$\boxed{\begin{aligned} x_1 &= \frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2 \\ x_2 &= \frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2 \end{aligned}}$$

~~Example 4 (if time permits)~~

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

DO Monday

Example 4 (if time permits)

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

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Nul$(A-\lambda I)$.

$$\boxed{\begin{matrix} P^T x = y \\ Py = x \end{matrix}}$$

Change of vars.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$p(\lambda) = \det(A - \lambda I)$

$\lambda_1 = 1$

$\lambda_2 = 1$

$\lambda_3 = -1$

$$Q_A(x_1, x_2, x_3) = x_1^2 + 2x_2x_3$$

"

$$Q_D(y_1, y_2, y_3) = 1y_1^2 + 1y_2^2 - 1y_3^2$$

MAX value constrained outputs

MIN value to $||x|| = 1$.

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

all give max output value

$$\vec{x} = P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Q_A(1,0,0) = 1$$

$$\vec{x} = P\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad Q_A\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0 + 2\cdot\frac{1}{2}$$

$$= 1.$$

Max value of Q if you plug in a vector of length 5? $||\vec{x}|| = 5$

$$\lambda_1 \cdot 5^2 = 1 \cdot 25 = 25.$$

$$Q_A(5x_1, 5x_2, 5x_3)$$

$$= (5x_1)^2 + 2(5x_2)(5x_3)$$

$$= 25x_1^2 + 2 \cdot 25 x_2 x_3 = 25(x_1^2 + 2x_2x_3)$$

$$Q_A(\vec{x}) = x^T A x$$

$$Q_A(c\vec{x}) = (c\vec{x})^T A (c\vec{x})$$

$$= c^2 \vec{x}^T A \vec{x} = c^2 Q_A(\vec{x})$$

7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2+6x_2^2+7x_3^2+4x_1x_2-4x_2x_3=9y_1^2+6y_2^2+3y_3^2$
2. $3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3=7y_1^2+4y_2^2$

Hint: $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for y_3^2.

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$ (See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3$ (See Exercise 2.)
5. $Q(\mathbf{x}) = x_1^2 + x_2^2 - 10x_1x_2$
6. $Q(\mathbf{x}) = 3x_1^2 + 9x_2^2 + 8x_1x_2$
7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 2, -1, and -4.]
8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 9 and -3.]
9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)
10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)
11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^T A\mathbf{x}$?
12. Let λ be any eigenvalue of a symmetric matrix A. Justify the statement made in this section that $m \le \lambda \le M$, where m and M are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^T A\mathbf{x}$.]
13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T\mathbf{x} = 1$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

[M] In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$
15. $4x_1^2-6x_1x_2-10x_1x_3-10x_1x_4-6x_2x_3-6x_2x_4-2x_3x_4$
16. $-6x_1^2-10x_2^2-13x_3^2-13x_4^2-4x_1x_2-4x_1x_3-4x_1x_4+6x_3x_4$
17. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$

A symmetric?

$(A)^T = (PDP^T)^T$ ✓ know ✓

$A^T = (P^T)^T D^T P^T$

$= PDP^T = A$