LINEAR ALGEBRA

Week 14 & 15

7.1 Exercises

Determine which of the matrices in Exercises 1–6 are symmetric.

1. $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

7. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

10. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

11. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$

12. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix P and a diagonal matrix D. To save you time, the eigenvalues in Exercises 17–22 are the following: (17) $-4, 4, 7$; (18) $-3, -6, 9$; (19) $-2, 7$; (20) $-3, 15$; (21) $1, 5, 9$; (22) $3, 5$.

13. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Then orthogonally diagonalize A.

In Exercises 25–32, mark each statement True or False (T/F). Justify each answer.

25. (T/F) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

26. (T/F) There are symmetric matrices that are not orthogonally diagonalizable.

27. (T/F) An orthogonal matrix is orthogonally diagonalizable.

28. (T/F) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

29. (T/F) For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

30. (T/F) If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

31. (T/F) An $n \times n$ symmetric matrix has n distinct real eigenvalues.

32. (T/F) The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

33. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

34. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB, and BB^T are symmetric matrices.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

39. Construct a spectral decomposition of A from Example 2.

40. Construct a spectral decomposition of A from Example 3.

41. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.
   a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.
   b. Show that B is a symmetric matrix and $B^2 = B$.
   c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

42. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.
   a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.
   b. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^\perp$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

Orthogonally diagonalize the matrices in Exercises 43–46. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue λ, find an orthonormal basis for $\text{Nul}(A - \lambda I)$, as in Examples 2 and 3.

T 43. $\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$

T 44. $\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$

T 45. $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

T 46. $\begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$

7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $3x_1^2 - 4x_1x_2 + 5x_2^2$ b. $3x_1^2 + 2x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $5x_1^2 + 16x_1x_2 - 5x_2^2$ b. $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$

b. $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 - 2x_2^2 + 5x_3^2 + 4x_1x_2 - 6x_1x_3$

b. $4x_3^2 - 2x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

8. Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct P using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$

10. $2x_1^2 + 6x_1x_2 - 6x_2^2$

11. $2x_1^2 - 4x_1x_2 - x_2^2$

12. $-x_1^2 - 2x_1x_2 - x_2^2$

13. $x_1^2 - 6x_1x_2 + 9x_2^2$

14. $3x_1^2 + 4x_1x_2$

15. [M] $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$

16. [M] $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$

17. [M] $11x_1^2 + 11x_2^2 + 11x_3^2 + 11x_4^2 + 16x_1x_2 - 12x_1x_4 + 12x_2x_3 + 16x_3x_4$

18. [M] $2x_1^2 + 2x_2^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.

b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.

c. The principal axes of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are eigenvectors of A.

d. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

e. If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.

f. A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is not a quadratic form.

b. If A is symmetric and P is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term.

c. If A is a 2×2 symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola.

d. An indefinite quadratic form is neither positive semidefinite nor negative semidefinite.

e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all positive.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of A.

23. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.

a. Q is positive definite if $\det A > 0$ and $a > 0$.

b. Q is negative definite if $\det A > 0$ and $a < 0$.

c. Q is indefinite if $\det A < 0$.

25. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

26. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.]

B works.]

27. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let A be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

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7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2+6x_2^2+7x_3^2+4x_1x_2-4x_2x_3=9y_1^2+6y_2^2+3y_3^2$
2. $3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3=7y_1^2+4y_2^2$

Hint: $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for y_3^2.

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$ (See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3$ (See Exercise 2.)
5. $Q(\mathbf{x}) = x_1^2 + x_2^2 - 10x_1x_2$
6. $Q(\mathbf{x}) = 3x_1^2 + 9x_2^2 + 8x_1x_2$
7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are $2, -1$, and -4.]
8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 9 and -3.]
9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)
10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)
11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^T A\mathbf{x}$?
12. Let λ be any eigenvalue of a symmetric matrix A. Justify the statement made in this section that $m \le \lambda \le M$, where m and M are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^T A\mathbf{x}$.]
13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T\mathbf{x} = 1$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

[M] In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$
15. $4x_1^2-6x_1x_2-10x_1x_3-10x_1x_4-6x_2x_3-6x_2x_4-2x_3x_4$
16. $-6x_1^2-10x_2^2-13x_3^2-13x_4^2-4x_1x_2-4x_1x_3-4x_1x_4+6x_3x_4$
17. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
30	26	17

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	60.0	0.2	12.00		Exam 1	30	/50
3	Exam (Mid)	52.0	0.2	10.40		Exam 2	26	/50
4	Exam (Lowest)	34.0	0.15	5.10		Exam 3	17	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	50.66667	0.2	10.13		Final Exam	48.67	/100
7		Final Grade:		62.63				
8		Final Letter Grade:		D		In-studio Quiz Total	9	/45
9						MyLab Total	172	/214
10						Total Exploration Points	18	/42
11								
12						MQE Total	70	
13						MQE Score	70	/70
14						MQE additional	0	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	0	
18						Total Final Exam Bonus	2	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
34	34	25

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	68.0	0.2	13.60		Exam 1	34	/50
3	Exam (Mid)	68.0	0.2	13.60		Exam 2	34	/50
4	Exam (Lowest)	50.0	0.15	7.50		Exam 3	25	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	64	0.2	12.80		Final Exam	62.00	/100
7		Final Grade:		72.50				
8		Final Letter Grade:		C		In-studio Quiz Total	9	/45
9						MyLab Total	172	/214
10						Total Exploration Points	18	/42
11								
12						MQE Total	70	
13						MQE Score	70	/70
14						MQE additional	0	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	0	
18						Total Final Exam Bonus	2	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
42	43	28

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	86.0	0.2	17.20		Exam 1	42	/50
3	Exam (Mid)	84.0	0.2	16.80		Exam 2	43	/50
4	Exam (Lowest)	56.0	0.15	8.40		Exam 3	28	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	77.33333	0.2	15.47		Final Exam	75.33	/100
7		Final Grade:		82.87				
8		Final Letter Grade:		B		In-studio Quiz Total	9	/45
9						MyLab Total	172	/214
10						Total Exploration Points	18	/42
11								
12						MQE Total	70	
13						MQE Score	70	/70
14						MQE additional	0	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	0	
18						Total Final Exam Bonus	2	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
34	28	34.5

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	69.0	0.2	13.80		Exam 1	34	/50
3	Exam (Mid)	68.0	0.2	13.60		Exam 2	28	/50
4	Exam (Lowest)	56.0	0.15	8.40		Exam 3	34.5	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	66.33333	0.2	13.27		Final Exam	64.33	/100
7	Final Grade:			74.07				
8	Final Letter Grade:			C		In-studio Quiz Total	9	/45
9						MyLab Total	172	/214
10						Total Exploration Points	18	/42
11								
12						MQE Total	70	
13						MQE Score	70	/70
14						MQE additional	0	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	0	
18						Total Final Exam Bonus	2	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
43	34	34.5

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	86.0	0.2	17.20		Exam 1	43	/50
3	Exam (Mid)	69.0	0.2	13.80		Exam 2	34	/50
4	Exam (Lowest)	68.0	0.15	10.20		Exam 3	34.5	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	79.85833	0.2	15.97		Final Exam	74.33	/100
7	Final Grade:			82.17				
8	Final Letter Grade:			B		In-studio Quiz Total	45	/45
9						MyLab Total	214	/214
10						Total Exploration Points	42	/42
11								
12						MQE Total	140.5	
13						MQE Score	70	/70
14						MQE additional	70.5	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	3.525	
18						Total Final Exam Bonus	5.525	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
44.5	45	41.5

grade_calculator

	A	B	C	D	E	F	G	H
1	Assignment	Grade (%)	Weight	Points				
2	Exam (Best)	90.0	0.2	18.00		Exam 1	44.5	/50
3	Exam (Mid)	89.0	0.2	17.80		Exam 2	45	/50
4	Exam (Lowest)	83.0	0.15	12.45		Exam 3	41.5	/50
5	MQE Score	100.0	0.25	25.00				
6	Final Exam	92.85833	0.2	18.57		Final Exam	87.33	/100
7	Final Grade:			91.82				
8	Final Letter Grade:			A		In-studio Quiz Total	45	/45
9						MyLab Total	214	/214
10						Total Exploration Points	42	/42
11								
12						MQE Total	140.5	
13						MQE Score	70	/70
14						MQE additional	70.5	
15								
16						Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17						MQE Bonus	3.525	
18						Total Final Exam Bonus	5.525	

Section 7.4 : The Singular Value Decomposition

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

4. $\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix} = A$

Steps to compute SVD of A:
*compute A^TA
*find eigenvalues of A^TA call them σ_i^2
*find orthonormal eigenvectors of A^TA call them v_i
*Compute u_i=1/σ_i Av_i
A=UΣV^T

U=[u1 u2 … um] V=[v1 v2 … vn] both orthogonal matrices
And Σ is a diagonal matrix with diagonal entries σ_i

σ_i^2 eigenvalues of A^TA.

$\vec{v}_i$ eigenvectors of A^TA w/ eigenvalues σ_i^2

$$\boxed{\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i}$$

SVD

$A = U\Sigma V^T$

orthogonal $m\times m$: $[u_1 u_2 \dots u_r u_{r+1} \dots u_m]$

$\begin{bmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ 0 & & & & 0 \ddots 0 \end{bmatrix}$ diagonal $m\times n$

orthogonal $n\times n$: $[v_1 \dots v_r v_{r+1} \dots v_n]$

$A = \begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}$

$73 * 9 - 24^2 = 81$

$A^TA = \begin{bmatrix} 3 & 8 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 73 & 24 \\ 24 & 9 \end{bmatrix}$

$p(\lambda) = \lambda^2 - 82\lambda + 81 = (\lambda - 81)(\lambda - 1) = 0$

$\lambda_1 = 81 \quad \lambda_2 = 1$

singular values.

$\sigma_1^2 = 81 \Rightarrow \boxed{\sigma_1 = 9} \qquad \sigma_2^2 = 1 \Rightarrow \boxed{\sigma_2 = 1}$

ATA.

$\lambda_1 = 81 \quad \cancel{A} - 81I = \begin{bmatrix} -8 & 24 \\ 24 & -73 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ S

$\vec{x} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} = \vec{v}_1$

$\lambda_2 = 1 \quad \vec{v}_2 = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$

$\vec{u}_1 = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9/\sqrt{10} + 0 \\ 24/\sqrt{10} + 3/\sqrt{10} \end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$\vec{u}_2 = \frac{1}{1}\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

$$\boxed{\begin{array}{cccc} A & U & \Sigma & V^T \\ \begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix} = & \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} & \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}^T \end{array}}$$

$$A=\begin{bmatrix}3 & -3\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

① compute A^TA

② Find e-vals/e-vectors of A^TA, normalize to get $\vec v_i$

③ $\vec u_i = \frac{1}{\sigma_i}A\vec v_i$ to get $\vec u_i$'s.

$A=[v_1 \cdots v_n]$

A^TA.

$$\begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}\begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix}\vec v_i \cdot \vec v_j\end{bmatrix}_{ij}$$

①
$$A^TA=\begin{bmatrix}3 & 0 & 1\\ -3 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & -3\\ 0 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}10 & -8\\ -8 & 10\end{bmatrix}$$

②
$$p(\lambda)=\lambda^2-20\lambda+36=(\lambda-18)(\lambda-2)=0$$

$\lambda_1=18$ $\quad$ $\lambda_2=2$

$\sigma_1=\sqrt{18}$ $\quad$ $\sigma_2=\sqrt{2}$

$\sigma_1=3\sqrt{2}$

$\lambda_1=18$ $\quad A^TA-18I$

$$=\begin{bmatrix}-8 & -8\\ -8 & -8\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

$\vec x = s\begin{bmatrix}-1\\ 1\end{bmatrix}$

$$v_1=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$$

$\lambda_2=2$

$$A^TA-2I=\begin{bmatrix}8 & -8\\ -8 & 8\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

$\vec x = s\begin{bmatrix}1\\ 1\end{bmatrix}$

$$v_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$$

$$u_i=\frac{1}{\sigma_i}Av_i$$

$$u_1=\frac{1}{\sigma_1}A\vec v_1=\frac{1}{3\sqrt{2}}\begin{bmatrix}3 & -3\\ 0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}=\frac{1}{3\sqrt{2}}\begin{bmatrix}-6/\sqrt{2}\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}=\vec u_1$$

$$\frac{1}{3\sqrt{2}}\cdot\frac{-6}{\sqrt{2}}=\frac{-6}{3\cdot 2}=-1 \checkmark$$

$$u_2=\frac{1}{\sigma_2}A\vec v_2=\frac{1}{\sqrt{2}}\begin{bmatrix}3 & -3\\ 0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 0\\ 2/\sqrt{2}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{2}}=\frac{2}{2}=1 \checkmark$$

$$\boxed{A=\begin{bmatrix}3 & -3\\ 0 & 0\\ 1 & 1\end{bmatrix}=\underbrace{\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}3\sqrt{2} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}^T}_{V^T}}$$

3×2 $\quad$ 3×2 $\quad$ 2×2

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\vec x = s\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$

u_3 must be a unit vector orthogonal to u_1 & u_2

Topics and Objectives

Topics

1. The Singular Value Decomposition (SVD) and some of its applications.

Learning Objectives

Example 1

The linear transform whose standard matrix is

$$A=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}2\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}=\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}$$

maps the unit circle in $\mathbb{R}^2$ to an ellipse, as shown below. Identify the unit

Topics and Objectives

Topics

1. The Singular Value Decomposition (SVD) and some of its applications.

Learning Objectives

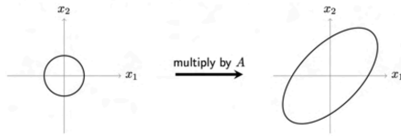
1. Compute the SVD for a rectangular matrix.
2. Apply the SVD to
   - estimate the rank and condition number of a matrix,
   - construct a basis for the four fundamental spaces of a matrix, and
   - construct a spectral decomposition of a matrix.

Example 1

The linear transform whose standard matrix is

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

maps the unit circle in $\mathbb{R}^2$ to an ellipse, as shown below. Identify the unit vector $\vec{x}$ in which $||A\vec{x}||$ is maximized and compute this length.



Singular Values

The matrix A^TA is always symmetric, with non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be the associated orthonormal eigenvectors. Then

$$\|A\vec{v}_j\|^2 =$$

If the A has rank r, then $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthogonal basis for Col$A$: For $1 \leq j < k \leq r$:

$$(A\vec{v}_j)^T A\vec{v}_k =$$

Definition: $\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \cdots \geq \sigma_n = \sqrt{\lambda_n}$ are the singular values of A.

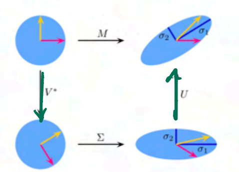
The SVD

Theorem: Singular Value Decomposition

A $m \times n$ matrix with rank r and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ has a decomposition $U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & \\ 0 & \sigma_2 & \dots & \vdots & \mathbf{0} \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \dots & \sigma_r & \\ & \mathbf{0} & & & \mathbf{0} \end{bmatrix}$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.



$M = U \cdot \Sigma \cdot V^*$

rank $A = r$

$$A = \begin{bmatrix} u_1 \dots & u_r & u_{r+1} \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & 0 \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots 0 \end{bmatrix} \begin{bmatrix} v_1 \dots v_r & v_{r+1} \dots v_n \end{bmatrix}$$

Algorithm to find the SVD of A

Suppose A is $m \times n$ and has rank $r \le n$.

1. Compute the squared singular values of A^TA, σ_i^2, and construct Σ.

A^TA eigenvalues are always ≥ 0.

2. Compute the unit singular vectors of A^TA, $\vec{v}_i$, use them to form V.

$Q(\vec{x}) = x^T \underbrace{A^TA}_{\text{symmetric}} x = (Ax)^T Ax = \|A\vec{x}\|^2 \ge 0$

positive semi definite.

3. Compute an orthonormal basis for Col$A$ using

$$\boxed{\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i,} \quad i = 1, 2, \ldots r$$

Extend the set ~~$\{\vec{u}_i\}$ to form~~ an orthonormal basis for $\mathbb{R}^m$, use the basis for form U.

Example 2: Write down the singular value decomposition for

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

CLAIMS about $\vec{u}_i$'s.

① $u_i \cdot u_j = 0$ since $\vec{v}_i \cdot \vec{v}_j = 0$ & $A^TAv_j = \sigma_j^2 v_j$ ✓

② $\|u_i\| = 1.$ ✓

$\|u_i\|^2 = u_i \cdot u_i = u_i^T u_i = (\frac{1}{\sigma_i} Av_i)^T \frac{1}{\sigma_i} Av_i$

$= \frac{1}{\sigma_i^2} v_i^T A^TAv_i$

$= \cancel{\frac{1}{\sigma_i^2}} v_i^T \cdot \cancel{\sigma_i^2} v_i = \|v_i\|^2 = 1.$

$u_i \cdot u_j = u_i^T u_j = (\frac{1}{\sigma_i} Av_i)^T \frac{1}{\sigma_j} Av_j$

$= \frac{1}{\sigma_i} \cdot \frac{1}{\sigma_j} v_i^T A^TA v_j$

$= \frac{1}{\sigma_i} \cdot \frac{1}{\sigma_j} * \sigma_j^2 v_i^T v_j$

$= \frac{\sigma_j}{\sigma_i} (\underbrace{v_i \cdot v_j}_{=0}) = 0.$

Example 3: Construct the singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$. (It has rank 1.)

META:

① compute A^TA

② Find λ_i's, eigenvalues/eigenvectors of A^TA

$\sigma_i = \sqrt{\lambda_i}$, normalize eigenvectors to get $\vec{v}_i$'s.

Get $\vec{u}_i$'s

③ $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$

(sometimes) "④" Fill in remaining columns of U s.t. U is orthogonal matrix

Soln.

① $A^TA = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

② $p(\lambda) = \lambda^2 - 18\lambda + 81 - 81 = \lambda^2 - 18\lambda$

$= \lambda(\lambda - 18) = 0$

$\lambda_1 = 18 > \lambda_2 = 0$

$\sigma_1 = \sqrt{18} = 3\sqrt{2}$ $\quad \sigma_2 = 0$

$\vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$A^TA - \lambda I$

$\lambda_1 = 18$ $\quad \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - \begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix}$

$= \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$

$A^TA \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 3: Construct the singular value decomposition of
$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.
(It has rank 1.)

META:

① compute A^TA

② Find λ_i's, eigenvalues/eigenvectors of A^TA

$\sigma_i = \sqrt{\lambda_i}$, normalize eigenvectors to get $\vec{v}_i$'s.

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Soln.

① $A^TA = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

② $p(\lambda) = \lambda^2 - 18\lambda + 81 - 81 = \lambda^2 - 18\lambda$
$= \lambda(\lambda - 18) = 0$

$A^TA - \lambda I$

$\lambda_1 = 18 \geq \lambda_2 = 0$

$\lambda_1 = 18$: $\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - \begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix} = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$: $A^TA \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\sigma_1 = \sqrt{18} = 3\sqrt{2}$ $\quad \sigma_2 = 0$

$\vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

③ $u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} -2/\sqrt{2} \\ 4/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix} = \frac{1}{3 \cdot 2} \begin{bmatrix} -2 \\ 4 \\ -4 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \leftarrow u_1$

$u_2 \overset{?}{=} \frac{1}{\sigma_2} Av_2 = \frac{1}{0} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = ??$

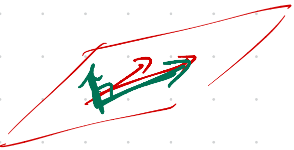
$$\boxed{\begin{aligned} &\underset{3\times3}{A = U} \; \underset{3\times2}{\Sigma} \; \underset{2\times2}{V^T} \qquad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &U = \begin{pmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{pmatrix} \end{aligned}}$$

ANS

$u_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$ need $\boxed{\left(\text{span}\{u_1\}\right)^{\perp}}$

orthonormal basis for this

$$u_1^T = \begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & \underset{s}{-2} & \underset{t}{2} \end{bmatrix}$$



$$X = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

not orthogonal!!

$u_2 \stackrel{?}{=} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ $u_3 \stackrel{?}{=} \cancel{\begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}}$

$$U \stackrel{?}{=} \begin{bmatrix} -1/3 & \cancel{2/\sqrt{5}} & \cancel{-2/\sqrt{5}} \\ 2/3 & \cancel{1/\sqrt{5}} & \cancel{0} \\ -2/3 & \cancel{0} & \cancel{1/\sqrt{5}} \end{bmatrix} ??$$

So Do G-S. $X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{w}_2 = X_2 - \frac{X_2 \cdot w_1}{w_1 \cdot w_1} w_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$\vec{w}_2 = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$

THEOREM **The Invertible Matrix Theorem (concluded)**

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$.
v. $(\text{Nul } A)^{\perp} = \mathbb{R}^n$.
w. $\text{Row } A = \mathbb{R}^n$.
x. A has n nonzero singular values.

IMT DLC

Final.

Singular value decomposition

From Wikipedia, the free encyclopedia

In linear algebra, the **singular value decomposition** (**SVD**) is a factorization of a real or complex matrix. It generalizes the eigendecomposition of a square normal matrix with an orthonormal eigenbasis to any $m \times n$ matrix. It is related to the polar decomposition.

Specifically, the singular value decomposition of an $m \times n$ complex matrix $\mathbf{M}$ is a factorization of the form $\mathbf{U\Sigma V^*}$, where $\mathbf{U}$ is an $m \times m$ complex unitary matrix, $\mathbf{\Sigma}$ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and $\mathbf{V}$ is an $n \times n$ complex unitary matrix. If $\mathbf{M}$ is real, $\mathbf{U}$ and $\mathbf{V}$ can also be guaranteed to be real orthogonal matrices. In such contexts, the SVD is often denoted $\mathbf{U\Sigma V^T}$.

The diagonal entries $\sigma_i = \Sigma_{ii}$ of $\mathbf{\Sigma}$ are known as the singular values of $\mathbf{M}$. The number of non-zero singular values is equal to the rank of $\mathbf{M}$. The columns of $\mathbf{U}$ and the columns of $\mathbf{V}$ are called the left-singular vectors and right-singular vectors of $\mathbf{M}$, respectively.

The SVD is not unique. It is always possible to choose the decomposition so that the singular values Σ_{ii} are in descending order. In this case, $\mathbf{\Sigma}$ (but not always $\mathbf{U}$ and $\mathbf{V}$) is uniquely determined by $\mathbf{M}$.

The term sometimes refers to the **compact SVD**, a similar decomposition $\mathbf{M} = \mathbf{U\Sigma V^*}$ in which $\mathbf{\Sigma}$ is square diagonal of size $r \times r$, where $r \le \min\{m, n\}$ is the rank of $\mathbf{M}$, and has only the non-zero singular values. In this variant, $\mathbf{U}$ is an $m \times r$ semi-unitary matrix and $\mathbf{V}$ is an $n \times r$ semi-unitary matrix, such that $\mathbf{U^*U} = \mathbf{V^*V} = \mathbf{I}_r$.

Mathematical applications of the SVD include computing the pseudoinverse, matrix approximation, and determining the rank, range, and null space of a matrix. The SVD is also extremely useful in all areas of science, engineering, and statistics, such as signal processing, least squares fitting of data, and process control.

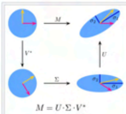


Illustration of the singular value decomposition $\mathbf{U\Sigma V^*}$ of a real 2×2 matrix $\mathbf{M}$.
Top: The action of $\mathbf{M}$, indicated by its effect on the unit disc D and the two canonical unit vectors e_1 and e_2.
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Safari

Applications of the SVD

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- Estimating the rank and condition number of a matrix
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Students are expected to be familiar with the 1^{st} two items in the list.

The Condition Number of a Matrix

If A is an invertible $n \times n$ matrix, the ratio

$$\frac{\sigma_1}{\sigma_n}$$

is the **condition number** of A.

Note that:

- The condition number of a matrix describes the sensitivity of a solution to $A\vec{x} = \vec{b}$ is to errors in A.
- We could define the condition number for a rectangular matrix, but that would go beyond the scope of this course.

$$\frac{1}{\sigma_i} A v_i = u_i$$

$$\boxed{Ax = \lambda x}$$

??

$$\boxed{Av_i = \sigma_i u_i}$$

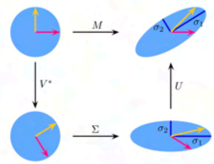
The SVD

Theorem: Singular Value Decomposition

A $m \times n$ matrix with rank r and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ has a decomposition $U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} = \begin{bmatrix} \begin{matrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \sigma_r \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.



$M = U \cdot \Sigma \cdot V^*$

$$\underset{m \times n}{A} = \underset{m \times m}{U}\ \underset{m \times n}{\Sigma}\ \underset{n \times n}{V^T}$$

diagonal.

U, V are square w/ orthonormal cols

FACTS. $\text{rank}(A) = \text{rank}(\Sigma)$
$= \#$ nonzero singular values

THEOREM **The Invertible Matrix Theorem (concluded)**

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

u. $(\text{Col } A)^\perp = \{\mathbf{0}\}$.
v. $(\text{Nul } A)^\perp = \mathbb{R}^n$.
w. Row $A = \mathbb{R}^n$.
x. A has n nonzero singular values.

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Singular value decomposition

From Wikipedia, the free encyclopedia

In linear algebra, the **singular value decomposition** (**SVD**) is a factorization of a real or complex matrix. It generalizes the eigendecomposition of a square normal matrix with an orthonormal eigenbasis to any $m \times n$ matrix. It is related to the polar decomposition.

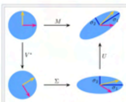
Specifically, the singular value decomposition of an $m \times n$ complex matrix $\mathbf{M}$ is a factorization of the form $\mathbf{U\Sigma V^*}$, where $\mathbf{U}$ is an $m \times m$ complex unitary matrix, $\mathbf{\Sigma}$ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and $\mathbf{V}$ is an $n \times n$ complex unitary matrix. If $\mathbf{M}$ is real, $\mathbf{U}$ and $\mathbf{V}$ can also be guaranteed to be real orthogonal matrices. In such contexts, the SVD is often denoted $\mathbf{U\Sigma V^T}$.

The diagonal entries $\sigma_i = \Sigma_{ii}$ of $\mathbf{\Sigma}$ are known as the singular values of $\mathbf{M}$. The number of non-zero singular values is equal to the rank of $\mathbf{M}$. The columns of $\mathbf{U}$ and the columns of $\mathbf{V}$ are called the left-singular vectors and right-singular vectors of $\mathbf{M}$, respectively.

The SVD is not unique. It is always possible to choose the decomposition so that the singular values Σ_{ii} are in descending order. In this case, $\mathbf{\Sigma}$ (but not always $\mathbf{U}$ and $\mathbf{V}$) is uniquely determined by $\mathbf{M}$.

The term sometimes refers to the **compact SVD**, a similar decomposition $\mathbf{M} = \mathbf{U\Sigma V^*}$ in which $\mathbf{\Sigma}$ is square diagonal of size $r \times r$, where $r \leq \min\{m, n\}$ is the rank of $\mathbf{M}$, and has only the non-zero singular values. In this variant, $\mathbf{U}$ is an $m \times r$ semi-unitary matrix and $\mathbf{V}$ is an $n \times r$ semi-unitary matrix, such that $\mathbf{U^*U} = \mathbf{V^*V} = \mathbf{I}_r$.

Mathematical applications of the SVD include computing the pseudoinverse, matrix approximation, and determining the rank, range, and null space of a matrix. The SVD is also extremely useful in all areas of science, engineering, and statistics, such as signal processing, least squares fitting of data, and process control.



$M = U \cdot \Sigma \cdot V^*$

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The SVD can also be used to construct the spectral decomposition for any matrix with rank r

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where $\vec{u}_s, \vec{v}_s$ are the s^{th} columns of U and V respectively.

For the case when $A = A^T$, we obtain the same spectral decomposition that we encountered in Section 7.2.

Similar to $A = PDP^T$

Symmetric A matrix

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $P = [u_1 \cdots u_n]$

eigenvalues of A.

orthonormal bases for eigenspaces of A.

$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$

$u_i u_i^T$ rank 1 matrices.

Example 2: Write down the singular value decomposition for

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \underset{U}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}$$

① $A^TA = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

$\lambda_1 = 9$ $\lambda_2 = 4$

$\sigma_1 = 3$ $\sigma_2 = 2$

$\lambda_1 = 9$

② $A^TA - 9I = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $x = s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ v_1

$\lambda_2 = 4$ $A^TA - 4I = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$x = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ v_2

$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ($\sigma_1 = 3$, $\sigma_2 = 2$)

③ Next $u_i = \frac{1}{\sigma_i} A v_i$

$\sigma_1 = 3$ $u_1 = \frac{1}{3}\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = u_1$

$\sigma_2 = 2$ $u_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = u_2$

Last two columns of $U = [u_1\ u_2\ u_3\ u_4]$

← need to complete $\{u_1, u_2\}$ to an orthonormal basis for $\mathbb{R}^4$.

Compute null space

$\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$x = r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (u_3, u_4)

(if not orthogonal then do G-S)

$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T$$

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$$= 3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} [0 \ \ 1] + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1 \ \ 0]$$

$$= 3 \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ $\sigma_1 = 3$, $\sigma_2 = 2$

$\det A \overset{?}{=} \sigma_1 \sigma_2$

$$\det A = \det(U \Sigma V^T)$$
$$= \det U \det \Sigma \det(V^T)$$
$$= \det U \det \Sigma \det V$$

$\det V = \pm 1$, $\det U = \pm 1$ U, V orthogonal

$$UU^T = I$$
$$\Rightarrow \det(UU^T) = \det I$$
$$\Rightarrow (\det U)^2 = 1.$$
$$\Rightarrow \det U = \pm 1 \checkmark$$

Example 4

For $A = U\Sigma V^*$, determine the rank of A, and orthonormal bases for Null$A$ and $(\text{Col}A)^\perp$.

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

First 3 rows of V^T are an orthonormal basis for Row A

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{.2} \\ 0 \\ 0 \\ 0 \\ \sqrt{.8} \end{bmatrix} \right\} \text{ basis for Row}A$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{.8} \\ 0 \\ 0 \\ 0 \\ \sqrt{.2} \end{bmatrix} \right\} \text{ basis for } (\text{Row } A)^\perp = \text{Nul } A$$

First 3 columns of U (b/c rank$(A)=3$ = # nonzero σ's) are an orthonormal basis for Col$A$.

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ basis for Col}A.$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ basis for } (\text{Col}A)^\perp = \text{Nul } A^T$$

The Four Fundamental Spaces

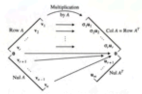

FIGURE 4 The four fundamental subspaces and the action of A.

1. $A\vec{v}_s = \sigma_s\vec{u}_s$.
2. $\vec{v}_1, \ldots, \vec{v}_r$ is an orthonormal basis for Row$A$.
3. $\vec{u}_1, \ldots, \vec{u}_r$ is an orthonormal basis for Col$A$.
4. $\vec{v}_{r+1}, \ldots, \vec{v}_n$ is an orthonormal basis for Null$A$.
5. $\vec{u}_{r+1}, \ldots, \vec{u}_m$ is an orthonormal basis for NullA^T.

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The SVD can also be used to construct the spectral decomposition for any matrix with rank r

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For the case when $A = A^T$, we obtain the same spectral decomposition that we encountered in Section 7.2.

$$A = \begin{pmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T}_{V^T}$$

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

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$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$$= 3\sqrt{2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= 3\sqrt{2} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

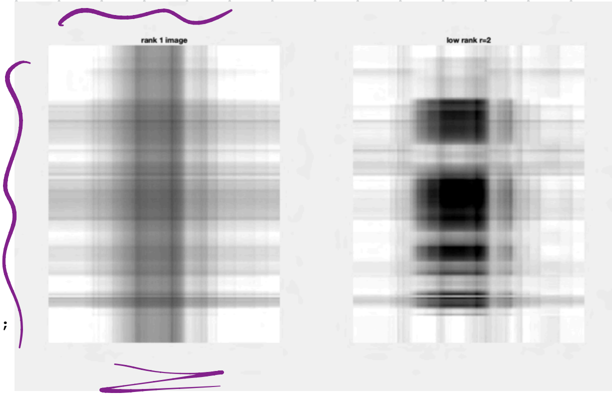
$$= \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

rank A=724

```
clc
RGB=imread('buzz.jpg');
gray=rgb2gray(RGB);
A=im2double(gray);
[U,S,V]=svd(A);
 sz=size(A)
 rank(A)
 Approx=zeros(sz);
 r=2
 for i=1:r
     u=U(:,i);
     s=S(i,i);
     v=V(:,i);
     Approx=Approx+s*u*v';
 end
 Approx;
% subplot(1,2,1),imshow(A),title('original');
 subplot(1,2,2),imshow(Approx),title(['low rank r=',num2str(r)]);
```

880

1,120

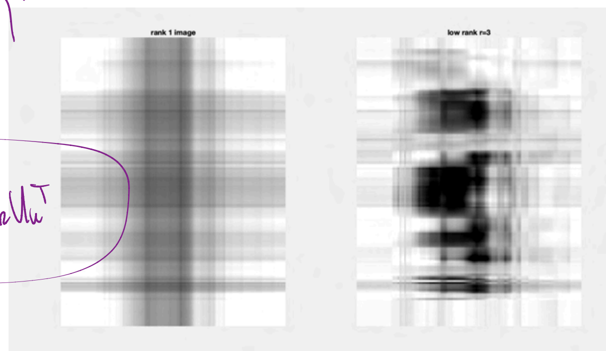


$$U\Sigma V^T = A$$

$$= \begin{bmatrix} u_1 u_2 \cdots u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & \cancel{\sigma_2}\ 0 & & 0 \\ & \ddots & & \\ & & \cancel{\sigma_{k+1}}\ 0 & \\ 0 & & & \ddots\ 0 \end{bmatrix} \begin{bmatrix} v_1 v_2 \cdots v_n \end{bmatrix}^T$$

$$= \boxed{\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_k u_k v_k^T}$$

rank k.



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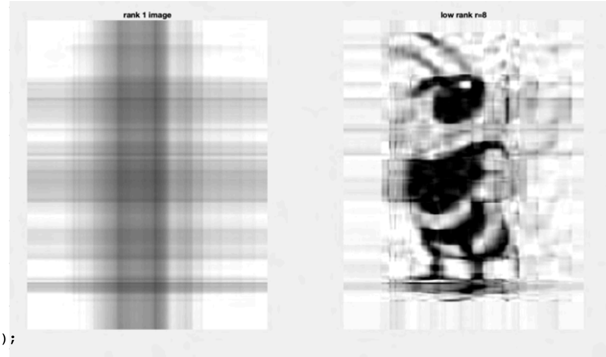

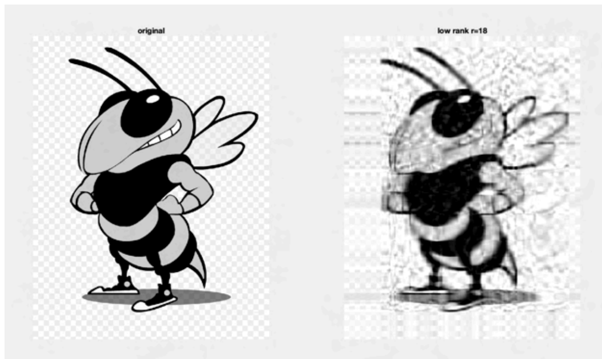

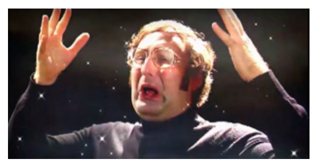

7.4 EXERCISES

Find the singular values of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ 2. $\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}$

Find an SVD of each matrix in Exercises 5–12. [*Hint:* In Exercise 11, one choice for U is $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$. In Exercise 12, one column of U can be $\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.]

5. $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ 6. $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ 8. $\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ 10. $\begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

13. Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ [*Hint:* Work with A^T.]

14. In Exercise 7, find a unit vector $\mathbf{x}$ at which $A\mathbf{x}$ has maximum length.

15. Suppose the factorization below is an SVD of a matrix A, with the entries in U and V rounded to two decimal places.

$$A = \begin{bmatrix} .40 & -.78 & .47 \\ .37 & -.33 & -.87 \\ -.84 & -.52 & -.16 \end{bmatrix} \begin{bmatrix} 7.10 & 0 & 0 \\ 0 & 3.10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} .30 & -.51 & -.81 \\ .76 & .64 & -.12 \\ .58 & -.58 & .58 \end{bmatrix}$$

a. What is the rank of A?

b. Use this decomposition of A, with no calculations, to write a basis for Col A and a basis for Nul A. [*Hint:* First write the columns of V.]

16. Repeat Exercise 15 for the following SVD of a 3×4 matrix A:

$$A = \begin{bmatrix} -.86 & -.11 & -.50 \\ .31 & .68 & -.67 \\ .41 & -.73 & -.55 \end{bmatrix} \begin{bmatrix} 12.48 & 0 & 0 & 0 \\ 0 & 6.34 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} .66 & -.03 & -.35 & .66 \\ -.13 & -.90 & -.39 & -.13 \\ .65 & .08 & -.16 & -.73 \\ -.34 & .42 & -.84 & -.08 \end{bmatrix}$$

In Exercises 17–24, A is an $m \times n$ matrix with a singular value decomposition $A = U\Sigma V^T$, where U is an $m \times m$ orthogonal matrix, Σ is an $m \times n$ "diagonal" matrix with r positive entries and no negative entries, and V is an $n \times n$ orthogonal matrix. Justify each answer.

17. Show that if A is square, then $|\det A|$ is the product of the singular values of A.

18. Suppose A is square and invertible. Find a singular value decomposition of A^{-1}.

19. Show that the columns of V are eigenvectors of A^TA, the columns of U are eigenvectors of AA^T, and the diagonal

entries of Σ are the singular values of A. [*Hint:* Use the SVD to compute A^TA and AA^T.]

20. Show that if P is an orthogonal $m \times m$ matrix, then PA has the same singular values as A.

21. Justify the statement in Example 2 that the second singular value of a matrix A is the maximum of $\|A\mathbf{x}\|$ as $\mathbf{x}$ varies over all unit vectors orthogonal to $\mathbf{v}_1$, with $\mathbf{v}_1$ a right singular vector corresponding to the first singular value of A. [*Hint:* Use Theorem 7 in Section 7.3.]

22. Show that if A is an $n \times n$ positive definite matrix, then an orthogonal diagonalization $A = PDP^T$ is a singular value decomposition of A.

23. Let $U = [\,\mathbf{u}_1 \ \cdots \ \mathbf{u}_m\,]$ and $V = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]$, where the $\mathbf{u}_i$ and $\mathbf{v}_i$ are as in Theorem 10. Show that

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T.$$

24. Using the notation of Exercise 23, show that $A^T\mathbf{u}_j = \sigma_j\mathbf{v}_j$ for $1 \le j \le r = \text{rank}\,A$.

25. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Describe how to find a basis $\mathcal{B}$ for $\mathbb{R}^n$ and a basis $\mathcal{C}$ for $\mathbb{R}^m$ such that the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$ is an $m \times n$ "diagonal" matrix.

[M] Compute an SVD of each matrix in Exercises 26 and 27. Report the final matrix entries accurate to two decimal places. Use the method of Examples 3 and 4.

26. $A = \begin{bmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{bmatrix}$

27. $A = \begin{bmatrix} 6 & -8 & -4 & 5 & -4 \\ 2 & 7 & -5 & -6 & 4 \\ 0 & -1 & -8 & 2 & 2 \\ -1 & -2 & 4 & 4 & -8 \end{bmatrix}$

28. **[M]** Compute the singular values of the 4×4 matrix in Exercise 9 in Section 2.3, and compute the condition number σ_1/σ_4.

29. **[M]** Compute the singular values of the 5×5 matrix in Exercise 10 in Section 2.3, and compute the condition number σ_1/σ_5.