LINEAR ALGEBRA

week 5

Section 2.3 : Invertible Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

Topics and Objectives

Topics
We will cover these topics in this section.

1. The invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize the invertibility of a matrix using the Invertible Matrix Theorem.
2. Construct and give examples of matrices that are/are not invertible.

Motivating Question
When is a square matrix invertible? Let me count the ways!

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through cour

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 – 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 – 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 – 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 – 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 – 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 – 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 – 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 – 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 – 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 – 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 – 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 – 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 – 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

$\boxed{\text{IMT}}$

A is invertible defn. if

$AB = I = BA$

for some B.

The Invertible Matrix Theorem

Invertible matrices enjoy a rich set of equivalent descriptions.

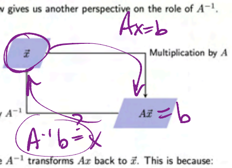
Theorem
Let A be an $n \times n$ matrix. These statements are all equivalent.

a) *A is invertible.*
b) *A is row equivalent to I_n.*
c) *A has n pivotal columns. (All columns are pivotal.)*
d) *$A\vec{x} = \vec{0}$ has only the trivial solution.*
e) *The columns of A are linearly independent.*
f) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
g) *The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.*
h) *The columns of A span $\mathbb{R}^n$.*
i) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is onto.*
j) *There is a $n \times n$ matrix C so that $CA = I_n$. (A has a left inverse.)*
k) *There is a $n \times n$ matrix D so that $AD = I_n$. (A has a right inverse.)*
l) *A^T is invertible.*

$$\binom{12}{2} = \frac{12 \cdot 11}{2} = \boxed{66}$$

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.

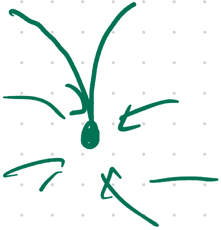


The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} = x.$$

singular $\Leftrightarrow$ non invertible

non-singular $\Leftrightarrow$ yes invertible



Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

does $Ax=0$ have only trivial soln?

$$[A|\vec{0}] \to \begin{bmatrix} 1 & \overset{\swarrow}{2} & \overset{\swarrow}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free var.

$$x = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$s=1, t=2$ $\boxed{x = \begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}}$ is a soln to $Ax=0$.

Check $A \cdot x \overset{?}{=} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8+2+6 \\ -8+2+6 \\ -6+2+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

$T(\vec{x}) = ?$

$$A\vec{x} = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

range of T is a line in $\mathbb{R}^3$.

The Invertible Matrix Theorem: Final Notes

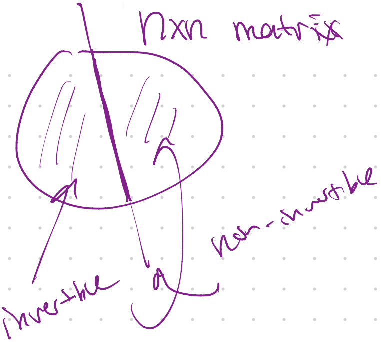
- Items j and k of the invertible matrix theorem (IMT) lead us directly to the following theorem.

Theorem

If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

4



Example 1

Is this matrix invertible? AKA non-singular?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

3 pivots.

So A is invertible.

How many solns to $Ax=b$ $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

$\boxed{1}$

Example 2

non-invertible.

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

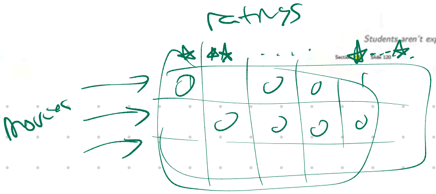
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \text{NP} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Matrix Completion Problems

- The previous example is an example of a matrix completion problem (MCP).
- MCPs are great questions for recitations, midterms, exams.
- the **Netflix Problem** is another example of an MCP.

Given a **ratings matrix** in which each entry (i, j) represents the rating of movie j by customer i if customer i has watched movie j, and is otherwise missing, predict the remaining matrix entries in order to make recommendations to customers on what to watch next.

Students aren't expected to be familiar with this material. It's presented to motivate matrix completion.



2.3 EXERCISES

Unless otherwise specified, assume that all matrices in these exercises are $n \times n$. Determine which of the matrices in Exercises 1–10 are invertible. Use as few calculations as possible. Justify your answers.

1. $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ **2.** $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$ **4.** $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

9. **[M]** $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$

10. **[M]** $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$

In Exercises 11 and 12, the matrices are all $n \times n$. Each part of the exercises is an *implication* of the form "If "statement 1", then "statement 2"." Mark an implication as True if the truth of "statement 2" *always* follows whenever "statement 1" happens to be true. An implication is False if there is an instance in which "statement 2" is false but "statement 1" is true. Justify each answer.

11. a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.
  b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.
  c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
  d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.
  e. If A^T is not invertible, then A is not invertible.

12. a. If there is an $n \times n$ matrix D such that $AD = I$, then there is also an $n \times n$ matrix C such that $CA = I$.
  b. If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.

  d. If the linear transformation $(\mathbf{x}) \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A has n pivot positions.
  e. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.

13. An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.

14. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.

15. Can a square matrix with two identical columns be invertible? Why or why not?

16. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

17. If A is invertible, then the columns of A^{-1} are linearly independent. Explain why.

18. If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^6$, is it possible that for some $\mathbf{v}$, the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution? Why or why not?

19. If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $D\mathbf{x} = \mathbf{b}$? Why?

20. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

21. If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y}$ in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$? Why or why not?

22. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Why?

23. If an $n \times n$ matrix K cannot be row reduced to I_n, what can you say about the columns of K? Why?

24. If L is $n \times n$ and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of L span $\mathbb{R}^n$? Why?

25. Verify the boxed statement preceding Example 1.

26. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent.

27. Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot *assume* that A and B are invertible. [*Hint:* There is a matrix W such that $ABW = I$. Why?]

28. Show that if AB is invertible, so is B.

29. If A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has more than

30. If A is an $n \times n$ matrix and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, what else can you say about this transformation? Justify your answer.

31. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Without using Theorems 5 or 8, explain why each equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

32. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

In Exercises 33 and 34, T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

33. $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$

34. $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$

35. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Explain why T is both one-to-one and onto $\mathbb{R}^n$. Use equations (1) and (2). Then give a second explanation using one or more theorems.

36. Let T be a linear transformation that maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} also one-to-one?

37. Suppose T and U are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$? Why or why not?

So, ... 1553 does not cover

Section 2.4 : Partitioned Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

$$I_4 = \begin{bmatrix} I_2 & O_{2\times2} \\ O_{2\times2} & I_2 \end{bmatrix}$$

Topics and Objectives

Topics
We will cover these topics in this section.

1. Partitioned matrices (or block matrices)

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Apply partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication.

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Math 1554 Linear Algebra

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7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

What is a Partitioned Matrix?

Example
This matrix:

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

Block matrices

can also be written as:

$$\begin{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix} = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right]$$

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Another Example of a Partitioned Matrix

Example: The reduced echelon form of a matrix. We can use a partitioned matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_4 & F \\ 0 & 0 \end{bmatrix}$$

This is useful when studying the **null space** of A, as we will see later in this course.

$$A = \begin{bmatrix} A_1 & 0 & 0 & & 0 \\ 0 & A_2 & 0 & & \vdots \\ 0 & 0 & A_3 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & A_k \end{bmatrix}$$

block diagonal.

Row Column Method

Recall that a row vector times a column vector (of the right dimensions) is a scalar. For example,

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix} =$$

This is the **row column** matrix multiplication method from Section 2.1.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\text{row}_i\, A \cdot \text{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions).

Example of Row Column Method

Recall, using our formula for a 2×2 matrix, $\begin{bmatrix}a & b\\0 & c\end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix}c & -b\\0 & a\end{bmatrix}$.

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{bmatrix}A & B\\ \mathbf{0} & C\end{bmatrix}$.

Idea: Suppose $\begin{bmatrix}A & B\\0 & C\end{bmatrix}^{-1} = \begin{bmatrix}X & Y\\Z & W\end{bmatrix}$.

Then

$$\begin{bmatrix}A & B\\0 & C\end{bmatrix}\begin{bmatrix}X & Y\\Z & W\end{bmatrix} = \begin{bmatrix}I_n & 0\\0 & I_n\end{bmatrix}$$

$$\boxed{\begin{bmatrix}A & B\\0 & C\end{bmatrix}^{-1} = \begin{bmatrix}A^{-1} & -A^{-1}BC^{-1}\\0 & C^{-1}\end{bmatrix}}$$

Ans

$$\overset{I_3}{\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}}$$

0

$$\boxed{AX + B\overset{0}{\cancel{Z}} = I_n} \Rightarrow \boxed{X = A^{-1}}$$

$$AY + BW = 0 \Rightarrow A\cdot Y + B\cdot C^{-1} = 0 \Rightarrow AY = -BC^{-1} \Rightarrow \boxed{Y = -A^{-1}BC^{-1}}$$

$$0X + \boxed{CZ = 0} \Rightarrow \overset{I}{C^{-1}\cdot C}Z = \overset{0}{C^{-1}\cdot 0} \Rightarrow \boxed{Z = 0}$$

$$0\cdot Y + \boxed{CW = I_n} \qquad \boxed{W = C^{-1}}$$

$$\begin{bmatrix}a & b\\0 & c\end{bmatrix}^{-1} = \begin{bmatrix}\frac{c}{ac} & \frac{-b}{ac}\\0 & \frac{a}{ac}\end{bmatrix} = \begin{bmatrix}\frac{1}{a} & \frac{-b}{ac}\\0 & \frac{1}{c}\end{bmatrix}$$

WARNING! $\xcancel{\begin{bmatrix}A & B\\0 & C\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{A} & \frac{-B}{AC}\\0 & \frac{1}{C}\end{bmatrix}}$

21. a. Verify that $A^2 = I$ when $A = \begin{bmatrix}1 & 0\\3 & -1\end{bmatrix}$.

b. Use partitioned matrices to show that $M^2 = I$ when

$$M = \begin{bmatrix}1 & 0 & 0 & 0\\3 & -1 & 0 & 0\\1 & 0 & -1 & 0\\0 & 1 & -3 & 1\end{bmatrix}$$

6. $\begin{bmatrix}X & 0\\Y & Z\end{bmatrix}\begin{bmatrix}A & 0\\B & C\end{bmatrix} = \begin{bmatrix}I & 0\\0 & I\end{bmatrix}$

$$\begin{bmatrix}1 & 0\\3 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\3 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2$$

$$M^2 = \begin{bmatrix}1 & 0 & 0 & 0\\3 & -1 & 0 & 0\\1 & 0 & -1 & 0\\0 & 1 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\3 & -1 & 0 & 0\\1 & 0 & -1 & 0\\0 & 1 & -3 & 1\end{bmatrix}$$

$$M^2 = \begin{bmatrix}A & 0\\I & -A\end{bmatrix}\begin{bmatrix}A & 0\\I & -A\end{bmatrix} = \begin{bmatrix}A^2+0 & 0+0\\A-A & 0+A^2\end{bmatrix} = \begin{bmatrix}I_2 & 0\\0 & I_2\end{bmatrix} = I_4$$

Q: Is M invertible?

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = I_4.$$

The Column Row Method (if time permits)

A column vector times a row vector is a matrix. For example,

$$\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix} =$$

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then,

$$AB = \begin{bmatrix}\text{col}_1 A & \cdots & \text{col}_n A\end{bmatrix}\begin{bmatrix}\text{row}_1 B\\ \vdots \\ \text{row}_n B\end{bmatrix}$$

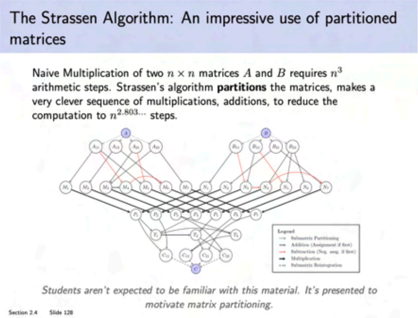
$$= \underbrace{\text{col}_1 A \,\text{row}_1 B + \cdots \text{col}_n A \,\text{row}_n B}_{m \times p \text{ matrices}}$$

This is the **Column Row Method** for matrix multiplication.

3×1 1×2 3×2

$$\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix} = \begin{bmatrix}1 & 3\\0 & 0\\2 & 6\end{bmatrix}$$

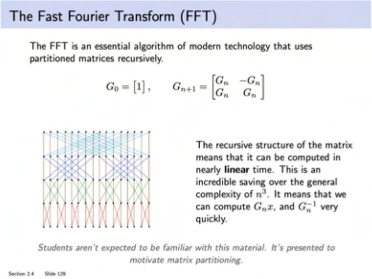
$$\vec{c}\,\vec{r}^T = \begin{bmatrix}c_1\vec{r}_1^T\\ c_2\vec{r}_2^T\\ \vdots\\ c_m\vec{r}_m^T\end{bmatrix}$$



The Strassen Algorithm: An impressive use of partitioned matrices

Naive Multiplication of two $n \times n$ matrices A and B requires n^3 arithmetic steps. Strassen's algorithm **partitions** the matrices, makes a very clever sequence of multiplications, additions, to reduce the computation to $n^{2.803\cdots}$ steps.

Students aren't expected to be familiar with this material. It's presented to motivate matrix partitioning.



The Fast Fourier Transform (FFT)

The FFT is an essential algorithm of modern technology that uses partitioned matrices recursively.

$$G_0 = [1], \qquad G_{n+1} = \begin{bmatrix}G_n & -G_n\\ G_n & G_n\end{bmatrix}$$

The recursive structure of the matrix means that it can be computed in nearly **linear** time. This is an incredible saving over the general complexity of n^3. It means that we can compute $G_n x$, and G_n^{-1} very quickly.

Students aren't expected to be familiar with this material. It's presented to motivate matrix partitioning.

2.4 EXERCISES

In Exercises 1–9, assume that the matrices are partitioned conformably for block multiplication. Compute the products shown in Exercises 1–4.

1. $\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

2. $\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

3. $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$

4. $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

In Exercises 5–8, find formulas for X, Y, and Z in terms of A, B, and C, and justify your calculations. In some cases, you may need to make assumptions about the size of a matrix in order to produce a formula. [*Hint:* Compute the product on the left, and set it equal to the right side.]

5. $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix}$

6. $\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

7. $\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

8. $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$

9. Suppose A_{11} is an invertible matrix. Find matrices X and Y such that the product below has the form indicated. Also, compute B_{22}. [*Hint:* Compute the product on the left, and set it equal to the right side.]

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$$

10. The inverse of $\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}$ is $\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix}$.

Find X, Y, and Z.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. If $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, with A_1 and A_2 the same sizes as B_1 and B_2, respectively, then $A + B = [A_1 + B_1 \quad A_2 + B_2]$.

b. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then the partitions of A and B are conformable for block multiplication.

12. a. The definition of the matrix–vector product $A\mathbf{x}$ is a special case of block multiplication.

b. If A_1, A_2, B_1, and B_2 are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, then the product BA is defined, but AB is not.

13. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B and C are square. Show that A is invertible if and only if both B and C are invertible.

14. Show that the block upper triangular matrix A in Example 5 is invertible if and only if both A_{11} and A_{22} are invertible. [*Hint:* If A_{11} and A_{22} are invertible, the formula for A^{-1} given in Example 5 actually works as the inverse of A.] This fact about A is an important part of several computer algorithms that estimate eigenvalues of matrices. Eigenvalues are discussed in Chapter 5.

15. Suppose A_{11} is invertible. Find X and Y such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \tag{7}$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$.. The matrix S is called the **Schur complement** of A_{11}. Likewise, if A_{22} is invertible, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is called the Schur complement of A_{22}. Such expressions occur frequently in the theory of systems engineering, and elsewhere.

16. Suppose the block matrix A on the left side of (7) is invertible and A_{11} is invertible. Show that the Schur complement S of A_{11} is invertible. [*Hint:* The outside factors on the right side of (7) are always invertible. Verify this.] When A and A_{11} are both invertible, (7) leads to a formula for A^{-1}, using S^{-1}, A_{11}^{-1}, and the other entries in A.

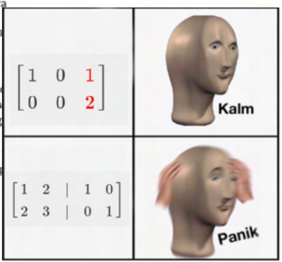
Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rathe
exercise of a combination of two facilities, which
ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear
areas of mathematics that Turing hel



Topics and Objectives

Topics

We will cover these topics in this section.

1. The LU factorization of a matrix
2. Using the LU factorization to solve a system
3. Why the LU factorization works

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute an LU factorization of a matrix.
2. Apply the LU factorization to solve systems of equations.
3. Determine whether a matrix has an LU factorization.

Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop.

Topics and Objectives

Topics

We will cover these topics in this section.

1. The LU factorization of a matrix
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For the topics covered in this section, students are expected to be able to do the following.

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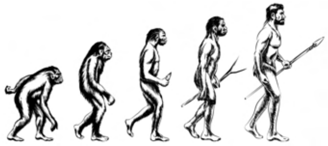
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3



In the beginning, …

with **integers** there were *prime* factorizations…

$$n = p_1^{k_1} \cdots p_r^{k_r}$$

then came the ***polynomial*** factorizations…

$$f(x) = (x-a)^n (x-b)^m$$

until finally, …

****matrix**** factorizations appeared!

lower

upper.

$$A = LU$$

$$A = QR.$$

$$A = U \Sigma V^T$$

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using
$$\vec{x} = A^{-1}\vec{b}$$
- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

$A\vec{x}_1 = \vec{b}_1$

$$[A \mid \vec{b}_1] \sim \cdots \sim \left[\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \middle| \vec{x}_1 \right]$$

$$[A \mid \vec{b}_2] \sim \cdots \sim \left[\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \middle| \vec{x}_2 \right]$$

$A\vec{x} = \vec{b}_2$

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

Triangular Matrices

- A rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$. Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$. Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Ask: Can you name a matrix that is both upper and lower triangular?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ diagonal matrix.}$$

The LU Factorization

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an echelon form of A.

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$A = LU = \underset{3\times3}{\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}}$$

L always square.

1's along diag.

U has same size as A

U upper triangular.

L lower triangular

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows. Then,

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

$$\underset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A = U$$

$-5R_2 + R_3 \to R_3$ $\quad -2R_1 + R_3 \to R_3$ $\quad 4R_1 + R_2 \to R_2$

$$\Rightarrow \overset{I}{\cancel{E_3^{-1} E_3}} E_2 E_1 A = E_3^{-1} U$$

$$\Rightarrow \overset{I}{\cancel{E_2^{-1} E_2}} E_1 A = E_2^{-1} E_3^{-1} U$$

$$\Rightarrow E_1^{-1} E_1 A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$\Rightarrow A = \underset{\uparrow L}{\boxed{E_1^{-1} E_2^{-1} E_3^{-1}}} U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm: construct $A = LU$, solve $A\vec{x} = LU\vec{x} = \vec{b}$ by:

1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

Example: Solve the linear system whose LU decomposition is given.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 16 \\ 2 \\ -4 \\ 6 \end{pmatrix}$$

Want $\vec{x}$ soln to $A\vec{x} = \vec{b}$

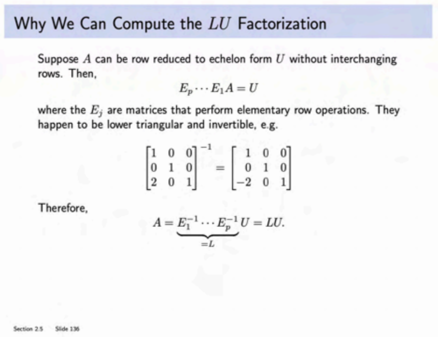
know $A = LU$.

So $LU\vec{x} = \vec{b}$ what is $\vec{x}$?

First solve $L\vec{y} = b$

Then solve $U\vec{x} = \vec{y}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$



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where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

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Therefore,

$$A = \underbrace{E_1^{-1}\cdots E_p^{-1}}_{=L} U = LU.$$

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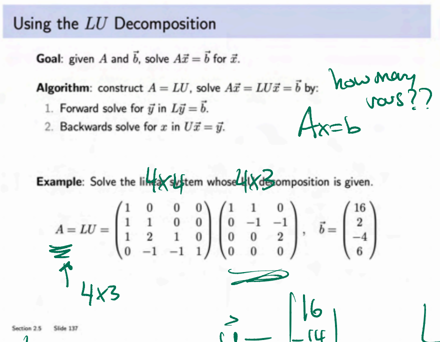

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

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1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

how many rows??

$Ax=b$

Example: Solve the linear system whose LU decomposition is given.

4x4 4x3

$$A = LU = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\0&-1&-1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&-1&-1\\0&0&2\\0&0&0\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}16\\2\\-4\\6\end{pmatrix}$$

$\equiv$ ↑ 4x3

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$U\vec{x}$

$$L\begin{bmatrix}16\\-14\\8\\0\end{bmatrix} = \begin{bmatrix}16\\2\\-4\\6\end{bmatrix}$$

① Solve $L\vec{y}=b$

$$\vec{y} = \begin{bmatrix}16\\-14\\8\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&0&0&0&16\\1&1&0&0&2\\1&2&1&0&-4\\0&-1&-1&1&6\end{array}\right] \sim \left[\begin{array}{cccc|c}1&0&0&0&16\\0&1&0&0&-14\\0&2&1&0&-20\\0&-1&-1&1&6\end{array}\right] \sim \left[\begin{array}{cccc|c}1&0&0&0&16\\0&1&0&0&-14\\0&0&1&0&8\\0&0&-1&1&-8\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1&0&0&0&16\\0&1&0&0&-14\\0&0&1&0&8\\0&0&0&1&0\end{array}\right]$$

② Solve $U\vec{x}=\vec{y}$

$$\left[\begin{array}{ccc|c}1&1&0&16\\0&-1&-1&-14\\0&0&2&8\\0&0&0&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&16\\0&-1&-1&-14\\0&0&1&4\\0&0&0&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&16\\0&-1&0&-10\\0&0&1&4\\0&0&0&0\end{array}\right]$$

$x_1=6$

$$\sim \left[\begin{array}{ccc|c}1&0&0&6\\0&1&0&10\\0&0&1&4\\0&0&0&0\end{array}\right]$$

$x_2=10$

$x_3=4$

$$\vec{x} = \begin{bmatrix}6\\10\\4\end{bmatrix}$$ ✓ ✓ ✓ ??

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note that

- In MATH 1554, the only row replacement operation we can use is to *replace a row with a multiple of a row above it.*
- More advanced linear algebra courses address this limitation.

Example: Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix}$$

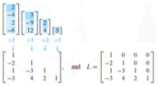
Another Explanation for How to Construct L

First compute the echelon form U of A. Highlight the entries that determine the sequence of row operations used to arrive at U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

The highlighted entries describe the row reduction of A. For each highlighted pivot column, divide entries by the pivot and place the result into L.



$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

÷2 ÷3 ÷2 ÷5

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \text{ and } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

only DOWNWARD row replacement

both 3x4

$$A = LU$$

3x3

$$A = \begin{bmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{bmatrix} \underset{\substack{4R_1+R_2 \to \\ -2R_1+R_3 \to}}{\sim} \begin{bmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -10 & 12 \end{bmatrix}$$

$$\underset{-5R_2+R_3 \to}{\sim} \begin{bmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\boxed{A = LU}$$

Summary

- To solve $A\vec{x} = LU\vec{x} = \vec{b}$,
  1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
  2. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.
- To compute the LU decomposition:
  1. Reduce A to an echelon form U by a sequence of downwards (only!) row replacement operations, if possible.
  2. Place entries in L such that the same sequence of row operations reduces L to I.
- The textbook offers a different explanation of how to construct the LU decomposition that students may find helpful.
- Another explanation on how to calculate the LU decomposition that students may find helpful is available from MIT OpenCourseWare: www.youtube.com/watch?v=rhNKncraJMk

Additional Example (if time permits)

Construct the LU decomposition of A.

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 9 & -5 & 15 \\ 15 & -1 & 10 \\ -6 & 2 & -4 \end{pmatrix}$$

$$A \sim \begin{array}{r} \\ -3R_1+R_2 \rightarrow \\ -5R_1+R_2 \rightarrow \\ 2R_1+R_3 \rightarrow \end{array} \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 3 \\ 0 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{array}{r} \\ \\ 2R_2+R_3 \rightarrow \\ 0R_2+R_4 \rightarrow \end{array} \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ \\ 1R_3+R_4 \rightarrow \end{array} \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & -2 & 1 & 0 \\ -2 & 0 & -1 & 1 \end{bmatrix}$$

2.5 EXERCISES

In Exercises 1–6, solve the equation $A\mathbf{x} = \mathbf{b}$ by using the LU factorization given for A. In Exercises 1 and 2, also solve $A\mathbf{x} = \mathbf{b}$ by ordinary row reduction.

1. $A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

2. $A = \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

3. $A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

4. $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

5. $A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \\ -4 & -1 & 9 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find an LU factorization of the matrices in Exercises 7–16 (with L unit lower triangular). Note that MATLAB will usually produce a permuted LU factorization because it uses partial pivoting for numerical accuracy.

7. $\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix}$

8. $\begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$

10. $\begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix}$

15. $\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$

17. When A is invertible, MATLAB finds A^{-1} by factoring $A = LU$ (where L may be permuted lower triangular), inverting L and U, and then computing $U^{-1}L^{-1}$. Use this method to compute the inverse of A in Exercise 2. (Apply the algorithm of Section 2.2 to L and to U.)

18. Find A^{-1} as in Exercise 17, using A from Exercise 3.

!!

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question
Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

X ← mathematicians really like to give structures to sets of objects.

Subsets of $\mathbb{R}^n$

Definition

A **subset** of $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

✓ $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ 1 "object" vector. in $\mathbb{R}^3$

✗ $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ 2 vectors in $\mathbb{R}^3$

✓ $\{\vec{x} \in \mathbb{R}^3 \mid x_3 = 0\}$ ∞-many vectors in $\mathbb{R}^2$

✓ span $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ ∞-many vectors in $\mathbb{R}^2$.

✓ The ~~image~~ range of $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotate by θ ccw.

"$\mathbb{R}^2$" infinity many vectors in $\mathbb{R}^2$.

Subspaces in $\mathbb{R}^n$

(0.) $\vec{0} \in H$ ✓

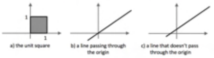
Definition

A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

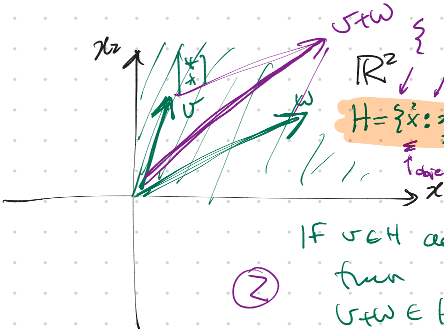
1. $c\vec{u} \in H$ ← closed under scalar mult.
2. $\vec{u} + \vec{v} \in H$ ← closed under vector addition.

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square b) a line passing through the origin c) a line that doesn't pass through the origin



{ What comes next is a set

$\mathbb{R}^2$ s.t.

$H = \{\vec{x} : x_1 \geq 0, x_2 \geq 0\}$

↑ object type ← conditions

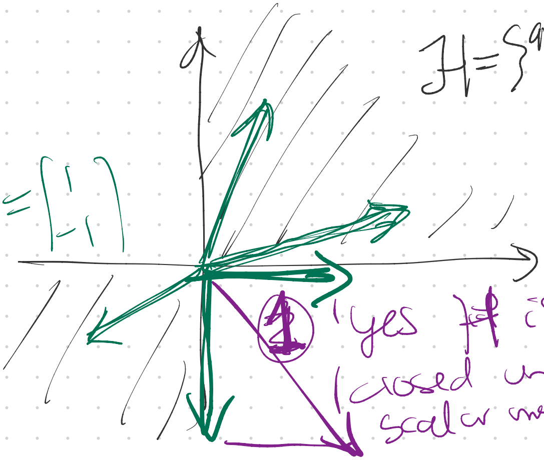
If $v \in H$ and $w \in H$

② then $v + w \in H$ ✓

H satisfies ② but not ①.

$H = \{\text{quadrant I or III}\}$

$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$



① yes H is closed under scalar mult.

② No not closed under vector add.

Ex. $H = \{ \vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1 \}$

Q1: Is it a subspace?

Q2: Can you give me 4 vectors in H?

$\vec{v}_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix} \in H$ ✓

$\vec{v}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \in H$ ✓

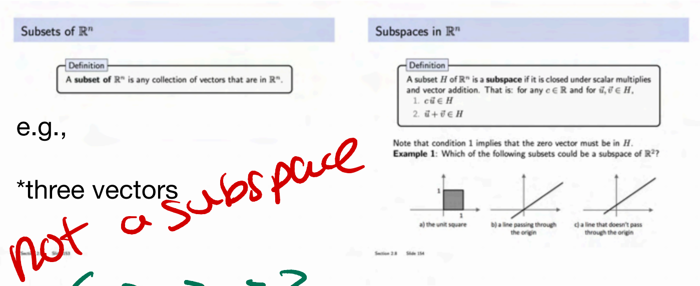
$\vec{v}_3 = \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} \in H$ ✓

$\vec{v}_4 = \begin{bmatrix}1\\0\\0\end{bmatrix} \in H$ ✓

~~(0.)~~ $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin H$ ✓

~~(1.)~~ $2\vec{v}_1 = \begin{bmatrix}2\\-2\\2\end{bmatrix} \overset{?}{\notin} \overset{?}{H}$

~~(2.)~~ $v_2 + v_4 = \begin{bmatrix}1\\0\\1\end{bmatrix} \overset{?}{\notin} H$



Subsets of $\mathbb{R}^n$

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

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Subspaces in $\mathbb{R}^n$

Definition

A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiples and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\vec{u} \in H$
2. $\vec{u} + \vec{v} \in H$

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?

Section 2.8 Slide 154

e.g.,

*three vectors

not a subspace

$\{\vec{u}, \vec{v}, \vec{w}\}$

The set of three vectors $\vec{u}, \vec{v}$, and $\vec{w}$

*the span of three vectors

Subspace!

$\text{span}\{\vec{u}, \vec{v}, \vec{\omega}\}$

The span of $\vec{u}, \vec{v}$, and $\vec{w}$

$$\vec{x} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$$

$$\vec{y} = d_1\vec{u} + d_2\vec{v} + d_3\vec{w}$$

$$\vec{x} + \vec{y} = (c_1\vec{u} + c_2\vec{v} + c_3\vec{\omega}) + (d_1\vec{u} + d_2\vec{v} + d_3\vec{\omega})$$

$$= (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v} + (c_3+d_3)\vec{\omega} \in \text{span}\{u, v, w$$

✓

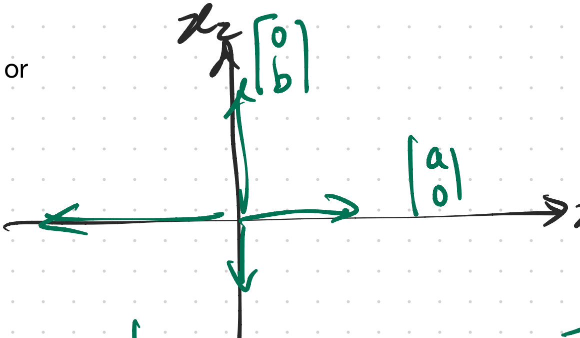
$$k\vec{x} = kc_1\vec{u} + kc_2\vec{v} + kc_3\vec{w}$$

*the set containing only the zero vector

Subspace!

$\{\vec{0}\}$

*all vectors in R^2 that are either on the x-axis or on the y-axis

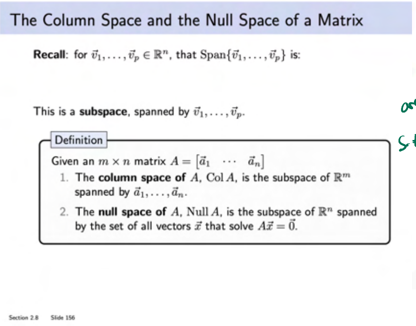


✓ (0.) $\vec{0} \in H$ ✓

✓ (1.) $c\vec{v} \in H$ ✓

~~(2.)~~ $v + w \notin \mathcal{H}$

$$\mathcal{H} = \{\vec{x} \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$$



The Column Space and the Null Space of a Matrix

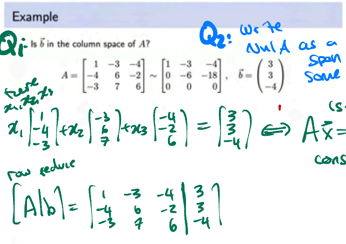
Recall: for $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is:

This is a **subspace**, spanned by $\vec{v}_1, \dots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$

1. The **column space of** A, $\text{Col}\,A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \dots, \vec{a}_n$.
2. The **null space of** A, $\text{Null}\,A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.



Example

Q1: Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

Q2: write Nul A as a span of some vectors.

are there x_1, x_2, x_3 s.t.

$$x_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \iff A\vec{x} = \vec{b} \text{ is consistent?}$$

row reduce

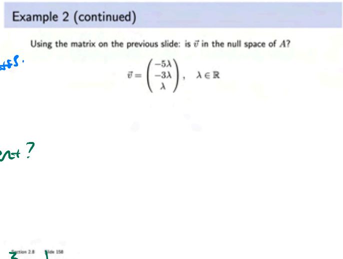
$$[A|b] = \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & 1 & 3 & -5/2 \\ 0 & 1 & 3 & -5/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} \textcircled{1} & -3 & -4 & 3 \\ 0 & \textcircled{1} & 3 & -5/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yes!

$b \in \text{Col}A$ ✓



Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$A\vec{x} = 0$

$$A \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad X = s \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Nul}\,A = \text{span}\left\{\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}\right\}}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\}$$

$$X = s\begin{bmatrix} \vdots \end{bmatrix} + t\begin{bmatrix} \vdots \end{bmatrix}$$

$$\text{Nul}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

Basis

① $\{u_1, u_2, \dots, u_p\}$ spans H

② $\{u_1, \dots, u_p\}$ lin ind.

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

$2 \quad 1 \quad 0 \quad 0$ ✓

The set $H = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0\}$ is a subspace.

a) H is a null space for what matrix A?

b) Construct a basis for H.

$H = \text{Nul}A = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$r \quad s \quad t$

$A = [1 \ 2 \ 1 \ 5]$

coeff matrix for

$1x_1 + 2x_2 + 1x_3 + 5x_4 = 0.$

$$X = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and a basis for H is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

sanity check?. Is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in H?

Example

$H = \text{span}\{u_1, \dots, v_p\}$ $\{u_1, \dots, u_p\}$ is a Spanning set for H.

Construct a basis for Nul$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Col}A = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\}$$

but $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}}, \cancel{\begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix}} \right\}$ is a linearly dependent set.

so these five vectors don't form a basis.

$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ is a basis for Col$A$.

Warnings!. ① don't take the columns of REF of A!!

② bases are NOT UNIQUE!!

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

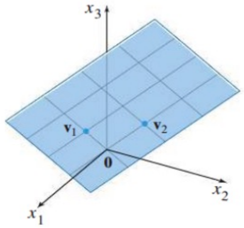


FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

Theorem

The pivotal columns a matrix A form $\boxed{\text{a basis}}$ for the Column space of A.

a good way to find a basis.

Use the pivotal columns of A, not the pivotal columns of the Echelon form.

Theorem

Suppose that the matrix A has reduced echelon form $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, in block matrix form. Then a basis of the Null space of A is given by the columns of $\begin{bmatrix} F \\ -I \end{bmatrix}$.

The assumption says that the first few columns are pivotal, and the last few are all free. This can be assumed, after the exchange of columns.

Use vectors in parametric vector form.

Additional Example (if time permits)

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$. Is V a subspace?

??

Not a subspace

e.g.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$

but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V$.

EXAMPLE 6 Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

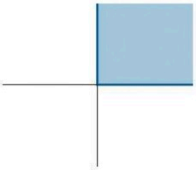
$$[A \quad \mathbf{0}] \sim \begin{bmatrix} [illegible] & [illegible] & 0 & -1 & [illegible] & 0 \\ 0 & 0 & 1 & 2 & -2 & [illegible] \\ 0 & 0 & 0 & 0 & 0 & [illegible] \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 \quad - \; x_4 + 3x_5 &= 0 \\ x_3 \; [illegible] \; 2 \, [illegible] &= 0 \\ 0 &= 0 \end{aligned}$$

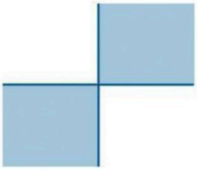
2.8 EXERCISES

Exercises 1–4 display sets in $\mathbb{R}^2$. Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of $\mathbb{R}^2$. (For instance, find two vectors in H whose sum is *not* in H, or find a vector in H with a scalar multiple that is not in H. Draw a picture.)

1.



2.



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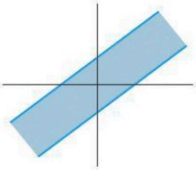
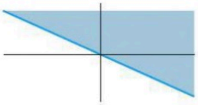


4.



5. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$. Determine if $\mathbf{u}$ is in the subspace of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$, and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Col A?

c. Is $\mathbf{p}$ in Col A? Why or why not?

8. Let $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$. Determine if $\mathbf{p}$ is in Col A, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

10. With $\mathbf{u} = (-2, 3, 1)$ and A as in Exercise 8, determine if $\mathbf{u}$ is in Nul A.

In Exercises 11 and 12, give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$.

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$

13. For A as in Exercise 11, find a nonzero vector in Nul A and a nonzero vector in Col A.

14. For A as in Exercise 12, find a nonzero vector in Nul A and a nonzero vector in Col A.

Determine which sets in Exercises 15–20 are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

15. $\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix}$

16. $\begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

19. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A subspace of $\mathbb{R}^n$ is any set H such that (i) the zero vector is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

b. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the same as the column space of the matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$.

c. The set of all solutions of a system of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^m$.

d. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

e. Row operations do not affect linear dependence relations among the columns of a matrix.

22. a. A subset H of $\mathbb{R}^n$ is a subspace if the zero vector is in H.

b. Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

c. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

d. The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

Exercises 23–26 display a matrix A and an echelon form of A. Find a basis for Col A and a basis for Nul A.

23. $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Col A, but $\mathbf{b}$ is not the same as any one of the columns of A.

28. Construct a nonzero 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is *not* in Col A.

29. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Nul A.

30. Suppose the columns of a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is a basis for Col A.

In Exercises 31–36, respond as comprehensively as possible, and justify your answer.

31. Suppose F is a 5×5 matrix whose column space is not equal to $\mathbb{R}^5$. What can you say about Nul F?

32. If R is a 6×6 matrix and Nul R is *not* the zero subspace, what can you say about Col R?

33. If Q is a 4×4 matrix and Col $Q = \mathbb{R}^4$, what can you say about solutions of equations of the form $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^4$?

34. If P is a 5×5 matrix and Nul P is the zero subspace, what can you say about solutions of equations of the form $P\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^5$?

35. What can you say about Nul B when B is a 5×4 matrix with linearly independent columns?

36. What can you say about the shape of an $m \times n$ matrix A when the columns of A form a basis for $\mathbb{R}^m$?

[M] In Exercises 37 and 38, construct bases for the column space and the null space of the given matrix A. Justify your work.

37. $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{bmatrix}$

WEB Column Space and Null Space

WEB A Basis for Col *A*