LINEAR ALGEBRA

week 7

Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

Course Schedule

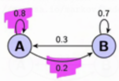
Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Example 1

- A small town has two libraries, A and B.
- After 1 month, among the books checked out of A,
  - 80% returned to A
  - 20% returned to B
- After 1 month, among the books checked out of B,
  - 30% returned to A
  - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is http://setosa.io/markov/index.html



$\mathrm{Nul}(A-I)$

from A

from B

@A.

$(.80)1000 + (.30)1000 = 800+300 = 1100$

$(.20)1000 + (.70)1000 = 200+700 = 900$

$1000\begin{bmatrix}.80\\.20\end{bmatrix} + 1000\begin{bmatrix}.30\\.70\end{bmatrix} = \begin{bmatrix}1100\\900\end{bmatrix}$

$\begin{bmatrix}.80 & .30\\.20 & .70\end{bmatrix}\begin{bmatrix}1000\\1000\end{bmatrix} = \begin{bmatrix}1100\\900\end{bmatrix}$

$(.80)1100 + (.30)900 = 1150$

$(.20)1100 + (.70)900 = 850$

$1100\begin{bmatrix}.80\\.20\end{bmatrix} + 900\begin{bmatrix}.30\\.70\end{bmatrix} = \begin{bmatrix}1150\\850\end{bmatrix}$

$A\vec{x}_0 = \vec{x}_1$

$A\vec{x}_1 = \vec{x}_2$

$A\vec{x}_2 = \vec{x}_3$

$\boxed{A^n\vec{x}_0 = \vec{x}_n}$

Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix}.5\\.5\end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

Columns tell you where you started

After k months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

$$P = \begin{bmatrix}.8 & .3\\.2 & .7\end{bmatrix}$$

rows tell you where you went.

stochastic matrix.

columns are probability vectors.


```
clc
format shortG
% library example
A=[.8 .3 ; .2 .7]
x0=[.5;.5]

% car rental example
%A=[.8 .1 .2 ; .2 .6 .3 ; 0 .3 .5]
%x0=[.2; .2; .6]

% election example
%A=[.7 .1 .3 ; .2 .8 .3 ; .1 .1 .4]
%x0=[.55;.4;.05]

% set value of n (largest index to compute)
%n=10;
for i=1:n
    % convert current index to string and create xi string
    index=string(i);
    s=strcat('x',index,'=');
    % compute xi value
    xi=A^i*x0;
    % display each xi=A^i*x0
    disp(s)
    disp(xi)
end
% display final xn=A^n*x0
format longG
n=4;
index=string(n);
s=strcat('x',index,'=');
xi=A^n*x0;
disp(s)
disp(xi)
```

Command Window

```
A =

      0.8       0.3
      0.2       0.7

x0 =

      0.5
      0.5

x1=
     0.55
     0.45

x2=
    0.575
    0.425

x3=
   0.5875
   0.4125

x4=
  0.59375
  0.40625

x4=
              0.59375
              0.40625

fx >>
```

Starting probability vector x_0

$x1 = Ax_0$

$x2 = Ax_1$

$x3 = Ax_2$

$x4 = Ax_3 = A^4x_0$

Markov Chains

A few definitions:

- A **probability vector** is a vector, $\vec{x}$, with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, P, whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $\vec{x}_k$, and a stochastic matrix P, such that:

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

- A **steady-state vector** for P is a vector $\vec{q}$ such that $P\vec{q} = \vec{q}$.

$I\vec{x} = \vec{x}$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

Example 2

Determine a steady-state vector for the stochastic matrix

$$P = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

Want $\vec{x} = \begin{bmatrix}a\\b\end{bmatrix}$ s.t. $P\vec{x} = \vec{x}$

Notice If $P\vec{x} = \vec{x}$

$\Rightarrow P\vec{x} = I\vec{x}$

$\Rightarrow P\vec{x} - I\vec{x} = \vec{0}$

$\Rightarrow (P-I)\vec{x} = \vec{0}$

So, row reduce $P-I$.

$P - I = \begin{bmatrix}.8&.3\\.2&.7\end{bmatrix} - \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$= \begin{bmatrix}-.2&.3\\.2&-.3\end{bmatrix}$

$\sim \begin{bmatrix}-2&3\\2&-3\end{bmatrix}$

$\sim \begin{bmatrix}2&-3\\0&0\end{bmatrix}$

$\sim \begin{bmatrix}1&-3/2\\0&0\end{bmatrix}$ (free variable s)

$\vec{x} = s\begin{bmatrix}3/2\\1\end{bmatrix}$

So I want a probability vector so I have to choose s-value so that the entries add up to 1.

$\frac{3}{2} + 1 = 5/2$

So $\vec{q} = \frac{2}{5}\begin{bmatrix}3/2\\1\end{bmatrix} = \begin{bmatrix}3/5\\2/5\end{bmatrix} = \begin{bmatrix}.6\\.4\end{bmatrix}$

Check?

$P\vec{q} = \begin{bmatrix}.8&.3\\.2&.7\end{bmatrix}\begin{bmatrix}.6\\.4\end{bmatrix}$

$= \begin{bmatrix}.48+.12\\.12+.28\end{bmatrix} = \begin{bmatrix}.6\\.4\end{bmatrix} = \vec{q}$ ✓ hooray!

library example

1200 @ A

800 @ B

Convergence

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

as $k \to \infty$.

Definition: a stochastic matrix P is **regular** if there is some k such that P^k only contains strictly positive entries.

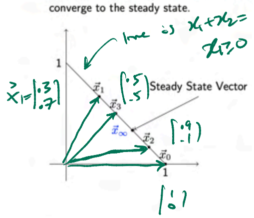
Theorem

If P is a regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

Stochastic Vectors in the Plane

The stochastic vectors in the plane are the line segment below, and a stochastic matrix maps stochastic vectors to themselves. Iterates $P^k\vec{x}_0$ converge to the steady state.

line is $x_1 + x_2 = 1$. $x_1 \ge 0$, $x_2 \ge 0$.



$P^k \longrightarrow [\vec{x}_\infty \ \ \vec{x}_\infty]$

$\begin{bmatrix}.8&.3\\.2&.7\end{bmatrix}$ is regular stochastic

but so is $P = \begin{bmatrix}0&.5\\1&.5\end{bmatrix}$ ✓ $P^2 = \begin{bmatrix}0&.5\\1&.5\end{bmatrix}\begin{bmatrix}0&.5\\1&.5\end{bmatrix} = \begin{bmatrix}.5&.25\\.5&.75\end{bmatrix}$

P^2 has no zeros.

Example 3

A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

		rented from		
		A	B	C
returned to	A	.8	.1	.2
	B	.2	.6	.3
	C	.0	.3	.5

table

There are 10 cars at each location today.

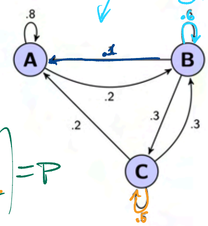
a) Construct a stochastic matrix, P, for this problem.

b) What happens to the distribution of cars after a long time? You may assume that P is regular.

From

$$\begin{array}{c} \text{to } A \\ \text{to } B \\ \\ \end{array} \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix} = P$$

transition diagram



stochastic matrix

$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Lets find $\vec{q}$ a steady state probability vector.

① form $P - I$ & row reduce

② get parametric vector form for $(P-I)\vec{x} = 0$

③ Choose a value of free parameter so that entries add up to 1.

$$P - I = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.2 & .1 & .2 \\ .2 & -.4 & .3 \\ 0 & .3 & -.5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 2 & -4 & 3 \\ 0 & 3 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 5 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 0 & 11/3 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11/6 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - 11/6\, x_3 = 0 \\ x_2 - 5/3\, x_3 = 0 \\ \quad x_3 = \text{free} \end{cases}$$

$$\vec{x} = t \begin{bmatrix} 11/6 \\ 5/3 \\ 1 \end{bmatrix} \qquad t = 6 \qquad \vec{x} = \begin{bmatrix} 11 \\ 10 \\ 6 \end{bmatrix}$$

$$\vec{q} = \begin{bmatrix} 11/27 \\ 10/27 \\ 6/27 \end{bmatrix}$$

EXAMPLE 1 Section 1.10 examined a model for population movement between a city and its suburbs. See Figure 1. The annual migration between these two parts of the metropolitan region was governed by the *migration matrix M*:

$$\begin{array}{cc} & \text{From:} \\ & \begin{array}{cc}\text{City} & \text{Suburbs}\end{array} \quad \text{To:} \\ M = & \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{array}{l}\text{City} \\ \text{Suburbs}\end{array} \end{array}$$

That is, each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. The columns of M are probability vectors, so M is a stochastic matrix. Suppose the 2014 population of the region is 600,000 in the city and 400,000 in the suburbs. Then the initial distribution of the population in the region is given by $\mathbf{x}_0$ in (1) above. What is the distribution of the population in 2015? In 2016?

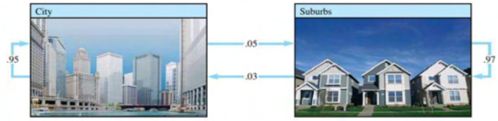


FIGURE 1 Annual percentage migration between city and suburbs.

EXAMPLE 4 The probability vector $\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ is a steady-state vector for the population migration matrix M in Example 1, because

$$M\mathbf{q} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}\begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .35625 + .01875 \\ .01875 + .60625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \mathbf{q}$$ ■

entries add to 1 & are non-negative

Q1: What is a Markov chain?

Sanity check

a sequence $\vec{X}_0, \vec{X}_1, \ldots, \vec{X}_n$ of probability vectors such that $\vec{X}_{k+1} = P\vec{X}_k$

where P is stochastic

cols are prob. vectors.

EXAMPLE 2 Suppose the voting results of a congressional election at a certain voting precinct are represented by a vector $\mathbf{x}$ in $\mathbb{R}^3$:

$$\mathbf{x} = \begin{bmatrix} \text{\% voting Democratic (D)} \\ \text{\% voting Republican (R)} \\ \text{\% voting Libertarian (L)} \end{bmatrix}$$

Suppose we record the outcome of the congressional election every two years by a vector of this type and the outcome of one election depends only on the results of the preceding election. Then the sequence of vectors that describe the votes every two years may be a Markov chain. As an example of a stochastic matrix P for this chain, we take

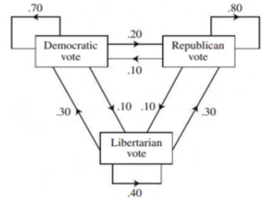


FIGURE 2 Voting changes from one election to the next.

$$\begin{array}{cccc} & \text{From:} & & \\ \text{D} & \text{R} & \text{L} & \end{array}$$

$$P = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{array}{l} \text{To:} \\ \text{D} \\ \text{R} \\ \text{L} \end{array}$$

SOLUTION The outcome of the next election is described by the state vector $\mathbf{x}_1$ and that of the election after that by $\mathbf{x}_2$, where

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} \begin{array}{l} \text{44\% will vote D.} \\ \text{44.5\% will vote R.} \\ \text{11.5\% will vote L.} \end{array}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix} \begin{array}{l} \text{38.7\% will vote D.} \\ \text{47.8\% will vote R.} \\ \text{13.5\% will vote L.} \end{array}$$

To understand why $\mathbf{x}_1$ does indeed give the outcome of the next election, suppose 1000 persons voted in the "first" election, with 550 voting D, 400 voting R, and 50 voting L. (See the percentages in $\mathbf{x}_0$.) In the next election, 70% of the 550 will vote D again, 10% of the 400 will switch from R to D, and 30% of the 50 will switch from L to D. Thus the total D vote will be

$$.70(550) + .10(400) + .30(50) = 385 + 40 + 15 = 440 \tag{2}$$

Thus 44% of the vote next time will be for the D candidate. The calculation in (2) is essentially the same as that used to compute the first entry in $\mathbf{x}_1$. Analogous calculations could be made for the other entries in $\mathbf{x}_1$, for the entries in $\mathbf{x}_2$, and so on. ■

EXAMPLE 3 Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consider a system whose state is described by the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$, for $k = 0, 1, \ldots$ What happens to the system as time passes? Compute the state vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{15}$ to find out.

SOLUTION

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

The results of further calculations are shown below, with entries rounded to four or five significant figures.

$$\mathbf{x}_4 = \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix}$$

$$\mathbf{x}_8 = \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, \quad \mathbf{x}_{11} = \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix}$$

$$\mathbf{x}_{12} = \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, \quad \mathbf{x}_{13} = \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, \quad \mathbf{x}_{14} = \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, \quad \mathbf{x}_{15} = \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix}$$

These vectors seem to be approaching $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$. The probabilities are hardly changing from one value of k to the next. Observe that the following calculation is exact (with no rounding error):

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

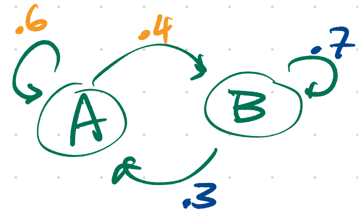
When the system is in state $\mathbf{q}$, there is no change in the system from one measurement to the next. ■

EXAMPLE 5 Let $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Find a steady-state vector for P.

$$P^2 = \begin{bmatrix} \boxed{.48} & .39 \\ \boxed{.52} & .61 \end{bmatrix}$$

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$



$$\vec{x} = s\begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \qquad s = 4 \qquad \vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\vec{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

$P^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ "First col of P^2"

$= \vec{x}_2$

For initial "every one start at A"

entry a_{ij} in P^n says the prob you ended in state i starting in state j after n steps.

SOLUTION First, solve the equation $P\mathbf{x} = \mathbf{x}$.

$$P\mathbf{x} - \mathbf{x} = \mathbf{0}$$
$$P\mathbf{x} - I\mathbf{x} = \mathbf{0} \qquad \text{Recall from Section 1.4 that } I\mathbf{x} = \mathbf{x}.$$
$$(P - I)\mathbf{x} = \mathbf{0}$$

For P as above,

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

To find all solutions of $(P - I)\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $x_1 = \frac{3}{4}x_2$ and x_2 is free. The general solution is $x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

Next, choose a simple basis for the solution space. One obvious choice is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ but a better choice with no fractions is $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (corresponding to $x_2 = 4$).

Finally, find a probability vector in the set of all solutions of $P\mathbf{x} = \mathbf{x}$. This process is easy, since every solution is a multiple of the solution $\mathbf{w}$ above. Divide $\mathbf{w}$ by the sum of its entries and obtain

$$\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

As a check, compute

$$P\mathbf{q} = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix}\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \mathbf{q} \quad ■$$

THEOREM 18 If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

4.9 EXERCISES

1. A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.

a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.

b. Give the initial state vector.

c. What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?

2. A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food on one trial, it will choose the same food on the next trial with a probability of 50%, and it will choose the other foods on the next trial with equal probabilities of 25%.

a. What is the stochastic matrix for this situation?

b. If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?

3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

a. What is the stochastic matrix for this situation?

b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?

c. If a student is well today, what is the probability that he or she will be well two days from now?

4. The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 60% chance the weather will be good tomorrow, a 30% chance the weather will be indifferent, and a 10% chance the weather will be bad. If the weather is indifferent today, it will be good tomorrow with probability .40 and indifferent with probability .30. Finally, if the weather is bad today, it will be good tomorrow with probability .40 and indifferent with probability .50.

a. What is the stochastic matrix for this situation?

b. Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?

c. Suppose the predicted weather for Monday is 40% indifferent weather and 60% bad weather. What are the chances for good weather on Wednesday?

In Exercises 5–8, find the steady-state vector.

5. $\begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$

6. $\begin{bmatrix} .8 & .5 \\ .2 & .5 \end{bmatrix}$

7. $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$

8. $\begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix}$

9. Determine if $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ is a regular stochastic matrix.

10. Determine if $P = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ is a regular stochastic matrix.

11. a. Find the steady-state vector for the Markov chain in Exercise 1.

b. At some time late in the day, what fraction of the listeners will be listening to the news?

12. Refer to Exercise 2. Which food will the animal prefer after many trials?

13. a. Find the steady-state vector for the Markov chain in Exercise 3.

b. What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?

14. Refer to Exercise 4. In the long run, how likely is it for the weather in Columbus to be good on a given day?

15. [M] The Demographic Research Unit of the California State Department of Finance supplied data for the following migration matrix, which describes the movement of the United

$$\begin{cases} x_1 + 3x_2 - 4x_3 = 2 \\ -x_1 - x_2 + 5x_2 = 3 \\ x_2 + 2x_2 = 1 \end{cases}$$

$AB = I_n$

$A^{-1}\vec{b} = \vec{x}$

Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

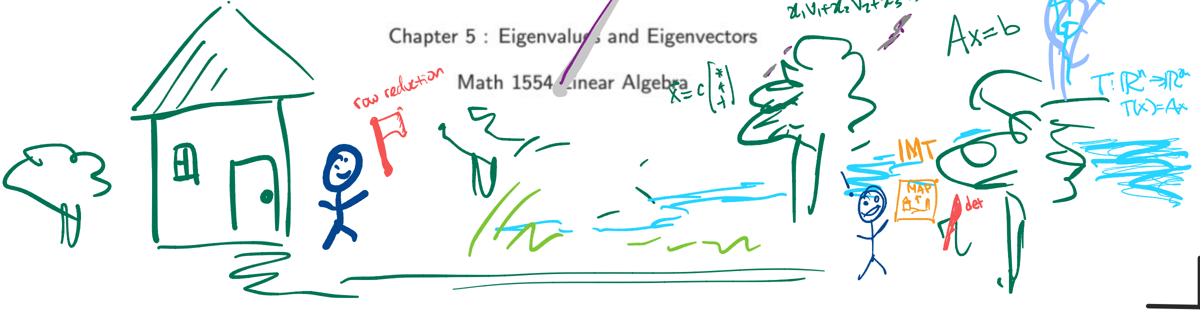
$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$

$Ax = b$

$\vec{x} = c\begin{bmatrix} * \\ * \\ * \end{bmatrix}$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(x) = Ax$



Topics and Objectives

Topics
We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

Eigenvectors and Eigenvalues

If $A \in \mathbb{R}^{n\times n}$, and there is a $\vec{v} \neq \vec{0}$ in $\mathbb{R}^n$, and

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

then $\vec{v}$ is an **eigenvector** for A, and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

Note that

- We will only consider square matrices.
- If $\lambda \in \mathbb{R}$, then
  - when $\lambda > 0$, $A\vec{v}$ and $\vec{v}$ point in the same direction
  - when $\lambda < 0$, $A\vec{v}$ and $\vec{v}$ point in opposite directions
- Even when all entries of A and $\vec{v}$ are real, λ can be complex (a rotation of the plane has no **real** eigenvalues.)
- We explore complex eigenvalues in Section 5.5.

5.1 EIGENVECTORS AND EIGENVALUES

Although a transformation $\mathbf{x} \mapsto A\mathbf{x}$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Figure 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, $\mathbf{v}$. ■

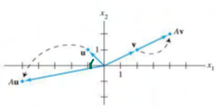


FIGURE 1 Effects of multiplication by A.

As another example, readers of Section 4.9 will recall that if A is a stochastic matrix, then the steady-state vector $\mathbf{q}$ for A satisfies the equation $A\mathbf{x} = \mathbf{x}$. That is, $A\mathbf{q} = 1 \cdot \mathbf{q}$.

5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

$$A\vec{u} = \overset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \overset{?}{=} \overset{\lambda\cdot\vec{u}}{\lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \lambda = 2$$

Example 1

Which of the following are eigenvectors of $A = \begin{pmatrix}1&1\\1&1\end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ yes

$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda=2$

b) $\vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix}$ $\lambda=0$

Nul(A) consists of the zero vector & all $\lambda=0$ eigenvectors of A.

c) $\vec{v}_3 = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \lambda\begin{bmatrix}0\\0\end{bmatrix}$ λ anything.

Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix}2&-4\\-1&-1\end{pmatrix}$.

If I can find $\vec{x}\neq\vec{0}$ s.t. $\boxed{A\vec{x}=3\vec{x}}$ then $\lambda=3$ is an eigenvalue.

$A\vec{x}=3\vec{x} \Rightarrow A\vec{x}-3\vec{x}=\vec{0}$

$\Rightarrow A\vec{x}-3I\vec{x}=\vec{0}$

$\Rightarrow (A-3I)\vec{x}=\vec{0}$

$A-3I = \begin{bmatrix}2&-4\\-1&-1\end{bmatrix} - \begin{bmatrix}3&0\\0&3\end{bmatrix}$

$= \begin{bmatrix}-1&-4\\-1&-4\end{bmatrix} \sim \begin{bmatrix}1&4\\0&0\end{bmatrix}$

$\boxed{\vec{x} = s\begin{bmatrix}-4\\1\end{bmatrix}}$ these are all eigenvectors if $s\neq 0$.

Check

$\underbrace{\begin{bmatrix}2&-4\\-1&-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}-4\\1\end{bmatrix}}_{\vec{x}} = \begin{bmatrix}-12\\3\end{bmatrix} = \underbrace{3\begin{bmatrix}-4\\1\end{bmatrix}}_{\lambda\vec{x}}$

Eigenspace

Nul(A - λI)

Definition

Suppose $A \in \mathbb{R}^{n\times n}$. The eigenvectors for a given λ span a subspace of $\mathbb{R}^n$ called the **λ-eigenspace** of A.

Note: the λ-eigenspace for matrix A is $\mathrm{Nul}(A-\lambda I)$.

Example 3

Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$\begin{pmatrix}5&-6\\3&-4\end{pmatrix},\quad \lambda=-1,2$$

Theorems

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.
2. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A.
3. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

IMT DLC

$\lambda=0$ eigenspace

$\mathrm{Nul}(A-0I) = \mathrm{Nul}(A)$.

WARNING!

e.g. $\begin{bmatrix}2&3&1\\0&1&4\\0&0&5\end{bmatrix}$ $\lambda_1=2$, $\lambda_2=1$, $\lambda_3=5$

$\lambda=-1$

$A-\lambda I = A-(-1)I$

$= A+I = \begin{bmatrix}6&-6\\3&-3\end{bmatrix} \sim \begin{bmatrix}1&-1\\0&0\end{bmatrix}$ $\vec{x}=s\begin{bmatrix}1\\1\end{bmatrix}$

basis for $\lambda=-1$ eigenspace is $\boxed{\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}}$

don't write span!

$\lambda=2$ $A-2I = \begin{bmatrix}3&-6\\3&-6\end{bmatrix} \sim \begin{bmatrix}1&-2\\0&0\end{bmatrix}$ $\vec{x}=s\begin{bmatrix}2\\1\end{bmatrix}$

basis for $\lambda=2$ eigenspace is $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

Warning!

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

Example: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_1 = 2$$

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda_2 = 0$$

- But the reduced echelon form of A is: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda = 1$, $\lambda_2 = 0$
- The reduced echelon form is triangular, and its eigenvalues are:

Additional Resource

3Blue1Brown
A beautiful, animated, and visual explanation of eigenvalues and eigenvectors.

http://bit.ly/2lXyJPg

When you do row operations
some eigenvalues could change
and some might stay the same.

DEFINITION An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

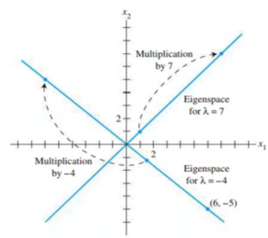


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

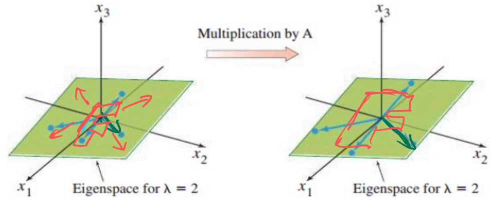


FIGURE 3 A acts as a dilation on the eigenspace.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\vec{x} = s\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{cases}$$

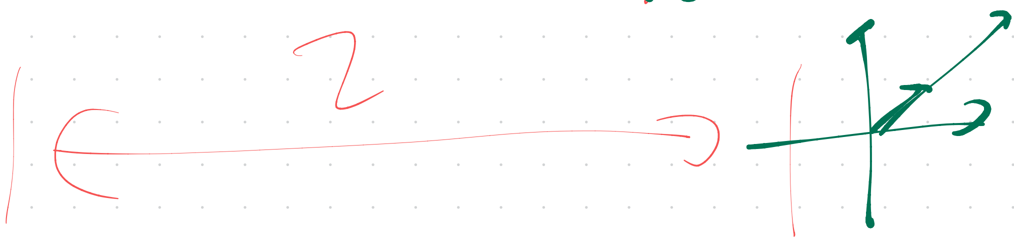
$x_1 = \frac{1}{2}s - 3t$
$x_2 = s$ (free)
$x_3 = t$ (free)

basis for $\lambda = 2$ eigenspace is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

THEOREM 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$



5.1 EXERCISES

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not?

3. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If so, find the eigenvalue.

4. Is $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$? If so, find the eigenvalue.

5. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.

6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

8. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

9. $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \lambda = 1, 5$

10. $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \lambda = 4$

11. $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \lambda = 10$

12. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \lambda = 1, 5$

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}, \lambda = -2$

15. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \lambda = 3$

16. $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

18. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

19. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify your answer.

In Exercises 21 and 22, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

21. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.
   b. A matrix A is not invertible if and only if 0 is an eigenvalue of A.
   c. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
   d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.
   e. To find the eigenvalues of A, reduce A to echelon form.

22. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.
   b. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
   c. A steady-state vector for a stochastic matrix is actually an eigenvector.
   d. The eigenvalues of a matrix are on its main diagonal.
   e. An eigenspace of A is a null space of a certain matrix.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

24. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

25. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

26. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

28. Use Exercise 27 to complete the proof of Theorem 1 for the case when A is lower triangular.

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Find an eigenvector.]

30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Use Exercises 27 and 29.]

In Exercises 31 and 32, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.

31. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

32. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

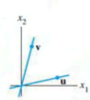
33. Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ and μ, and let c_1 and c_2 be scalars. Define

$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v} \quad (k = 0, 1, 2, \ldots)$$

   a. What is $\mathbf{x}_{k+1}$, by definition?
   b. Compute $A\mathbf{x}_k$ from the formula for $\mathbf{x}_k$, and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$. This calculation will prove that the sequence $\{\mathbf{x}_k\}$ defined above satisfies the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$.

34. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$ if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. [*Hint:* How might you relate $\mathbf{x}_0$ to eigenvectors of A?]

35. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure, and suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.



36. Repeat Exercise 35, assuming $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A that correspond to eigenvalues -1 and 3, respectively.

[M] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

37. $\begin{bmatrix} 8 & -10 & -5 \\ 2 & 17 & 2 \\ -9 & -18 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$

39. $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$

40. $\begin{bmatrix} -4 & -4 & 20 & -8 & -1 \\ 14 & 12 & 46 & 18 & 2 \\ 6 & 4 & -18 & 8 & 1 \\ 11 & 7 & -37 & 17 & 2 \\ 18 & 12 & -60 & 24 & 5 \end{bmatrix}$

Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

$A\vec{x}=\lambda\vec{x}$ has a non-zero soln for $\vec{x}$

$\iff (A-\lambda I)\vec{x}=\vec{0}$ has a free var.

The Characteristic Polynomial

Recall:

λ is an eigenvalue of A $\Leftrightarrow$ $(A-\lambda I)$ is not invertible

Therefore, to calculate the eigenvalues of A, we can solve

$$\det(A-\lambda I) = 0$$

The quantity $\det(A-\lambda I)$ is the **characteristic polynomial** of A. deg is n.

The quantity $\det(A-\lambda I) = 0$ is the **characteristic equation** of A.

The roots of the characteristic polynomial are the eigenvalues of A.

Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

trace, det

$\lambda^2 - 6\lambda + 5 - 4$

$= \boxed{\lambda^2 - 6\lambda + 1}$

$$p(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(1-\lambda) - 4$$

$$= 5 - \lambda - 5\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 6\lambda + 1$$

So the eigenvalues of A are:

$\sqrt{32} = \sqrt{8 \cdot 4}$

$= \sqrt{8} \cdot 2$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{6 \pm \sqrt{36-4}}{2}$$

$$\lambda_{1,2} = 3 \pm \frac{\sqrt{32}}{2} = 3 \pm \frac{2\sqrt{8}}{2}$$

$$\boxed{\lambda_1 = 3+2\sqrt{2}, \quad \lambda_2 = 3-2\sqrt{2}}$$

Characteristic Polynomial of 2×2 Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when M is singular?

$$\det(M - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - \underbrace{tr(M)}_{\substack{\text{sum} \\ \lambda_1+\lambda_2+\cdots+\lambda_n}}\lambda + \underbrace{\det M}_{\substack{\text{product} \\ \lambda_1\cdot\lambda_2\cdots\lambda_n}}$$

Algebraic Multiplicity

Definition

The **algebraic multiplicity** of an eigenvalue
as a root of the characteristic polynomial.

Example

Compute the algebraic multiplicities of the eigenva

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Ex. $A = \begin{bmatrix} 4 & -3 & 1 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ compute the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ eigenvalues.

Soln.

$$p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -3 & 1 & 2 \\ 0 & 2-\lambda & 1 & 2 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 4-\lambda & -3 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)\begin{vmatrix} 4-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(3-\lambda)[(4-\lambda)(2-\lambda) - 3\cdot 0]$$

$(a-b) = -(b-a)$

$$= (3-\lambda)(3-\lambda)(4-\lambda)(2-\lambda)$$

$$= \cancel{(-1)}^4 \underbrace{(\lambda-3)^2(\lambda-4)(\lambda-2)}_{p(\lambda)} = 0$$

roots are $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 4$, $\lambda_4 = 2$

Characteristic Polynomial of 2×2 Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when M is singular?

$$\det(M-\lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

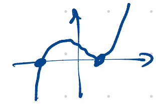
$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - \underbrace{\mathrm{tr}(M)}_{\text{sum } \lambda_1+\lambda_2+\cdots+\lambda_n}\lambda + \underbrace{\det M}_{\text{product } \lambda_1\cdot\lambda_2\cdots\lambda_n}$$

Algebraic Multiplicity

$$p(\lambda) = (-1)^n (\lambda-c_1)^{k_1}(\lambda-c_2)^{k_2}\cdots(\lambda-c_\ell)^{k_\ell}$$



Definition

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\lambda_1 = 1, \ \lambda_2 = 0, \ \lambda_3 = 0, \ \lambda_4 = -1}$ 3 distinct roots.

one root has mult 2
others have mult 1.

Geometric Multiplicity

$\dim \mathrm{Nul}(A-\lambda I) = \#$ free cols in $A-\lambda I$.

Definition

The **geometric multiplicity** of an eigenvalue λ is the dimension of $\mathrm{Null}(A-\lambda I)$.

1. Geometric multiplicity is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

alg mult. 2.

$\boxed{\lambda_1 = 0 \quad \lambda_2 = 0}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$A - 0I = A$ # free cols $= 1$.

$\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

$\dim \mathrm{Nul}(A-\lambda I) = 1$.

so geo mult of $\lambda = 0$ is 1.

Example

Give an example of a 4×4 matrix with $\lambda = 3$ the only eigenvalue, but the geometric multiplicity of $\lambda = 3$ is one.

Hint: A can be upper tri.

$$A = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

take-home

Construct a matrix A w/ any eigenvalues w/ any mult. For alg $\geq$ geo (s.t. geo $\leq$ alg)

FACT geo = alg ✓ or geo < alg ✓ both possible

~~geo > alg~~ not possible

Ex. $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \underbrace{\lambda_1 = 2, \ \lambda_2 = 2}_{\text{alg } 2}, \ \underbrace{\lambda_3 = 1}_{\text{alg } 1} \leftarrow$ roots are.

$$p(\lambda) = -(\lambda-2)^2(\lambda-1) = 0$$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

check? $A\vec{x} = 2\vec{x}$?

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = x_1$ free
$x_2 = 0$
$x_3 = 0$

Recall: Long-Term Behavior of Markov Chains

Recall:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

as $k \to \infty$.

- If P is regular, then there is a unique steady state which is the long term trend of every Markov chain $P^k\vec{x}_0 = \vec{x}_k$

Now lets ask:

- If we don't know whether P is regular, what else might we do to describe the long-term behavior of the system?
- What can eigenvalues tell us about the behavior of these systems?

row reduce $P - I \sim \dots$

to get some vector in $\mathrm{Nul}(P-I)$

(notice q is a $\lambda = 1$ eigenvector)

$$P\vec{q} = 1 \cdot \vec{q}$$

Example: Eigenvalues and Markov Chains

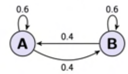
$(.6)^2 - (.4)^2 = .36 - .16 = .2$

Note: the textbook has a similar example that you can review.

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}\vec{x}_k, \quad k = 0, 1, 2, 3, \dots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:



Goal: use eigenvalues to describe the long-term behavior of our system.

$p(\lambda) = \lambda^2 - \mathrm{trace}(A)\lambda + \det A$

What are the eigenvalues of P?

$$p(\lambda) = \lambda^2 - 1.2\lambda + .2$$
$$= (\lambda - 1)(\lambda - .2) = 0$$
$$\lambda_1 = 1, \ \lambda_2 = .2$$

What are the corresponding eigenvectors of P?

$\lambda = 1$: $P - I = \begin{bmatrix} -.4 & .4 \\ .4 & -.4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\vec{x} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = .2$: $P - .2I = \begin{bmatrix} .4 & .4 \\ .4 & .4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\vec{x} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Use the eigenvalues and eigenvectors of P to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

write $\vec{x}_0$ as a linear comb of eigenvectors.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}_0 = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{x}_1 = P\vec{x}_0 = P\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}P\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}P\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

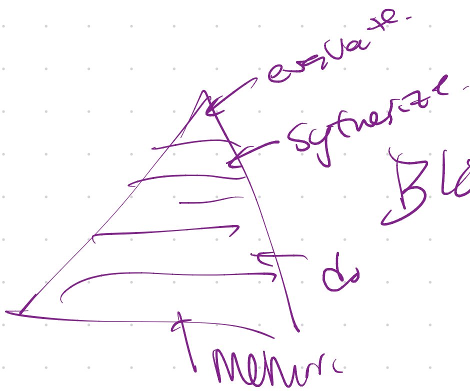
$$= \frac{1}{2}\underset{\lambda_1 = 1}{(1)}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\underset{\lambda_2 = .2}{(.2)}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{x}_1$$

$$\vec{x}_2 = P^2\vec{x}_0 = P\vec{x}_1 = \frac{1}{2}P\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}(.2)P\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}(.2)(.2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}(.2)^2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{x}_k = P^k\vec{x}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}(.2)^k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

goes to zero as $k \to \infty$.

$$\boxed{\begin{gathered} P^k\vec{x}_0 = \lambda_1^k c_1\vec{v}_1 + \lambda_2^k c_2\vec{v}_2 \\ P\vec{v}_1 = \lambda_1\vec{v}_1 \\ P\vec{v}_2 = \lambda_2\vec{v}_2 \end{gathered}}$$



Bloom's taxonomy

$\det A = \det(PBP^{-1}) = \cancel{\det P} \det B\, \cancel{\det P^{-1}}$
$= \det B.$

Similar Matrices

Definition

Two $n \times n$ matrices A and B are **similar** if there is a matrix P so that $A = PBP^{-1}$.

Theorem

$PIP^{-1} = PP^{-1} = I.$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

If A and B similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section 5.2, but doesn't have exercises on this concept until 5.3.
- Two matrices, A and B, do not need to be similar to have the same eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A & B similar

$\Rightarrow$ The same eigenvalues
w/ same alg mult
w/ same geo mult.

Additional Examples (if time permits)

1. True or false.
   a) If A is similar to the identity matrix, then A is equal to the identity matrix.
   b) A row replacement operation on a matrix does not change its eigenvalues.
2. For what values of k does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$

~~geo~~

5.2 Exercises

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix using expansion across a row or down a column. [*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 6 & 7 & -4 \end{bmatrix}$

For the matrices in Exercises 15–17, list the eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$

In Exercises 21–30, A and B are $n \times n$ matrices. Mark each statement True or False (T/F). Justify each answer.

21. (T/F) If 0 is an eigenvalue of A, then A is invertible.

22. (T/F) The zero vector is in the eigenspace of A associated with an eigenvalue λ.

23. (T/F) The matrix A and its transpose, A^T, have different sets of eigenvalues.

24. (T/F) The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

25. (T/F) If 2 is an eigenvalue of A, then $A - 2I$ is not invertible.

26. (T/F) If two matrices have the same set of eigenvalues, then they are similar.

27. (T/F) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

28. (T/F) The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

29. (T/F) The eigenvalue of the $n \times n$ identity matrix is 1 with algebraic multiplicity n.

30. (T/F) The matrix A can have more than n eigenvalues.