Handwritten Homework Assignments - Exploration for MATH 1554

For each assignment, complete the questions by hand on a separate sheet of paper. Write neatly and use complete sentences where necessary. **You must submit original work**, but I'm okay with you all working together to share ideas. Handwritten homework assignments are due on Fridays in Gradescope, and no late submissions are accepted.

Week 1: *Practice sketching in 3D.* In $\mathbb{R}^3$ (so using three coordinate axes) sketch the following, making a new sketch for each part (i)-(v):

(a) a horizontal plane,

(b) a plane which is not horizontal,

(c) a line passing through the origin,

(d) the plane defined by $x_1 - x_2 = 0$.

In each case, you are practicing drawing an accurate, representative graph of the plane of points which satisfy the given equation in the variables x_1, x_2, and x_3.

Week 2: *Practice with span, linear combination, and inconsistent systems.* ***(a)*** Choose two vectors v, w in $\mathbb{R}^2$ and a third vector b also in $\mathbb{R}^2$, and express b as a linear combination of v, w by finding scalars c_1, c_2 such that $c_1v+c_2w = b$. Sketch the situation in $\mathbb{R}^2$ with an illustration that uses the parallelogram rule. ***(b)*** Repeat part *(a)* with vectors in $\mathbb{R}^3$ such that the augmented matrix $[v\ w \mid b]$ gives a consistent system, again illustrating by graphing but this time in $\mathbb{R}^3$. ***(c)*** Why is it harder to find a consistent system for part *(b)* compared to part *(a)*? Explain your idea clearly using complete sentences.

Warning!

PLEASE NOTE: Your submissions for this and future exploration assignments need to contain vectors which are *general looking*. Choosing vectors which are scalar multiples of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, or $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, or have too many zeros or ones, or are otherwise too simple and miss the point of the exploration will receive a deduction of points.

Please, do NOT ask on Piazza if your vectors are general enough to get full credit. The explorations are assignments which require you to make a *judgement call*, to **explore** a particular concept of the course and NOT to come up with the simplest example which satisfies the minimum requirements of the assignment; if in our judgement your submission doesn't meet the bar then we will let you know, by deducting points. If you lose points on an exploration (for any reason), please make up the lost points with other MQE's. As a reminder, exploration assignments are *not eligible* for regrade requests.

Week 3: *Learn some basics of MATLAB.* See Sal's personal website for links to the MATLAB #1 Exploration. (requires MATLAB installation)

Week 4: *Practice with transformations.* For each matrix A below, ***(a)*** state the domain and codomain of T_A, ***(b)*** find $T_A(e_1), T_A(e_2)$, ***(c)*** find $T_A(v), T_A(w)$, ***(d)*** describe in a few words what the transformation is doing, and ***(e)*** state whether the associated transformation is one-to-one/onto, and ***(f)*** give the matrix an appropriate "name" (fine to be silly name like, e.g., "the x-zero-er" for projection to y-axis"). For the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ \text{-3} \end{bmatrix}$$

(i) $A = \begin{bmatrix} \text{-1} & 0 \\ 0 & \text{-1} \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & \text{-1} \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

(v) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(vi) $A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \text{-}\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

(vii) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(viii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Next, for the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \\ \text{-2} \end{bmatrix}$$

(i) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sal says: For the "name your matrix" this is a bit silly, and that's ok. Just come up with a creative name that makes sense to you, and don't worry about it too much!

Week 5: *Practice matrix algebra "fake truths".* For full credit, correctly indicate which problem you are solving by writing the statement you are answering (like "$AB = 0$ and $A \neq 0$,$B \neq 0$"). For grading purposes, please try to write the problems in the same order as listed here. The matrix 0 is the zero matrix and the matrix I is the identity matrix.

For each problem find square matrices which satisfy the given conditions. You don't have to justify *how* you found the matrices for each problem, but you *must verify the equality with calculations* in each case. Just show the matrices A, B, C and the given products.

The following restrictions are required for each problem:
No matrix A, B, or C can be diagonal, none can be equal or a scalar multiple of each other, and no product can be the zero matrix (except **(iv)**) or scalar multiple of the identity matrix (except **(v)**). All of the below are possible with these restrictions.

(i) $AB = BA$ but neither A nor B is 0 nor I, and $A \neq B$.

(ii) $AB \neq BA$.

(iii) $AB = AC$ but $B \neq C$, and the matrix A has *no zeros entries.*

(iv) $AB = 0$ but neither A nor B is 0.

(v) $AB = I$ but neither A nor B is I.

Week 6: **Step 1: Pick a matrix and find nul(A).** Pick a matrix A of size no smaller than 3×5 (to get a good feel for the problem). Choose entries not all positive, and not too many zeros, and your matrix shouldn't be rref (ideally, but it's ok to pick a matrix in rref if you want). Find the null space nul(A) by finding the parametric vector form of the general solution x to $A\mathrm{x} = 0$, and use these vectors to express nul(A) as their span. Call the vectors $v_1, \ldots, v_n$.

Step 2: An example that nul(A) is closed under vector addition. Choose two vectors $\mathrm{w}_1, \mathrm{w}_2$ in the span nul$(A) = \text{span}\{v_1, \ldots, v_n\}$ from step 1. Do this by taking two or more vectors in the basis from step 1 and adding them to each other using some scalars, *i.e.* chose a random linear combination of the vectors $v_1, \ldots, v_n$ from step 1. Do this twice with different weights each time, once to get w_1 and once to get w_2. Add these vectors together to get $\mathrm{z} = \mathrm{w}_1 + \mathrm{w}_2$. Check that z is in the null space of A by multiplying A times z and verifying that $A\mathrm{z} = 0$. *Question: what are the weights of z and how are these weights related to the weights of w_1, w_2?*

Step 3: An example that nul(A) is closed under scalar multiplication. Chose a vector w in the null space of A. Choose a random scalar c. Check that cw is in the null space of A by verifying $Aw = 0$.

Step 4: The general case. Try to convince yourself that no matter how A is chosen, nul(A) is always closed under scalar multiplication and vector addition. *Hint: one way is to see that $A(x + y) = Ax + Ay$ and $A(cx) = c(Ax)$, and another option is to think about span and how if x, y are in span$\{v_1, \ldots, v_n\}$ the weights of $x + y$ are related to the weights of x, y and the weights of cx are related to the weights of x.*

Week 7: For the following exploration, your matrix A needs to be relatively random looking. So, in particular it should have:

- not too many 1's or 0's,
- not be in REF or RREF,
- not be a scalar multiple of the identity matrix, or have too many columns which are scalar multiples of e_i the standard basis vectors, and additionally
- should not have a very ugly answer to *(Part i)*.

Part (i) is not related to Parts (ii) and (iii), but Parts (ii) and (iii) are related to each other.

Part i) Write down a system of three equations in three variables, whose augmented matrix $[A|b]$ would then be 3×4. Find the determinant of the matrix A whose entries are the coefficients of the system. Ensure the determinant is non-zero and find the inverse of the matrix and also calculate $A^{-1}b$ *(you may use matlab or an online calculator for this part).* Describe how $A^{-1}b$ relates to the system $[A|b]$ in a few words.

Part ii) Using either the matrix A you wrote down in *part (i)* or a new matrix A which also satisfies the bullet points of being relatively random looking, row reduce the matrix A to rref. What is the rref of A? Following the row operations you made to reduce A to rref, state the determinant of each elementary matrix. That is, if I is the rref of A and $A \sim E_1A \sim E_2E_1A \sim \cdots \sim E_n \cdots E_2E_1A = I$ are the matrices you got when you row reduced A to I, then calculate or otherwise find the determinant det A, det E_1, det E_2, $\ldots$, det E_n, and det I.

Part iii) Compute the determinant of each of the intermediate matrices E_1A, E_2E_1A, $\ldots$, $E_n \cdots E_2E_1A$. Compare your result to what you did in *part (ii)*. Try to state the relationship between A and the intermediate row equivalent matrices, in terms of the determinants of the E_i's. Formulate a relationship between det(A) and the product of the det(E_i)'s.

Week 8: Two separate parts.

1. Find a 2×2 matrix A *with real entries* with **no real eigenvalues** and show that it is correct by finding the characteristic polynomial and explaining why it has no real roots.
2. Find all eigenvalues and a corresponding eigenvector for each matrix below **without calculations** by thinking it out using the linear transformation's **geometric interpretation**. Write a few words (like 5) in each case explaining why your eigenvector/eigenvalue pair works. For each matrix, graph each eigenvector and it's image after the transformation as well as a random NON-eigenvector. Check your graph is accurate by showing the matrix multiplication.

   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A = id.$

   (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A =$projection onto x-axis.

   (c) $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, $T_A =$rotation by θ.
   (and state the values of θ for which A has eigenvectors)

   (d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T_A =$reflect about the line "$y = x$".

   (e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A =$stretch in x-direction.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, T_A =shear.

(g) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, T_A =stretch z-axis.

(h) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, T_A =rotate by 90° about the x-axis.

Week 9: *Note: Use MATLAB to generate a bunch of examples and check them using the* eig(A) *command to determine if eigenvalues have changed, and then write out* ***by handwriting*** *the examples that you find which satisfy the requirements below.*

Show that there is no relationship between any kind of row operation and the eigenvalues of the matrices involved as follows. For each of the **three types** of row operation $cR_i + R_j \to R_j$, $cR_j \to R_j$, and $R_i \leftrightarrow R_j$ which are *adding two rows*, *multiplying a row by a scalar*, and *switching two rows*: Find four matrices (for a **total of 12 matrices**) A, B, C, D such that $A \neq B$ and $C \neq D$ and also $A \sim B$ and $C \sim D$, the matrix A is row equivalent to B via exactly **one** row operation of the appropriate type, and also C is row equivalent to D via exactly **one** row operation of the appropriate type, and such that A and B have the exact same eigenvalues but C and D have different eigenvalues, and **be sure to state all eigenvalues**. Conclude that there is no relationship whatsoever between two matrices "being row equivalent" and "have the same eigenvalues".

Week 10: **Part 1:** Let λ be a real eigenvalue of a matrix with real entries A. Show that the set $V_\lambda = \{x \ : \ Ax = \lambda x\}$ is a subspace of $\mathbb{R}^n$. If you reduce your solution to a question about null spaces (or spans), be sure to include prove that null spaces (or spans) are subspaces; it's fine if you want to do it that way so long as your argument is clear.
Hint: check the definition of subspace. **Note: Do NOT use Chegg on this problem.**

Part 2: Explain why the set of eigenvectors of A corresponding to a particular eigenvalue λ is NOT a subspace? Why does this not contradict what you were asked to do in Part 2?
Hint: check the definition of eigenvector.

Week 11: UPDATED! This Exploration 11 this week is worth **double MQE points**.

NOTE: **Please do not use any calculators, notes, textbook, or outside help from instructors, TAs, Chegg, or other students for these problems.** If you have no idea how to solve them, please do the best you can to clearly indicate any of the terms or definitions that you do understand what they mean in order to attain some partial credit. We have been asked to assess the 1554 students ability to solve these two problems this semester as part of an initiative by the School of Mathematics. All submissions will be graded *by completion rather than by accuracy* so just make a serious attempt at both questions for full credit, please.

Problem 1: Find all eigenvalues, and an associated eigenvector for each eigenvalue, of the matrix A below.

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

Problem 2: Let H be the subspace shown below. Find an orthonormal basis of the subspace.

$$H = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: Please leave your answer with radicals. The numbers are actually not terrible except for the radical.

Week 12: NOTE: This week's assignment doesn't have to be handwritten. (For 1pt) Write a paragraph which explains how eigenvectors/eigenvalues or some other topic from the course are used in a field which interests you. Be specific and put some thought into this. (For 2pt) If your paragraph is not essentially the thing on wikipedia about how bridges have something to do with eigenvalues, but actually give some details or original content then you get 2pts instead of 1pt. (For 3pt) If you also support your research with real math or alternately something creative. This has to include *some mathematical content* but can take *any form whatsoever.* Last year's submissions included a poem, several posters, a few slide-show presentations like using PowerPoint, etc., and quite a few of them actually were pretty decent research project results that I was quite impressed by, but I remember the poem the best; it was funny and it used the right math ideas about linear algebra to *be* funny, which essentially forces that the person *understood* the concepts. It was brilliant.

Please understand the point of this exercise, should you choose to do it: Pick any scientific discipline. I mean ANY. If it is scientific it's ok: how to build a bridge, how do design a new chemical, how to solve some hard algorithmic problem using computers (like how many stars the Netflix algorithm should predict for your enjoyment of the 1977 original version of Disney's Pete's Dragon, for example). Take 5 steps into your chosen scientific field and you will bump into linear algebra. That's the exercise. 1pt is essentially "write down in your own words what wiki has to say about it", 2pts is essentially "do something a little better but without any real math content", and then 3pts is "a pretty good job explaining how a specific concept from linear algebra is used in a scientific field you are interested in", where I will collect and grade these myself so it is up to my subjective expert opinion if what you say is a good job with the math explaining.

Week 13: *This exercise will ask you to explore the equality* $Null(A^T) = (Col(A))^{\perp}$. Pick two linearly independent vectors u and v, and try to find a vector w which is orthogonal to both u and v by hand. Next, write down a system of linear equations using the entries of u, v as coefficients and the entries of w as variables; solve this system of equations to find w such that w is perpendicular to both u and v. Check that $u \cdot w = 0$ and $v \cdot w = 0$ by hand. Then, answer the following questions:

1. How many equations did you need to solve in order to find w? and how many variables?
2. Why is any vector in Col($[u\ v]$) perpendicular to w?
3. Why is a vector in Nul($[u\ v]^T$) orthogonal to both u and v?

Week 14: MATLAB #3 (see other document): *Basis of eigenvectors* and *the SVD exploration.*