Math 1554 Linear Algebra Spring 2023

Midterm 1

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

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Section Number (e.g. A3, G2, etc.) ________ TA Name __________________________

Circle your instructor:

Prof Kim Prof Barone Prof David/Schroeder Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

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You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
●	○	If a vector $\vec{b}$ can be written **uniquely** as a linear combination of vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ then there is a pivot in the first three columns of the matrix $\begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{pmatrix}$.
●	○	If $A\vec{x} = \vec{b}$ is consistent, then $\vec{b}$ is in the span of the columns of A.
●	○	If $\vec{v}$ and $\vec{w}$ are solutions to an inhomogeneous system $A\vec{x} = \vec{b}$, then $\vec{v} - \vec{w}$ is a solution to $A\vec{x} = \vec{0}$.
○	●	If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.
○	●	If A is size 3×4 and none of the rows of A consist entirely of zeros, then A has 3 pivots.
○	●	If A and B are square $n \times n$ matrices, then $A^2 - B^2 = (A - B)(A + B)$.
○	●	If A is size $m \times n$ with $m \neq n$ and the columns of A are linearly independent, then the transformation $T(\vec{x}) = A\vec{x}$ is onto.
●	○	If the coefficient matrix A for a system of linear equations has a pivot in every row, then the system $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^m$.

You do not need to justify your reasoning for questions on this page.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	●	An $m \times n$ matrix A with a pivot in its last column such that $A\vec{x} = \vec{0}$ is inconsistent.
○	●	Two nonzero vectors $\vec{v}_1, \vec{v}_2$ such that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\{\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2\}$ is linearly dependent.
●	○	A matrix A of size 4×3 with linearly dependent columns.
○	●	A transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ that is onto.

(c) (2 points) If A is an $m \times 5$ matrix and $A\vec{x} = 0$ has a unique solution, then which of the following is true. *Select only one.*

● $m \geq 5$
○ $m = 5$
○ $m \leq 5$
○ m can be any natural number

↑ no free vars

$$\begin{array}{cc} & \overbrace{\qquad\qquad\qquad}^{5} \\ m \left\{ \right. & \begin{bmatrix} * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{bmatrix} \end{array}$$

$m \geq 5$

If $m < 5$ then not every col has a pivot

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You do not need to justify your reasoning for questions on this page.

(d) (3 points) For the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Which of the following sets are linearly independent? *Select all that apply.*

- ● $\{\vec{v}_1, \vec{v}_2\}$
- ● $\{\vec{v}_2, \vec{v}_3\}$
- ○ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

2 pivots

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\{v_1, v_2, v_3\}$ lin dependent

(e) (2 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Which of following accurately describes the transformation that $T(\vec{x}) = A\vec{x}$? *Select only one.*

- ○ Rotation by $\frac{\pi}{2}$ radians around the x axis.
- ○ Rotation by $\frac{\pi}{2}$ radians around the z axis.
- ○ Reflection across the $x = 0$ plane.
- ● Reflection across the $y = z$ plane. ← must be option 4

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A is NOT a rotation by $\pi/2$ (since $A^2 = I$). option 1&2 not correct

reflection across $x=0$ sends $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, so option 3 also not correct.

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You do not need to justify your reasoning for questions on this page.

2. (2 points) If possible, fill in the missing element of the vector $\vec{w}$ with a number so that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. If it is not possible write NP in the space.

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$-3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & h \\ 3 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & h-10 \\ 0 & -6 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & h-10 \\ 0 & 0 & -12-2(h-10) \end{bmatrix}$$

If $h=4$ $\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$$-12-2(h-10)=0 \Rightarrow 8-2h=0 \Rightarrow \underline{\underline{h=4}}$$

$$\sim \begin{array}{c} \qquad\quad \checkmark t \\ \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$x = t\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

3. (4 points) Find b and c such that $AB = BA$.

$$A = \begin{pmatrix} 2 & b \\ -3 & c \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -5 \\ 3 & 5 \end{pmatrix}$$

$b = \boxed{5}$ $\qquad c = \boxed{1}$

$$AB = \begin{bmatrix} 2 & b \\ -3 & c \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \boxed{8+3b} & -10+5b \\ \boxed{-12+3c} & 15+5c \end{bmatrix}$$

same

same

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & c \end{bmatrix} = \begin{bmatrix} \boxed{23} & 4b-5c \\ \boxed{-9} & 3b+5c \end{bmatrix}$$

$$8+3b=23 \Rightarrow 3b=15 \Rightarrow b=5$$

$$-12+3c=-9 \Rightarrow 3c=3 \Rightarrow c=1$$

Midterm 1. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

4. (8 points) Let T be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_2 \\ 2x_1 - 4x_2 \end{bmatrix}.$$

(i) What is domain and codomain of T?

domain is $\boxed{\mathbb{R}^2}$

codomain is $\boxed{\mathbb{R}^3}$

(ii) What is the standard matrix of T?

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -4 \end{bmatrix}}$$

(iii) Find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{b}$, where $\vec{b} = \begin{bmatrix} -2 \\ 9 \\ -4 \end{bmatrix}$.

$\vec{x} = \boxed{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}$

$$\left[\begin{array}{cc|c} 1 & -2 & -2 \\ 0 & 3 & 9 \\ 2 & -4 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

check

$$T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 - 2\cdot 3 \\ 3\cdot 3 \\ 2\cdot 4 - 4\cdot 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ -4 \end{bmatrix} ✓$$

(iv) Is T one-to-one? ● yes ○ no

(v) Is T onto? ○ yes ● no

5. (5 points) **Show all work for problems on this page.**

For what value(s) of h is the following set of vectors linearly independent?

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ h \\ h^2 \end{pmatrix} \right\}$$

$$h = \boxed{2, 1}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & h \\ 0 & 1 & h^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & h+2 \\ 0 & 1 & h^2 \end{bmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & h+2 \\ 0 & 0 & h^2-h-2 \end{pmatrix}$$

$$h^2 - h - 2 = 0 \Rightarrow (h-2)(h+1) = 0$$

$$h = 2 \text{ or } h = -1.$$

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6. (5 points) **Show your work in the space below and put your answer in the box.** Provide a dependence relation on $\vec{v}_1, \vec{v}_2, \vec{v}_3$, *i.e.*, a nontrivial linear combination demonstrating that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent. For full credit, show how the dependence relation is obtained by row reducing the appropriate coefficient matrix.

   You may leave your answer in terms of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\boxed{\vec{v}_1 + 4\vec{v}_2 + \vec{v}_3 = \vec{0}}$$

Need to find scalars c_1, c_2, c_3 such that (not all c_1, c_2, c_3 are ~~zero~~)

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 5 \\ 4 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 5 \\ 0 & -6 & 24 \\ 0 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{ccc} & & \downarrow t \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad X = t\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

So

$$\vec{v}_1 + 4\vec{v}_2 + \vec{v}_3 = \vec{0} \quad \text{Check} \quad \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

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7. **Show your work in the space below the first box and put your answers in the boxes.**

(a) (5 points) Write the parametric vector form for the general solution to the inhomogeneous equation $A\vec{x} = \vec{b}$.

$$A = \begin{pmatrix} 2 & 1 & 0 & -4 & 1 \\ 5 & 3 & 0 & -10 & 4 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$$

$$X = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccccc|c} 2 & 1 & 0 & -4 & 1 & 7 \\ 5 & 3 & 0 & -10 & 4 & 18 \end{array}\right] \underset{-2R_1+R_2 \to}{\sim} \left[\begin{array}{ccccc|c} 2 & 1 & 0 & -4 & 1 & 7 \\ 1 & 1 & 0 & -2 & 2 & 4 \end{array}\right]$$

$$\sim \updownarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 2 & 4 \\ 2 & 1 & 0 & -4 & 1 & 7 \end{array}\right] \underset{-2R_1+R_2}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 2 & 4 \\ 0 & -1 & 0 & 0 & -3 & -1 \end{array}\right]$$

$$\sim \begin{array}{c} R_2+R_1 \to \\ -R_2 \to \end{array} \begin{array}{cccccc} & & \downarrow r & \downarrow s & \downarrow t & \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} -2 \\ 0 \end{array} & \begin{array}{c} -1 \\ 3 \end{array} & \left|\begin{array}{c} 3 \\ 1 \end{array}\right] \end{array}$$

$x_1 = 2s + t + 3$
$x_2 = -3t + 1$
$x_3 = r$ (free)
$x_4 = s$ (free)
$x_5 = t$ (free)

$$X = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) (2 points) For the homogeneous system with the same coefficient matrix A as part (a) above, write down the general solution to $A\vec{x} = \vec{0}$. *Hint: use your answer from part (a).*

$$X = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **NOT** ***be detached*** *from your exam booklet at any time.*