Math 1554 Linear Algebra Spring 2023

Midterm 2

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: Key GTID Number: ____________

Student GT Email Address: ____________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ____________

Circle your instructor:

Prof Kim Prof Barone Prof Schroeder Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

4:02

4:26

Midterm 2. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	●	If $A, B \in \mathbb{R}^{n \times n}$ and $AB\vec{x} = \vec{0}$ has a non-trivial solution, then A is not invertible.
●	○	If A has LU-factorization $A = LU$, then $\det(L) = 1$.
●	○	If A and B share an eigenvector $\vec{x}$ corresponding to eigenvalue λ, so that λ is an eigenvalue of both A and B for the same eigenvector $\vec{x}$, then 2λ must be an eigenvalue of the matrix $A + B$.
●	○	If A is $m \times n$ and $A\vec{x} = b$ has a solution for every $\vec{b} \in \mathbb{R}^m$, then Col$(A) = \mathbb{R}^m$.
○	●	If $\det(A) = 1$ and $\det(B) = 0$, then $AB = BA$.
●	○	If A is $n \times n$ and 0 is an eigenvalue of A, then the transformation $T(\vec{x}) = A\vec{x}$ is not onto.
●	○	If $\vec{x}$ and $\vec{y}$ are probability vectors, then $\frac{1}{3}\vec{x} + \frac{2}{3}\vec{y}$ is a probability vector.
○	●	If A is 3×3, then $\det(2A) = 2\det(A)$.

You do not need to justify your reasoning for questions on this page.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
●	○	$A \in \mathbb{R}^{6\times 6}$, and $\text{rank}(A) = \dim \text{Nul}(A)$.
○	●	A 3×3 matrix whose nullspace is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right\}$ and whose column space is spanned by $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.
●	○	A 4×6 matrix A with a null space of dimension 5.
○	●	An $n \times n$ matrix A with $\det(AA^T) = -1$.

(c) (2 points) If A is the standard matrix for the transformation that projects vectors in $\mathbb{R}^3$ to the xy-plane, then what is the dimension of the null space of A? *Select only one.*

○ 0
● 1
○ 2
○ 3

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

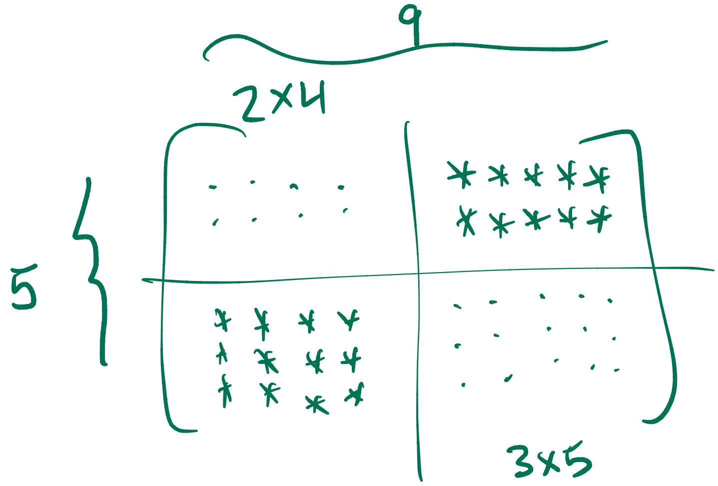
(d) (2 points) Suppose an 3×3 matrix A can be row reduced to reduced row echelon form (RREF) using only row replacement row operations (without any row swaps/scaling). Among the options listed below, which are possible values for $\det(A)$?
Select all that apply.

- ○ -1
- ● 0
- ● 1
- ○ 3

2. (3 points) Suppose B is a 2×5 matrix and C is 3×4 matrix. Find the dimensions of the matrices A, D, and M for the block matrix

$$M = \begin{pmatrix} A & \overset{2\times5}{B} \\ \underset{3\times4}{C} & D \end{pmatrix}.$$

A has	2	rows and	4	columns.
D has	3	rows and	5	columns.
M has	5	rows and	9	columns.



Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

3. (2 points) Find the dimension of the subspace S consisting of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ which satisfy the conditions that

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ x_1 + 3x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 + 4x_3 + 4x_3 + 3x_4 &= 0 \end{aligned}$$

$$\dim(S) = \boxed{1}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & -1 & 2 \\ 2 & 4 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 4 & -2 & 3 \\ 0 & 2 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & -6 & -6 \end{bmatrix}$$

$\dim \operatorname{Col} A = 3$

$\dim \operatorname{Nul} A = 1$

$S = \{x \mid Ax = 0\} = \operatorname{Nul} A$

4. (4 points) Suppose $\det \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{M} = 4$. Find the determinant of the matrices below.

$$A = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \qquad B = \begin{pmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{pmatrix} \qquad C = \begin{pmatrix} a & a - c & c \\ d & d - f & f \\ g & g - i & i \end{pmatrix}$$

$$\det(A) = \boxed{4} \qquad \det(B) = \boxed{8} \qquad \det(C) = \boxed{0}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \sim \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} \sim \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

2 row swaps

$\det A = \det M * -1 * -1$
$= 4 * (-1)^2$
$= 4$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \sim_{2R_2} \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} \sim_{R_2 + R_1} \begin{bmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{bmatrix}$$

row scaling & row replacement

$\det A = \det M * 2 * 1$
$= 8$

$$\begin{bmatrix} a & a-c & c \\ d & d-f & f \\ g & g-i & i \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ So $\det C = 0$

Midterm 2. Your initials: _______________

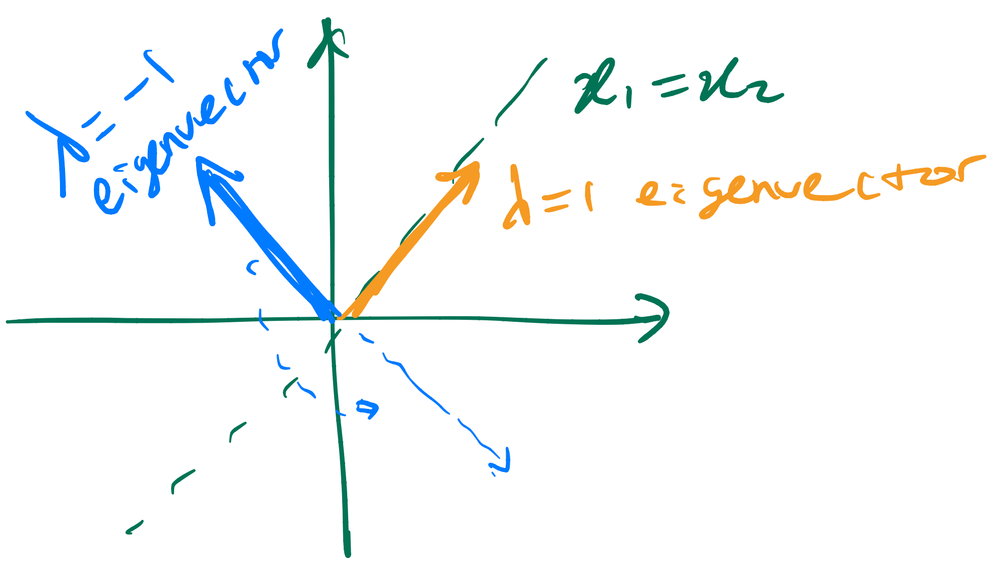
You do not need to justify your reasoning for questions on this page.

5. (3 points) Give a matrix A in RREF whose column space is spanned by $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ and whose null space is spanned by $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}\right\}$. If this is not possible, write `NP` in the box.

$$A = \boxed{\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}}$$

6. (2 points) Consider the transformation $T(\vec{x}) = A\vec{x}$ which reflects vectors in $\mathbb{R}^2$ across the line $x_1 = x_2$. List in the box the real eigenvalues of the matrix A, or write `NP` in the box if there are no real eigenvalues.

$$\boxed{1, -1}$$



$$A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

7. (4 points) **Show all work for problems on this page.**

   Find an eigenvector $\vec{v}$ for the eigenvalue $\lambda = 3$ of A. *Hint: check your answer.*

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 4 & -1 \\ 1 & 0 & 4 \end{bmatrix}$$

$\underline{\lambda = 3}$

$$A - 3I = \begin{bmatrix} 4-3 & 0 & 1 \\ -2 & 4-3 & -1 \\ 1 & 0 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad t \qquad x = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\vec{v} = \boxed{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$

$\underline{\text{Check}}$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 4 & -1 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+1 \\ 2-4-1 \\ -1+4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \checkmark$$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 5 & 6 \\ -1 & 1 & 2 \\ 2 & 7 & 8 \end{pmatrix}.$$

$$A = \begin{bmatrix} 1 & 5 & 6 \\ -1 & 1 & 2 \\ 2 & 7 & 8 \end{bmatrix} \sim \begin{matrix} \\ 1R_1 + R_2 \\ -2R_1 + R_3 \end{matrix} \begin{bmatrix} 1 & 5 & 6 \\ 0 & 6 & 8 \\ 0 & -3 & -4 \end{bmatrix} \sim \begin{matrix} \\ \\ \frac{1}{2}R_2 + R_3 \end{matrix} \begin{bmatrix} 1 & 5 & 6 \\ 0 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}$$

$L = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}}$

$U = \boxed{\begin{bmatrix} 1 & 5 & 6 \\ 0 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}}$

9. (4 points) **Show all work for problems on this page.**

Find all possible values of k such that the matrix A is singular.
Hint: use cofactor expansion to compute the determinant.

$$A = \begin{pmatrix} 1 & -3 & k \\ 7 & 2 & -3 \\ -1 & 2 & 5 \end{pmatrix}$$

$$k = \boxed{-7}$$

$$\det A = 1 \begin{vmatrix} 2 & -3 \\ 2 & 5 \end{vmatrix} - (-3) \begin{vmatrix} 7 & -3 \\ -1 & 5 \end{vmatrix} + k \begin{vmatrix} 7 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= (10+6) + 3(35-3) + k(14+2)$$

$$= 16 + 3 * 32 + 16k = 0$$

$$\Rightarrow 1 + 3 \cdot 2 + k = 0 \qquad \text{(divide by 16)}$$

$$\Rightarrow 7 + k = 0$$

$$\Rightarrow k = -7$$

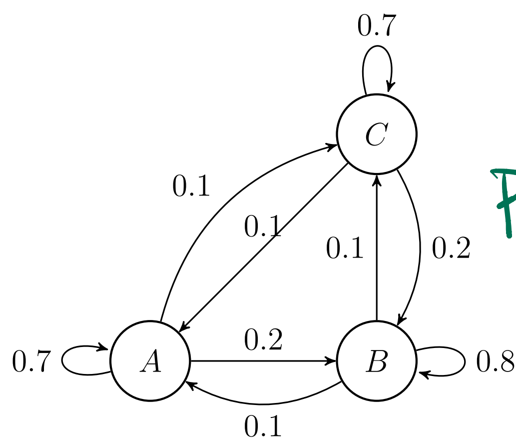
10. (6 points) **Show your work for part (c) on this page.**

Use the following Markov chain diagram to answer the questions.

(a) Find the stochastic matrix P of the Markov chain.

(b) What is $\det(P - I)$?

(c) Find the unique steady state probability vector $\vec{q}$ of P.



(a)

$$P = \begin{array}{cc} & \text{from} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{to} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix} \end{array}$$

(b)

$$P - I = \begin{bmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix}$$

$P = \boxed{\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}}$

$$\sim \begin{bmatrix} 1 & 1 & -3 \\ -3 & 1 & 1 \\ 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 4 & -8 \\ 0 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det(P - I) = \boxed{0}$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad X = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\vec{q} = \boxed{\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}}$

(c)

$$q = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$P - I$ singular $\Rightarrow \det(P - I) = 0$

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **NOT** ***be detached*** *from your exam booklet at any time.*