LINEAR ALGEBRA

Week 13

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices
2. Orthogonal diagonalization
3. Spectral decomposition

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.
2. Construct a spectral decomposition of a matrix.

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course m

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 - 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 - 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/5 - 2/9	2.3,2.4	WS2.2-2.4	2.5	WS2.5	2.8
6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
7	2/19 - 2/23	3.3	WS3.2	4.9	WS3.3,4.9	5.1
8	2/26 - 3/1	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/4 - 3/8	5.3	WS5.3	5.5	WS5.5	6.1
10	3/11 - 3/15	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/18 - 3/22	Break	Break	Break	Break	Break
12	3/25 - 3/29	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	4/1 - 4/5	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/8 - 4/12	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/15 - 4/19	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/22 - 4/24	Last lecture	Last Studio	Reading Period		
17	4/25 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 30th at 6:00pm**				

Symmetric Matrices

Definition

Matrix A is **symmetric** if $A^T = A$.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

✓ $A = [*]$ $\qquad B = \begin{bmatrix} * & \star \\ \star & * \end{bmatrix}$ $\qquad C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

✗ $D = \begin{bmatrix} \star & \star \\ 0 & 0 \end{bmatrix}$ $\qquad$ ✗ $E = \begin{bmatrix} 4 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix}$ ✓

$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ ✓

$B = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ ✓

$B = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$ ✗

$B^T = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \neq B$

$E^T = E$

same size is requirement of symmetric.

$E^T = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 4 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} = E$

A^TA is Symmetric

A very common example: For **any** matrix A with columns $a_1, \ldots, a_n$,

$$A^TA = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \cdots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \cdots & a_2^Ta_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \cdots & a_n^Ta_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$$

$v_1 \quad v_2 \quad v_3$

$$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 14 & 18 \\ 14 & 20 & 26 \\ 18 & 26 & 34 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$A^TA = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{bmatrix}$$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. $\boxed{\text{Then } \vec{v}_1 \text{ and } \vec{v}_2 \text{ are orthogonal.}}$

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

Suppose $Av_1 = \lambda_1 v_2$, $Av_2 = \lambda_2 v_2$ where $\lambda_1 \neq \lambda_2$ ②

and $A^T = A$. (A symmetric) ①

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (a\ b\ c) * \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Notice

$\star = Av_1 \cdot v_2 = \lambda_1 v_1 \cdot v_2 = \boxed{\lambda_1 (v_1 \cdot v_2)}$

also

$\star = Av_1 \cdot v_2 = (Av_1)^T * v_2 = v_1^T A^T v_2$

$\overset{①}{=} v_1^T A v_2 = v_1 \cdot Av_2 = \boxed{\lambda_2 (v_1 \cdot v_2)}$

$Av_2 = \lambda_2 v_2$

So

$\lambda_1(v_1 \cdot v_2) - \lambda_2(v_1 \cdot v_2) = 0$

$\Rightarrow (\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$

$\overset{②}{\Rightarrow} \boxed{v_1 \cdot v_2 = 0}$

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$ only eigenvalues of A.

$$A = PDP^T$$

$$P = \begin{bmatrix} +1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$P^T = P^{-1}$

P orthogonal matrix w/ orthonormal cols.

$\lambda_1 = -1$

$$A - \lambda I = A + I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1$

$$A - I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$V_1 \cdot V_2 = 0$ ✓

$V_1 \cdot V_3 = -1 + 1 = 0$ ✓

$V_2 \cdot V_3 = 0$ ✓

$\{V_1, V_2, V_3\}$ orthogonal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

FACT. If $A = PDP^T$ & P orthogonal, D diagonal

Then $A = A^T$

Proof:

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A. \quad \blacksquare$$

Spectral Theorem

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real.
2. The dimension of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity. geo = alg.
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is orthogonal (columns are unit length & orthogonal)

Proof (if time permits):

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank 1.

Motivate Chapter 7.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

(Spectral Theorem)

THM: If A is symmetric ($A^T = A$)

then

(1) A diag'ble. (all λ's real, geo = alg)

(2) If v_1, v_2 eigenvectors of A w/ $\lambda_1 \neq \lambda_2$

then $v_1 \cdot v_2 = 0$

Spectrum of a matrix is just the set of eigenvalues.

(3) $A = PDP^T$ orthogonal diagonalization

where P orthogonal (orthonormal columns)

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\boxed{u_1 u_1^T \vec{y} = \text{proj}_{u_1}(\vec{y})}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$= 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\text{Span}\{(-1, 1)\}$

$\text{Span}\{(1, 1)\} = L$

$$= 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$T(\vec{x}) = \text{proj}_L(\vec{x})$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}}$$

7.1 Exercises

Determine which of the matrices in Exercises 1–6 are symmetric.

1. $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

7. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

10. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

11. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$

12. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix P and a diagonal matrix D. To save you time, the eigenvalues in Exercises 17–22 are the following: (17) −4, 4, 7; (18) −3, −6, 9; (19) −2, 7; (20) −3, 15; (21) 1, 5, 9; (22) 3, 5.

13. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Then orthogonally diagonalize A.

In Exercises 25–32, mark each statement True or False (T/F). Justify each answer.

25. (T/F) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

26. (T/F) There are symmetric matrices that are not orthogonally diagonalizable.

27. (T/F) An orthogonal matrix is orthogonally diagonalizable.

28. (T/F) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

29. (T/F) For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

30. (T/F) If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

31. (T/F) An $n \times n$ symmetric matrix has n distinct real eigenvalues.

32. (T/F) The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

33. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

34. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB, and BB^T are symmetric matrices.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

39. Construct a spectral decomposition of A from Example 2.

40. Construct a spectral decomposition of A from Example 3.

41. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.
    a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.
    b. Show that B is a symmetric matrix and $B^2 = B$.
    c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

42. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.
    a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.
    b. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

Orthogonally diagonalize the matrices in Exercises 43–46. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue λ, find an orthonormal basis for $\mathrm{Nul}(A - \lambda I)$, as in Examples 2 and 3.

T 43. $\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$

T 44. $\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$

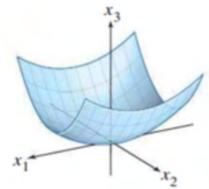
T 45. $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

T 46. $\begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$

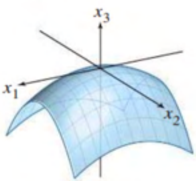
Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

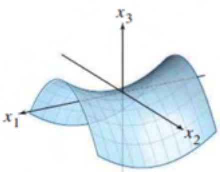
Math 1554 Linear Algebra



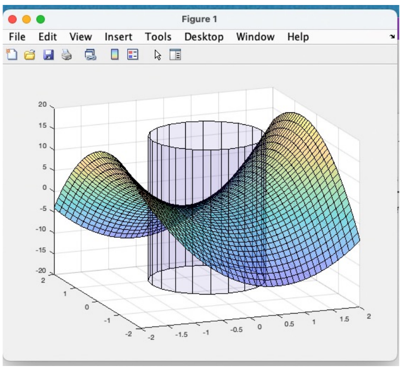
Positive definite

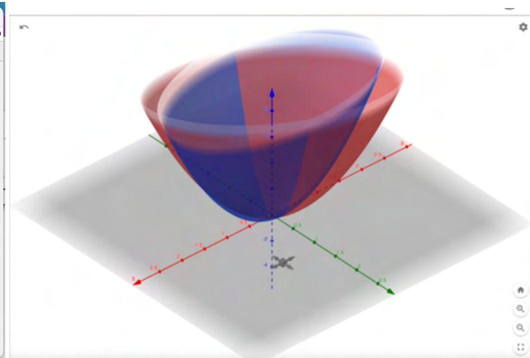


Negative definite



Indefinite





Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course

		Mon	Tue	Wed	Thu	Fri
Week	**Dates**	**Lecture**	**Studio**	**Lecture**	**Studio**	**Lecture**
1	1/8 - 1/12	1.1	WS1.1	1.2	WS1.2	1.3
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3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/5 - 2/9	2.3,2.4	WS2.2-2.4	2.5	WS2.5	2.8
6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
7	2/19 - 2/23	3.3	WS3.2	4.9	WS3.3,4.9	5.1
8	2/26 - 3/1	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/4 - 3/8	5.3	WS5.3	5.5	WS5.5	6.1
10	3/11 - 3/15	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/18 - 3/22	Break	Break	Break	Break	Break
12	3/25 - 3/29	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	4/1 - 4/5	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/8 - 4/12	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/15 - 4/19	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/22 - 4/24	Last lecture	Last Studio	Reading Period		
17	4/25 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 30th at 6:00pm**				

Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A\vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question Does this inequality hold for all x, y?

$$x^2 \mathbin{\cancel{-}}+ 6xy + 9y^2 \geq 0$$

$$Q(x,y) = \boxed{x^2 + \underset{2}{\color{red}6}xy + \underset{1}{} y^2} \overset{?}{\geq} 0$$

$$Q(1,-1) = (1)^2 + 6(1)(-1) + (-1)^2$$

$$= 1 - 6 + 1 = -4.$$

Quadratic Forms

Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.

$$x^T A x \qquad x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{pmatrix} x + 3y \\ 3x + y \end{pmatrix}$$

$$= x(x+3y) + y(3x+y)$$

$$= x^2 + 3yx + 3xy + y^2$$

$$= \boxed{x^2 + 6xy + y^2}$$

Exam 3 vs. expectations

https://strawpoll.com/GJn478JBbyz

next time.

Example 1

Compute the quadratic form $\vec{x}^T A\vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$Q_A(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{pmatrix} 4x_1 \\ 3x_2 \end{pmatrix}$$

$$= 4x_1^2 + 3x_2^2$$

$$\uparrow 0x_1x_2$$

$$Q_B(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

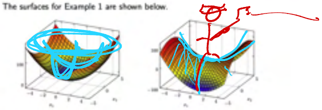
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$= 4x_1^2 + x_1x_2 + x_1x_2 - 3x_2^2$$

$$= 4x_1^2 + 2x_1x_2 - 3x_2^2$$

Example 1 - Surface Plots

The surfaces for Example 1 are shown below.

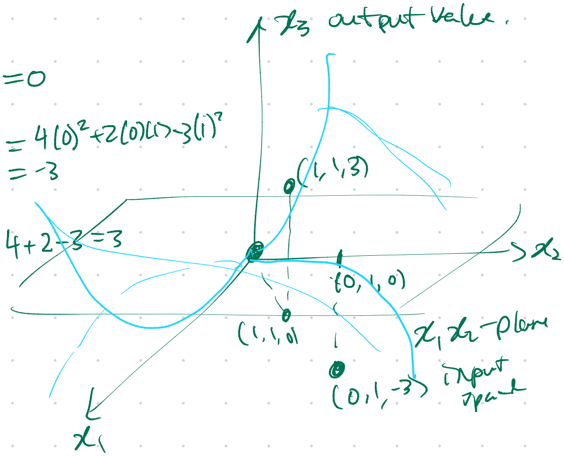


Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

$$Q_B(0,0) = 0$$

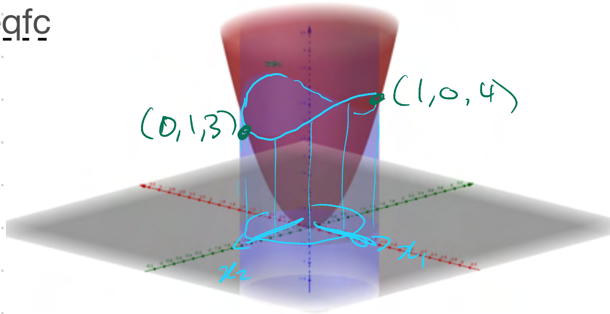
$$Q_B(0,1) = 4(0)^2 + 2(0)(1) - 3(1)^2$$
$$= -3$$

$$Q_B(1,1) = 4 + 2 - 3 = 3$$



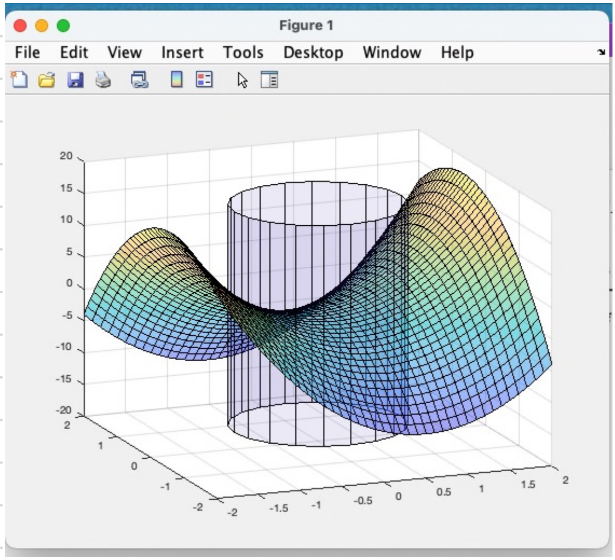
https://www.geogebra.org/m/pbzpeqfc

$Q_A(\vec{x}) = 4x_1^2 + 3x_2^2$




```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on

%% no errors check
1+1
```

$Q(\vec{x}) = 4x_1^2 + 2x_1x_2 - 3x_2^2$



$$\boxed{Q(\vec{x}) = x^T A x}$$

Example 2

Write Q in the form $\vec{x}^T A \vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q(\vec{x}) = 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3$$

+ not −

A size 3x3.

Change of Variable

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A \vec{x}$ becomes:

$$Q(x_1, x_2, x_3) = 5x_1^2 - 1x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3$$

$0x_1x_2$

Same.

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}$$

$$[x_1\ x_2\ x_3] \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1\ x_2\ x_3] \begin{bmatrix} 5x_1 + 3x_3 \\ -x_2 - 6x_3 \\ 3x_1 - 6x_2 + 3x_3 \end{bmatrix}$$

$$= 5x_1^2 + 3x_1x_3 - x_2^2 - 6x_2x_3 + 3x_1x_3 - 6x_2x_3 + 3x_3^2$$

$$= 5x_1^2 + 6x_1x_3 - x_2^2 - 12x_2x_3 + 3x_3^2$$

harder to understand.

$$= c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2$$

easier to understand

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A \vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$Py = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T x$$

$P^T x = y$

$\Rightarrow PP^T x = Py$

$\Rightarrow x = Py$

$$Q_A(\vec{x}) = x^T A x = 3x_1^2 + 4x_1x_2 + 6x_2^2$$

$$Q_D(\vec{y}) = y^T D y = 2y_1^2 + 7y_2^2$$

Same after change of variables.

replace A w/ $PDP^T = A$.

$x^T (P^T)^T = x^T P$.

$$Q_A(\vec{x}) = x^T A x = x^T (PDP^T) x$$

$$= (x^T P) D (P^T x) = (P^T x)^T D P^T x$$

does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$
$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$
$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\boxed{\begin{array}{l} y = P^T x \\ x = P y \end{array}}$$ formula.

$$Q_A(\vec{x}) = x^T A x = 3x_1^2 + 4x_1x_2 + 6x_2^2$$
$$Q_D(\vec{y}) = y^T D y = 2y_1^2 + 7y_2^2$$

Inputs for $\vec{x}$ vs. $\vec{y}$.

$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ plug into $Q_D(\vec{y})$.

$$Q_D(\underbrace{1,-1}_{y}) = 2(1)^2 + 7(-1)^2 = 2+7 = \boxed{9}$$

$$P\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \vec{x} = \begin{bmatrix} 3/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

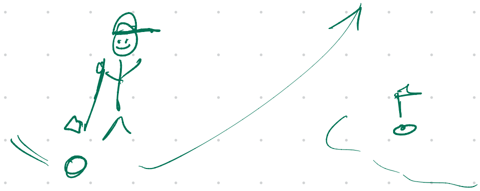
$$Q_A\left(\frac{1}{\sqrt{5}}, \frac{-3}{\sqrt{5}}\right) = 3\left(\frac{1}{\sqrt{5}}\right)^2 + 4\left(\frac{1}{\sqrt{5}}\right)\left(\frac{-3}{\sqrt{5}}\right) + 6\left(\frac{-3}{\sqrt{5}}\right)^2$$
$$= \frac{3}{5} - \frac{12}{5} + \frac{54}{5} = \frac{45}{5} = 9$$

$$Q_A(\vec{x}) = Q_D(\vec{y}) \text{ if } \boxed{\begin{array}{l} y = P^T x \\ x = P y \end{array}}$$

Principle Axes Theorem

Theorem

If A is a ____________________ matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{x}^T D\vec{x}$ with no cross-product terms.



Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

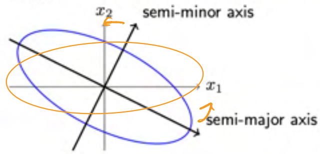
$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

Example 5

Compute the quadratic form $Q = \vec{x}^T A\vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.



If A symmetric then

$A = PDP^T$ 7.1

$Q_A(\vec{x}) = Q_D(\vec{y})$ 7.2 $\longleftrightarrow$ quadratic forms

cross terms

no cross terms

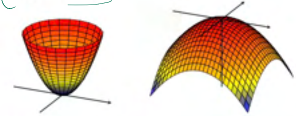
Change of vars.

$$\boxed{\begin{aligned} P^T\vec{x} &= y \\ Py &= x \end{aligned}}$$

$Q_A(\vec{x}) = Q_D(\vec{y})$ but w/ $\left.\begin{aligned} \|\vec{x}\| &= 1 \\ \|\vec{y}\| &= 1 \end{aligned}\right\}$ constrained optimization

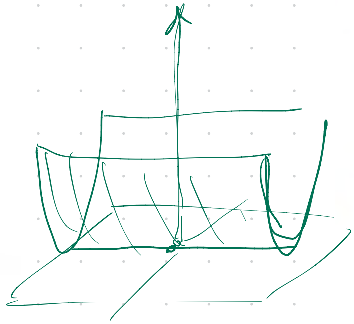
Classifying Quadratic Forms

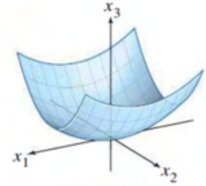
$Q = x_1^2 + x_2^2$ $\qquad$ $Q = -x_1^2 - x_2^2$



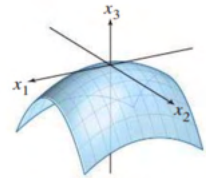
Definition

A quadratic form Q is
1. **positive definite** if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.
2. **negative definite** if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.
3. **positive semidefinite** if $Q(\vec{x}) \geq 0$ for all $\vec{x}$.
4. **negative semidefinite** if $Q(\vec{x}) \leq 0$ for all $\vec{x}$.
5. **indefinite** if sometimes positive & sometimes negative.

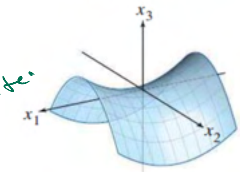




Positive definite



Negative definite



Indefinite

PSD
positive semi-definite.

Quadratic Forms and Eigenvalues

Theorem

If A is a symmetric matrix with eigenvalues λ_i, then $Q = \vec{x}^T A \vec{x}$ is
1. **positive definite** iff $\lambda_i > 0$
2. **negative definite** iff $\lambda_i < 0$
3. **indefinite** iff λ_i some positive & some negative.

PSD $Q_A(\vec{x})$ iff $\lambda_i \geq 0$

NSD $Q_A(\vec{x})$ iff $\lambda_i \leq 0$.

if and only if.

P if Q $\qquad$ $Q \Rightarrow P$

P only if Q $\qquad$ $P \Rightarrow Q$

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y?

$$x^2 - 6xy + 9y^2 \geq 0$$

T/F?

Q_1: What is A?

Q_2: Is $Q_A(\vec{x})$ PSD, NSD, PD, ND, Indefinite?

Q_3: What are eigenvalues of A?

PSD yes $\qquad$ PD no

NSD no $\qquad$ ND no

Q_1:
$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

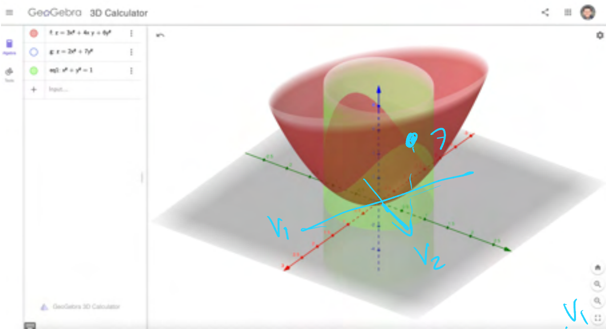
Q_3:
$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$
$$\lambda_1 = 10 \quad \lambda_2 = 0$$

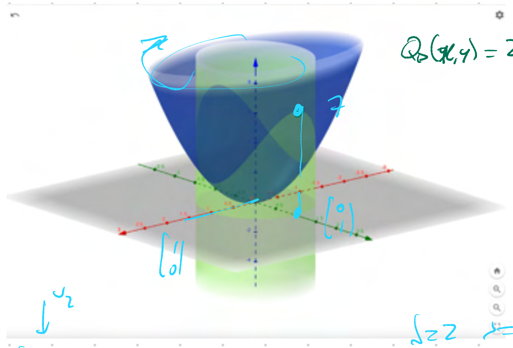
Example 6 (BONUS!)

$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$ $\quad \lambda_1 = 2 \quad \lambda_2 = 7$

PSD yes $\quad$ PD yes

NSD no $\quad$ ND no





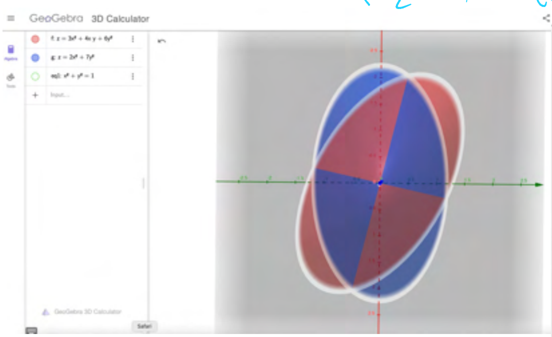
$Q_D(x,y) = 2x^2+7y^2$

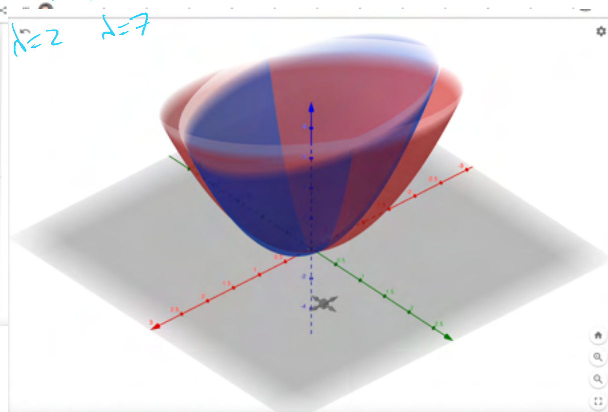
$$\begin{pmatrix}3 & 2\\ 2 & 6\end{pmatrix} \quad \overset{v_1}{\begin{pmatrix}*\\ *\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}*\\ *\end{pmatrix}}$$

$\lambda=2 \quad \lambda=7$

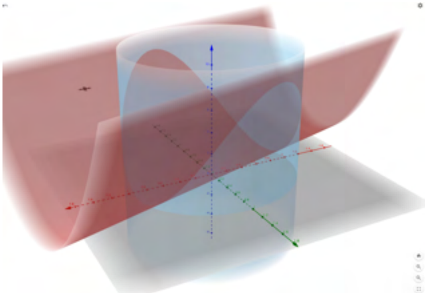
$$\begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix} \quad \begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}$$

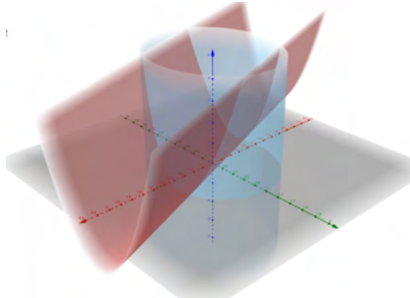
$\lambda=2 \quad \lambda=7$





https://www.geogebra.org/m/c6yg2agh



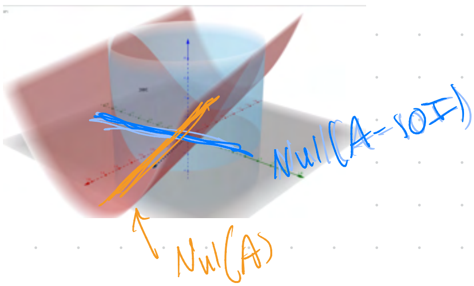


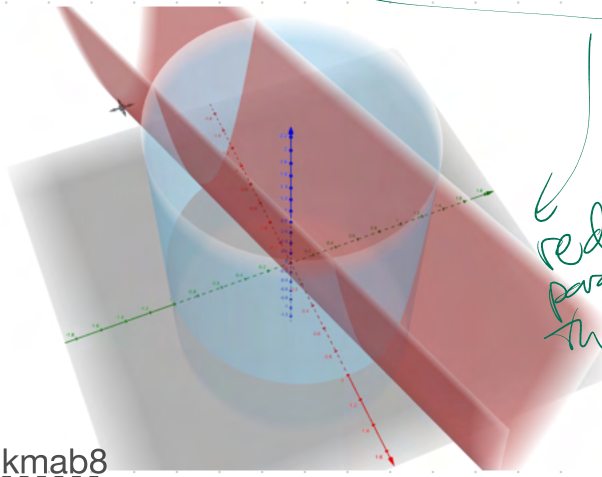
f: $z = x^2 - 6x\,y + 9y^2$

eq1: $x^2 + y^2 = 1$

$u = \text{Vector}\left(\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right) = \begin{pmatrix} 0.95 \\ 0.32 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

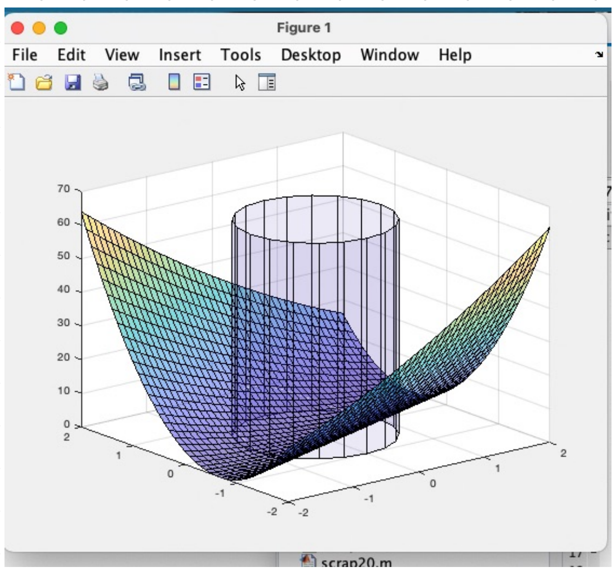




red parabola thing

https://www.geogebra.org/m/akbkmab8

```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
30	26	17

	A	B	C	D
1	Assignment	Grade (%)	Weight	Points
2	Exam (Best)	60.0	0.2	12.00
3	Exam (Mid)	52.0	0.2	10.40
4	Exam (Lowest)	34.0	0.15	5.10
5	MQE Score	100.0	0.25	25.00
6	Final Exam	50.66667	0.2	10.13
7	Final Grade:			62.63
8	Final Letter Grade:			D

	F	G	H
2	Exam 1	30	/50
3	Exam 2	26	/50
4	Exam 3	17	/50
6	Final Exam	48.67	/100
8	In-studio Quiz Total	9	/45
9	MyLab Total	172	/214
10	Total Exploration Points	18	/42
12	MQE Total	70	
13	MQE Score	70	/70
14	MQE additional	0	
16	Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17	MQE Bonus	0	
18	Total Final Exam Bonus	2	

	A	B	C	D
1	Assignment	Grade (%)	Weight	Points
2	Exam (Best)	60.0	0.2	12.00
3	Exam (Mid)	52.0	0.2	10.40
4	Exam (Lowest)	34.0	0.15	5.10
5	MQE Score	100.0	0.25	25.00
6	Final Exam	92.43	0.2	18.49
7	Final Grade:			70.99
8	Final Letter Grade:			C

	F	G	H	I
2	Exam 1	30	/50	*students - only modify the highlighted cells
3	Exam 2	26	/50	
4	Exam 3	17	/50	
6	Final Exam	90	/100	
8	Quiz (In-studio + Quiz 0) Total	9	/45	
9	Total Exploration Points	18	/42	
10	MyMathLab Total	172	/183	1pt MML = 0.3pt MQE
11	WeBWorK Total	0	/183	1pt WeBWorK = 0.3pt MQE
13	MyMathLab/WeBWorK MQE	51.6		convert homework total to MQE and take larger number
14	***MQE Total (calculated)***	***78.6***	1	*<--- put a 0 or 1 to use **calculated** or Canvas total*
15	***MQE Total (Canvas)***	***88***	0	*<--- put a 0 or 1 to use calculated or **Canvas** total*
16	***MQE Score***	***70***	/70	
17	MQE additional	8.6		
19	Final Exam 2% Bonus? (CIOS incentive)	TRUE		
20	MQE Bonus	0.43		
21	Total Final Exam Bonus	2.43		

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
34	34	25

	A	B	C	D
1	Assignment	Grade (%)	Weight	Points
2	Exam (Best)	68.0	0.2	13.60
3	Exam (Mid)	68.0	0.2	13.60
4	Exam (Lowest)	50.0	0.15	7.50
5	MQE Score	100.0	0.25	25.00
6	Final Exam	64	0.2	12.80
7	Final Grade:			73.50
8	Final Letter Grade:			C

	F	G	H
2	Exam 1	34	/50
3	Exam 2	34	/50
4	Exam 3	25	/50
6	Final Exam	62.00	/100
8	In-studio Quiz Total	9	/45
9	MyLab Total	172	/214
10	Total Exploration Points	18	/42
12	MQE Total	70	
13	MQE Score	70	/70
14	MQE additional	0	
16	Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17	MQE Bonus	0	
18	Total Final Exam Bonus	2	

Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
42	43	28

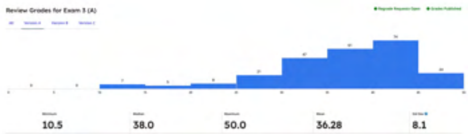
	A	B	C	D
1	Assignment	Grade (%)	Weight	Points
2	Exam (Best)	86.0	0.2	17.20
3	Exam (Mid)	84.0	0.2	16.80
4	Exam (Lowest)	56.0	0.15	8.40
5	MQE Score	100.0	0.25	25.00
6	Final Exam	77.33333	0.2	15.47
7	Final Grade:			82.87
8	Final Letter Grade:			B

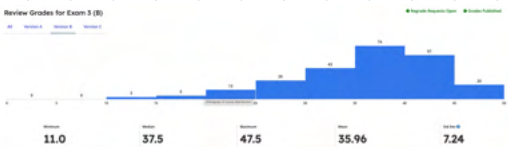
	F	G	H
2	Exam 1	42	/50
3	Exam 2	43	/50
4	Exam 3	28	/50
6	Final Exam	75.33	/100
8	In-studio Quiz Total	9	/45
9	MyLab Total	172	/214
10	Total Exploration Points	18	/42
12	MQE Total	70	
13	MQE Score	70	/70
14	MQE additional	0	
16	Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17	MQE Bonus	0	
18	Total Final Exam Bonus	2	

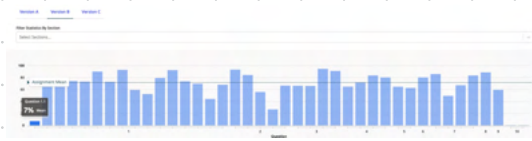
Exam 1 Out of 50 ...	Exam 2 Out of 50	Exam 3 Out of 50
43	34	34.5

	A	B	C	D
1	Assignment	Grade (%)	Weight	Points
2	Exam (Best)	86.0	0.2	17.20
3	Exam (Mid)	69.0	0.2	13.80
4	Exam (Lowest)	68.0	0.15	10.20
5	MQE Score	100.0	0.25	25.00
6	Final Exam	79.85833	0.2	15.97
7	Final Grade:			82.17
8	Final Letter Grade:			B

	F	G	H
2	Exam 1	43	/50
3	Exam 2	34	/50
4	Exam 3	34.5	/50
6	Final Exam	74.33	/100
8	In-studio Quiz Total	45	/45
9	MyLab Total	214	/214
10	Total Exploration Points	42	/42
12	MQE Total	140.5	
13	MQE Score	70	/70
14	MQE additional	70.5	
16	Final Exam 2% Bonus? (CIOS incentive)	TRUE	
17	MQE Bonus	3.525	
18	Total Final Exam Bonus	5.525	







15%

1(a)iii VerA / 1(a)i VerB

true false

If A is $n \times n$ and A has n ~~distinct~~ real eigenvectors, then A is diagonalizable.

lin ind

if "lin ind"

79%

1(a)viii VerA / 1(a)v VerB

If A has QR-factorization $A = QR$, then $\mathrm{Nul}(A^T) = \mathrm{Nul}(Q^T)$.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}, \ldots$

51%

1(b)iii VerA / 1(b)ii VerB

A 3×3 matrix in RREF that has exactly one pivot and is not diagonalizable.

1(c) Version B shown

(c) (2 points) Which of the following are examples of a matrix A which satisfy the following property. The matrix A is not diagonalizable while the matrix A^2 is diagonalizable. *Select all that apply.*

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ 73%
- $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ 69%
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 44%
- $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ 68%

Midterm 3 (B). Your initials: ____________

You do not need to justify your reasoning for questions on this page.

$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}(b/a)$

2. (5 points) Consider $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sketch (a) the vector $\vec{u}$ and (b) the vector $A\vec{u}$. Then, fill in the blanks for the following statements to make the statements true.

93%

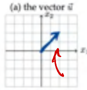
87%

47%

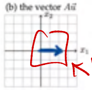
29%

66%

(a) the vector $\vec{u}$



(b) the vector $A\vec{u}$



length 2.

(c) The transformation defined by $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is a rotation-dilation where the rotation is through $\boxed{7\pi/8}$ (degrees or radians counter-clockwise) and the scaling factor is $\boxed{\sqrt{2}}$.

!!

$-45°$, $-\frac{\pi}{4}$, $\frac{7\pi}{8}$, or $315°$ all acceptable

(d) An eigenvector associated to the eigenvalue $\lambda = 1-i$ of A is the vector $\vec{x} = \begin{pmatrix} \boxed{i} \\ 1 \end{pmatrix}$.

$A - \lambda I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad x = s\begin{bmatrix} i \\ 1 \end{bmatrix}$

3. (3 points) Fill in the blanks so that the following statements are true. Let $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) An eigenvector of A corresponding to eigenvalue $\lambda = 2$ is $\vec{x} = \begin{pmatrix} \boxed{4} \\ 2 \\ \boxed{6} \end{pmatrix}$. 64%

(b) The vector $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to eigenvalue $\lambda = \boxed{1}$. 65%

91%

(c) The dimension of the $\lambda = 1$ eigenspace of A is equal to $\boxed{2}$. 88%

7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and

a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $3x_1^2 - 4x_1x_2 + 5x_2^2$ b. $3x_1^2 + 2x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.

a. $5x_1^2 + 16x_1x_2 - 5x_2^2$ b. $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$

b. $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $3x_1^2 - 2x_2^2 + 5x_3^2 + 4x_1x_2 - 6x_1x_3$

b. $4x_3^2 - 2x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

8. Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct P using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$ **10.** $2x_1^2 + 6x_1x_2 - 6x_2^2$

11. $2x_1^2 - 4x_1x_2 - x_2^2$ **12.** $-x_1^2 - 2x_1x_2 - x_2^2$

13. $x_1^2 - 6x_1x_2 + 9x_2^2$ **14.** $3x_1^2 + 4x_1x_2$

15. **[M]** $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$

16. **[M]** $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$

17. **[M]** $11x_1^2 + 11x_2^2 + 11x_3^2 + 11x_4^2 + 16x_1x_2 - 12x_1x_4 + 12x_2x_3 + 16x_3x_4$

18. **[M]** $2x_1^2 + 2x_2^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.

b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.

c. The principal axes of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are eigenvectors of A.

d. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

e. If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.

f. A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is not a quadratic form.

b. If A is symmetric and P is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term.

c. If A is a 2×2 symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola.

d. An indefinite quadratic form is neither positive semidefinite nor negative semidefinite.

e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all positive.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of A.

23. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.

a. Q is positive definite if $\det A > 0$ and $a > 0$.

b. Q is negative definite if $\det A > 0$ and $a < 0$.

c. Q is indefinite if $\det A < 0$.

25. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

26. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.]

27. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let A be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

SG Mastering: Diagonalization and Quadratic Forms 7-7

$\underbrace{Q_A(\vec{x}) = Q_D(\vec{y})}_{\text{old part}}$ $\boxed{\text{w/ } \|\vec{x}\| = 1}$ new part

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

$Q(\vec{x}) = 3x_1^2 + 7x_2^2$

$Q(0, \pm 1) = 7$

$Q(\pm 1, 0) = 3$

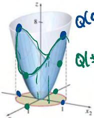


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

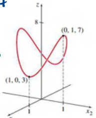
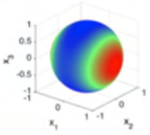


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.



10	3/11 - 3/15	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/18 - 3/22	Break	Break	Break	Break	Break
12	3/25 - 3/29	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	4/1 - 4/5	6.6	WS6.5,6.6	Exam 3, Review	Cancelled	PageRank
14	4/8 - 4/12	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/15 - 4/19	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/22 - 4/24	Last lecture	Last Studio	Reading Period		
17	4/25 - 5/2	Final Exams: MATH 1554 Common Final Exam Tuesday, April 30th at 6:00pm				

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

Example 1

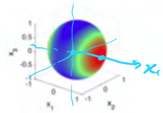
The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = ||\vec{x}||^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$

Find the largest and smallest values of Q on the surface of the sphere.



unit vectors.

$$\vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$Q(1,0,0) = 9(1)^2 + 4(0)^2 + 3(0)^2 = 9$$

$$Q(0,0,1) = 9(0)^2 + 4(0)^2 + 3(1)^2 = 3$$

$$Q\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) = 9\left(\tfrac{1}{\sqrt{2}}\right)^2 + 4(0)^2 + 3\left(\tfrac{-1}{\sqrt{2}}\right)^2 = \frac{9+0+3}{2} = \frac{12}{2} = 6 \checkmark$$

$$Q\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) = 9\left(\tfrac{1}{3}\right)^2 + 4\left(\tfrac{2}{3}\right)^2 + 3\left(\tfrac{2}{3}\right)^2 = \frac{9(1)+4(4)+3(4)}{9} = \frac{37}{9} \approx 4.\overline{11} \checkmark$$

Ex. Find the largest output z-value with restricted input ||x||=1 where z is given by:

$$z = 3x_1^2 + 7x_2^2.$$

$Q_A(\pm 1, 0) = 3$ MIN

$Q_A(0, \pm 1) = 7$ MAX

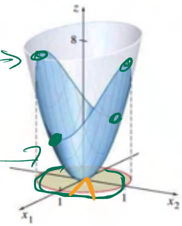


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

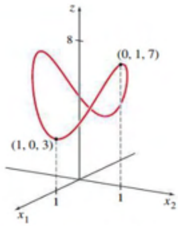


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.

EXAMPLE 3 Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of the quadratic form $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, and find a unit vector at which this maximum value is attained.

SOLUTION By Theorem 6, the desired maximum value is the greatest eigenvalue of A. The characteristic equation turns out to be

$$0 = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 6)(\lambda - 3)(\lambda - 1)$$

The greatest eigenvalue is 6.

$\lambda = 6, 3, 1$

$Q_D(\vec{y}) = 6y_1^2 + 3y_2^2 + y_3^2$

MAX is 6 @ $(\pm 1, 0, 0)$

MIN is 3 @ $(0, 0, \pm 1)$

$Q_A(\vec{x}) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

PD

(i.e. PSD)

please tell me min value?
subject to $\|\vec{x}\| = 1$?

& give me an $\vec{x}$ input which attains this minimum output value?

last col of P.

$P\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

unit length eigenvector of A.

If we know $Q_D(0,0,1) = 1$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{y}$ attains the MIN VALUE of Q_D, then

correspondingly $P\vec{y} = \vec{x}$ $\vec{x}$ attains same value for Q_A.

$x = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$x^T A x$

$A - I = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$Q_A\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = 3\left(\frac{-1}{\sqrt{2}}\right)^2 + 3\left(\frac{1}{\sqrt{2}}\right)^2 + 4(0)^2$

$+ 4\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + 0$

$= \frac{3}{2} + \frac{3}{2} - \frac{4}{2} = 3 - 2 = \boxed{1}$

EXAMPLE 5 Let A be the matrix in Example 3 and let $\mathbf{u}_1$ be a unit eigenvector corresponding to the greatest eigenvalue of A. Find the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the conditions

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T\mathbf{u}_1 = 0 \qquad (4)$$

Suppose $A\vec{v} = \lambda \vec{v}$ & $Q_A(\vec{x}) = x^T A x$.

quadratic form.

What is the value of

$$Q_A(\vec{v}) = v^T A v$$

number.

$$= v^T \lambda v$$

$$= \lambda v^T v$$

$$= \lambda \, v \cdot v$$

$$= \lambda \|v\|^2$$

If you plug in a unit length eigenvector?

$\longrightarrow$ output is λ!

$$A \overset{①}{=} PDP^T$$

$$x^T D x$$

$$\Rightarrow x^T A \boxed{x} \overset{①}{=} x^T PDP^T x$$

$$= (P^T x)^T D \, (P^T x)$$

$\boxed{P^T x}$

$$(x^T A) x = x^T (A x)$$

??

A Symmetric

$Q(\vec{x}) = x^T A x$ ← a number

If $\vec{x}$ is an eigenvector (of any length)

$Q(\vec{x}) = x^T A x = x^T \lambda x = \lambda x^T x = \boxed{\lambda \|x\|^2}$

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

subject to

$$\|\vec{x}\| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : \|\vec{x}\| = 1\}$$
$$M = \max\{Q(\vec{x}) : \|\vec{x}\| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $\|\vec{x}\| = 1$,

- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm\vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm\vec{u}_n$.

If $\|x\| = 1$ and and $x \in \mathbb{R}^n$

then $\lambda_{small} \leq Q(\vec{x}) \leq \lambda_{BIG}$

λ_{small} = "smallest eigenvalue of A"

λ_{BIG} = "BIGGEST eigenvalue of A"

Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $\|\vec{x}\| = 1$, and identify points where these values are obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

Q1: Give me A? ✓

Q2: λ's for A?

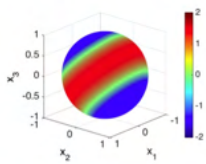
Q3: corresponding unit eigenvector for each λ?

Q4: MAX/MIN value of $Q(\vec{x})$. subject to $\|\vec{x}\| = 1$, also a vector where the MIN & MAX occur.

Q5: characterize $Q(\vec{x})$ as PD/ND/PSD/NSD/none

Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.



Q1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

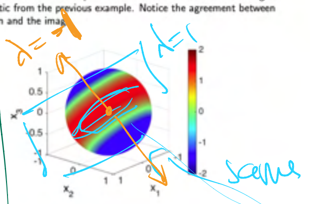
Example 2

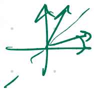
Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

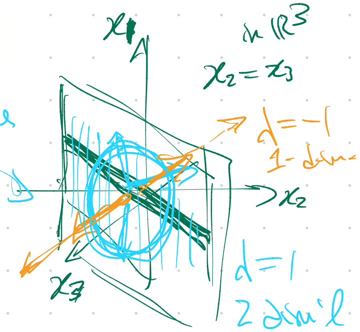
$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.







Q1: Give me A? ✓

Q2: λ's for A?

Q3: corresponding unit eigenvector for each λ?

Q4: MAX/MIN value of $Q(\vec{x})$. subject to $||\vec{x}|| = 1$, also a vector where the MIN & MAX occur.

Q5: characterize $Q(\vec{x})$ as PD/ND/PSD/NSD/none

Q1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Q2: $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)(\lambda^2 - 1)$

$= -(\lambda-1)(\lambda-1)(\lambda+1)$

Q3: $\text{Nul}(A - \lambda I)$ $\lambda_1 = 1$, $\lambda_2 = -1$.

$\lambda = 1$ alg 2 $\lambda_2 = -1$ alg 1.

$\lambda_1 = 1$

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{array}{ccc} & s & t \\ \left[\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} -1 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

$\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

∞ many choices!

✓ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, ✓ $\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, ✓ $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$s = 1$, $t = 0$ $s = 0$, $t = 1/\sqrt{2}$ $s = 1/3$, $t = 2/3$

$\lambda_2 = -1$

$$A + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{array}{ccc} & & s \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right] \end{array}$$

$\vec{x} = s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

✓ $\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, ✓ $\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

only two choices!

Q4: Max value of $Q(x_1, x_2, x_3) = x_1^2 + 2x_2x_3$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$ is $\boxed{1}$ @ e.g. $\begin{bmatrix} 1/2 \\ 2/3 \\ 2/3 \end{bmatrix}$

Min value " ——— " is $\boxed{-1}$ @ $\pm\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$Q(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0 + 2(\frac{1}{\sqrt{2}})(\frac{-1}{\sqrt{2}}) = -1$ ✓

$Q(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

$= (\frac{1}{3})^2 + 2(\frac{2}{3})(\frac{2}{3})$

$= \frac{1}{9} + \frac{8}{9} = 1.$

An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_*$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$ and to $\vec{x} \cdot \vec{u}_1 = 0$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

<u>answer</u> pick a vector in the $\lambda = 1$ eigenspace orthogonal to $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Example 4 (if time permits)

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2+6x_2^2+7x_3^2+4x_1x_2-4x_2x_3=9y_1^2+6y_2^2+3y_3^2$

2. $3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3=7y_1^2+4y_2^2$

Hint: $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for y_3^2.

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$ (See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3$ (See Exercise 2.)

5. $Q(\mathbf{x}) = x_1^2 + x_2^2 - 10x_1x_2$

6. $Q(\mathbf{x}) = 3x_1^2 + 9x_2^2 + 8x_1x_2$

7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are $2, -1$, and -4.]

8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 9 and -3.]

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^T A\mathbf{x}$?

12. Let λ be any eigenvalue of a symmetric matrix A. Justify the statement made in this section that $m \le \lambda \le M$, where m and M are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^T A\mathbf{x}$.]

13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T\mathbf{x} = 1$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

[M] In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$

15. $4x_1^2-6x_1x_2-10x_1x_3-10x_1x_4-6x_2x_3-6x_2x_4-2x_3x_4$

16. $-6x_1^2-10x_2^2-13x_3^2-13x_4^2-4x_1x_2-4x_1x_3-4x_1x_4+6x_3x_4$

17. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$