# LINEAR ALGEBRA

## week 2

## Section 1.4 : The Matrix Equation

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

*"Mathematics is the art of giving the same name to different things."*
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

## 1.4 : Matrix Equation $A\vec{x} = \vec{b}$

**Topics**

We will cover these topics in this section.

1. Matrix notation for systems of equations.
2. The matrix product $A\vec{x}$.

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Compute matrix-vector products.
2. Express linear systems as vector equations and matrix equations.
3. Characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots.

Office Hours

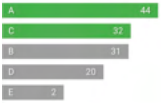
Itempool

What are the pivot locations for the given matrix?

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

*Select all that apply!*

- A. $\begin{bmatrix} \blacksquare & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$
- B. $\begin{bmatrix} * & * & * & * \\ \blacksquare & * & * & * \\ * & * & * & * \end{bmatrix}$
- C. $\begin{bmatrix} * & * & * & * \\ * & * & * & \blacksquare \\ * & * & * & * \end{bmatrix}$
- D. $\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & \blacksquare \end{bmatrix}$
- E. Another location not shown above

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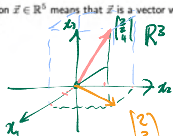
Office Hours

Itempool

## Notation   $\vec{x} \in \mathbb{R}^3$   "$\vec{x}$ belongs to $\mathbb{R}^3$"

| symbol | meaning |
|---|---|
| $\in$ | belongs to |
| $\mathbb{R}^n$ | the set of vectors with $n$ real-valued elements |
| $\mathbb{R}^{m\times n}$ | the set of real-valued matrices with $m$ rows and $n$ columns |

**Example**: the notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.




$\begin{bmatrix}2\\2\\0\end{bmatrix} \in \mathbb{R}^3$ ✓

$\begin{bmatrix}2\\2\end{bmatrix} \in \mathbb{R}^2$ ✓

$\begin{bmatrix}2\\2\\0\end{bmatrix}$ is in "the floor" of $\mathbb{R}^3$

set of vectors w/ $x_3 = 0$.



## Linear Combinations   STAR EQUATION

**Definition**
$A$ is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and $x \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of $A$:

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Note that $A\vec{x}$ is in the span of the columns of $A$.

**Example**
The following product can be written as a linear combination of vectors:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -3 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$



① $\vec{b} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$ is $\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}}_{A}$ times $\underbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}}_{\vec{x}}$

② $\vec{b} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$ is in the span of the columns of $A$ $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

③ The system w/ augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & -3 & 3 & 12 \end{array}\right]$ is consistent w/ solution $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$.

## Solution Sets

**Theorem**
*If $A$ is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and $x \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to*

$$A\vec{x} = \vec{b}$$

*has the same set of solutions as the vector equation*

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

*which as the same set of solutions as the set of linear equations with the augmented matrix*

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$



## Existence of Solutions

**Theorem**
*The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of $A$.*

ie $\vec{b} \in$ span{ "cols of $A$" }

## Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\underbrace{\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix}}_{A} \vec{x} = \vec{b}$$

augmented column $\vec{b}$

when is there a pivot in $\vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right] \sim_{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & -2b_1+b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & b_3 \\ 0 & 2 & -4 & -2b_1+b_2 \end{array}\right]$$

$$\sim_{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & b_3 \\ 0 & 0 & 0 & -2b_1+b_2-2b_3 \end{array}\right]$$



## The Row Vector Rule for Computing $A\vec{x}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$A\vec{x} = \vec{b}$ is consistent

$$\Leftrightarrow x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is true for some choices of $x_1, x_2, x_3$.



If $-2b_1 + b_2 - 2b_3 = 0$ then the system $A\vec{x} = \vec{b}$ is consistent. Otherwise if $-2b_1 + b_2 - 2b_3 \neq 0$, then the system $A\vec{x} = \vec{b}$, is inconsistent

in a plane in $\mathbb{R}^3$

## Summary

We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:
$$2x_1 + 3x_2 = 7$$
$$x_1 - x_2 = 5$$

2. An augmented matrix:
$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

3. A vector equation:
$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation:
$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Each representation gives us a different way to think about linear systems.

**DEFINITION**

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of $A$ and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of $A$ using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & -5 & 3 & b_2 \end{array}\right]$$

**EXAMPLE 1**

a. $$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}}_{\vec{x}} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The system $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is consistent and $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ is a solution

b. $$\underbrace{\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 7 \end{bmatrix}}_{\vec{x}} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}}_{\vec{b}}$$

$$\begin{cases} 2x_1 - 3x_2 = -13 \\ 8x_1 + 0x_2 = 32 \\ -5x_1 + 2x_2 = -6 \end{cases}$$

$\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ is a solution to this system

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

**EXAMPLE 3** Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $b_1, b_2, b_3$?

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 7 & 5 & 3b_1+b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 3b_1+b_3 \\ 0 & 14 & 10 & 4b_1+b_2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 3b_1+b_3 \\ 0 & 0 & 0 & \underbrace{-6b_1-2b_3+4b_1+b_2} \end{array}\right]$$

$\hookrightarrow$ $-2b_1 + b_2 - 2b_3 = 0$

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent

**SOLUTION** Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

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**Theorem:** The system $A\vec{x} = \vec{b}$ is consistent for every possible vector $\vec{b}$ $\underbrace{\text{if and only if}}_{\text{(exactly when)}}$ the matrix $A$ has **a pivot in every row**.

**Proof.** If $A$ doesn't have a row of zeros after row reducing, then you can never have a pivot in the augmented column of $[A \mid b]$. ∎

## 1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

7. $x_1\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

8. $z_1\begin{bmatrix} 4 \\ -2 \end{bmatrix} + z_2\begin{bmatrix} -4 \\ 5 \end{bmatrix} + z_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} + z_4\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. $\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$

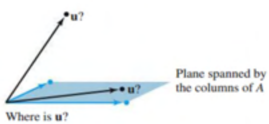
10. $\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$

Given $A$ and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

11. $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane $\mathbb{R}^3$ spanned by the columns of $A$? (See the figure.) Why or why not?




14. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of $A$? Why or why not?

15. Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

16. Repeat Exercise 15: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Exercises 17–20 refer to the matrices $A$ and $B$ below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

**23.** a. The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

b. A vector $\mathbf{b}$ is a linear combination of the columns of a matrix $A$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

c. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row.

d. The first entry in the product $A\mathbf{x}$ is a sum of products.

e. If the columns of an $m \times n$ matrix $A$ span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

f. If $A$ is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then $A$ cannot have a pivot position in every row.

**24.** a. Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

b. Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix $A$ and vector $\mathbf{x}$.

c. The solution set of a linear system whose augmented matrix is $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}\,]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$.

d. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of $A$.

e. If the augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

* Quiz 0 on Gradescope
* Exploration #2.



## 1.5 : Solution Sets of Linear Systems

**Topics**

We will cover these topics in this section.

1. Homogeneous systems
2. Parametric **vector** forms of solutions to linear systems

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Express the solution set of a linear system in parametric vector form.
2. Provide a geometric interpretation to the solution set of a linear system.
3. Characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



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## Homogeneous Systems

Definition
Linear systems of the form $A\vec{x}=\vec{0}$ are **homogeneous**.

Linear systems of the form $A\vec{x}=\vec{b}$ $(\vec{b}\neq\vec{0})$ are **inhomogeneous**.

Because homogeneous systems always have the **trivial solution**, $\vec{x}=\vec{0}$, the interesting question is whether they have non-trivial. solutions.

Observation

$A\vec{x}=\vec{0}$ has a nontrivial solution
$\iff$ there is a free variable
$\iff$ $A$ has a column with no pivot.



## Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$x_1 + 3x_2 + x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0$$
$$x_1 - 2x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -1 & -5 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \sim \begin{array}{l} \\ -2R_1+R_2 \rightarrow \\ -R_1+R_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$



$x_1$ & $x_2$ pivot variables

$x_3$ free var.

parametric equation form

$$x_1 - 2x_3 = 0$$
$$x_2 + x_3 = 0$$
$$x_3 = t \quad \text{(free)}$$

$$\begin{cases} x_1 = 2t \\ x_2 = -t \\ x_3 = t \quad \text{(free)} \end{cases}$$

NEXT parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{x} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

## Parametric Forms, Homogeneous Case

In the example on the previous slide we expressed the solution to a system using a vector equation. This is a **parametric form** of the solution.

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.



## Example 2 (non-homogeneous system)

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ 2x_1 - x_2 - 5x_3 &= 11 \\ x_1 - 2x_3 &= 6 \end{aligned}$$

(Note that the left-hand side is the same as Example 1).



Solutions to $A\vec{x} = 0$ were $\boxed{\vec{x} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}$

same $A$ new $\vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & -1 & -5 & 11 \\ 1 & 0 & -2 & 6 \end{array}\right] \sim \cdots \sim \begin{array}{c} \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$t$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

Solns to $A\vec{x} = b$

$$\boxed{\vec{x} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}$$

parametric equation form

$$\begin{aligned} x_1 - 2x_3 &= 6 \\ x_2 + x_3 &= 1 \\ x_3 &= t \text{ (free)} \end{aligned} \longrightarrow \begin{cases} x_1 = 6 + 2t \\ x_2 = 1 - t \\ x_3 = t \end{cases}$$

part w/ no t

part w/ t

next, parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6+2t \\ 1-t \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

**Notice** the solutions to $A\vec{x} = \vec{b}$ are the same as the solutions to $A\vec{x} = \vec{0}$ but you have to add $\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$

## 1.5 EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

**1.**
$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

**2.**
$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

**3.**
$$\begin{aligned} -3x_1 + 5x_2 - 7x_3 &= 0 \\ -6x_1 + 7x_2 + x_3 &= 0 \end{aligned}$$

**4.**
$$\begin{aligned} -5x_1 + 7x_2 + 9x_3 &= 0 \\ x_1 - 2x_2 + 6x_3 &= 0 \end{aligned}$$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

**5.**
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

**6.**
$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

In Exercises 7–12, describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where $A$ is row equivalent to the given matrix.

**7.** $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$  **8.** $\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$

**9.** $\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$  **10.** $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

#9a solve $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$.

#9b solve $A\vec{x} = 0$.

**11.** $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**12.** $\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**13.** Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with $x_3$ free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

**14.** Suppose the solution set of a certain system of linear equations can be described as $x_1 = 3x_4$, $x_2 = 8 + x_4$, $x_3 = 2 - 5x_4$, with $x_4$ free. Use vectors to describe this set as a "line" in $\mathbb{R}^4$.

**15.** Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

**16.** As in Exercise 15, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

**17.** Describe and compare the solution sets of $x_1 + 9x_2 - 4x_3 = 0$ and $x_1 + 9x_2 - 4x_3 = -2$.

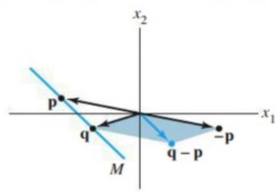
**18.** Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

In Exercises 19 and 20, find the parametric equation of the line through **a** parallel to **b**.

**19.** $\mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$  **20.** $\mathbf{a} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$

In Exercises 21 and 22, find a parametric equation of the line $M$ through **p** and **q**. [*Hint:* $M$ is parallel to the vector $\mathbf{q} - \mathbf{p}$. See the figure below.]

**21.** $\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$  **22.** $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$




The line through **p** and **q**.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

**23.** a. A homogeneous equation is always consistent.

b. The equation $A\mathbf{x} = \mathbf{0}$ gives an explicit description of its solution set.

c. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through **v** parallel to **p**.

e. The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.

**24.** a. If **x** is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in **x** is nonzero.

b. The equation $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$, with $x_2$ and $x_3$ free (and neither **u** nor **v** a multiple of the other), describes a plane through the origin.

c. The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

d. The effect of adding **p** to a vector is to move the vector in a direction parallel to **p**.

e. The solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$.

In Exercises 29–32, (a) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and (b) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible **b**?

**29.** $A$ is a $3 \times 3$ matrix with three pivot positions.

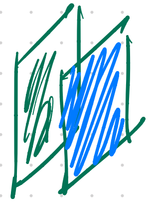
**30.** $A$ is a $3 \times 3$ matrix with two pivot positions.

**31.** $A$ is a $3 \times 2$ matrix with two pivot positions.

**32.** $A$ is a $2 \times 4$ matrix with two pivot positions.

**33.** Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection. [*Hint:* Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

Ex. $A = \begin{bmatrix} 3 & -9 & 6 & 3 \\ -1 & 3 & -2 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$



Solve (a) $A\vec{x} = b$ and (b) $A\vec{x} = 0$

Soln. Solve $A\vec{x} = b$ by row reducing $[A | \vec{b}]$

$$\left[\begin{array}{cccc|c} 3 & -9 & 6 & 3 & 9 \\ -1 & 3 & -2 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & -1 & 3 \\ 3 & -9 & 6 & 3 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & -1 & 3 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right]$$

$$\begin{array}{cccc} & r & t & \\ & \downarrow & \downarrow & \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 - 3x_2 + 2x_3 = 3$

$x_2 = r$ (free)

$x_3 = t$ (free)

$x_4 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Solns to $A\vec{x} = 0$

$$\begin{cases} x_1 = 3 + 3r - 2t \\ x_2 = r \\ x_3 = t \\ x_4 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 3+3r-2t \\ r \\ t \\ 0 \end{bmatrix} = \cancel{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \begin{bmatrix} 3r \\ r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ t \\ 0 \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

Theorem: If the general solutions to $A\vec{x}=\vec{0}$ are $\vec{x}_h$ (homogeneous solns)

Then the general solutions to $A\vec{x}=\vec{b}$ are

$$\vec{x} = \underset{\substack{\uparrow \\ \text{one particular soln.}}}{\vec{x}_p} + \underset{\substack{\uparrow \\ \text{all homogeneous solutions.}}}{\vec{x}_h}$$

## 1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations.

| Dates | Mon Lecture | Tue Studio | Wed Lecture | Thu Studio | Fri Lecture |
|---|---|---|---|---|---|
| 8/21 - 8/25 | 1.1 | WS1.1 | 1.2 | WS1.2 | 1.3 |
| 8/28 - 9/1 | 1.4 | WS1.3,1.4 | 1.5 | WS1.5 | 1.7 |
| 9/4 - 9/8 | Break | WS1.7 | 1.8 | WS1.8 | 1.9 |
| 9/11 - 9/15 | 2.1 | WS1.9,2.1 | Exam 1, Review | Cancelled | 2.2 |

**DEFINITION** An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$


Equivalent defn. * $Ax=0$ has only the trivial soln $A=[v_1\, v_2 \dots v_p]$

* $A$ has a pivot in every col. $A=[v_1 \dots v_p]$

FACTS: If $A$ is $m\times n$ $A=[v_1 \dots v_n]$

* If $n>m$ then $\{v_1,\dots,v_n\}$ lin dep
* If $\{v_1,\dots,v_p\}$ are lin ind then $m \geq n$.
* $Ax=0$ has a free var $\Rightarrow \{v_1,\dots,v_n\}$ lin dep.

Ex. Which of the following sets of vectors are lin ind/lin dep

① $\boxed{\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}}$

linearly dependent.

$$\left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\1&3&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&1&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

$t$ free

$$\sim \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\end{bmatrix}$$

$t$

$$X = t\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

Solns to $A\vec{x}=0$

IF $t=5$ $\vec{x} = \begin{bmatrix}10\\-5\\5\end{bmatrix}$

$A \quad \vec{x} \quad = 0$

$$\begin{bmatrix}1&2&0\\0&1&1\\1&3&1\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = 10\begin{bmatrix}1\\0\\1\end{bmatrix} + (-5)\begin{bmatrix}2\\1\\3\end{bmatrix} + (5)\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

② $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix} \right\}$

$$\left[\begin{array}{cc|c}1&2&0\\2&4&0\\1&2&0\end{array}\right] \sim$$

2 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

3 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\}$

4 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

at the end of this section.

**THEOREM 7**

**Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

**THEOREM 8**

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

**THEOREM 9** If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Ex. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ lin dep? lin ind?

Ex.

In Exercises 11–14, find the value(s) of $h$ for which the vectors are linearly *dependent*. Justify each answer.

**11.** $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$ **12.** $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$

## 1.7 EXERCISES

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

**1.** $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

**3.** $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ **4.** $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

**5.** $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$

**7.** $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

In Exercises 9 and 10, (a) for what values of $h$ is $\mathbf{v}_3$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, and (b) for what values of $h$ is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*? Justify each answer.



**9.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

**10.** $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$

In Exercises 11–14, find the value(s) of $h$ for which the vectors are linearly *dependent*. Justify each answer.

**11.** $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$ **12.** $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$

**13.** $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

**15.** $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ **16.** $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$

**17.** $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$ **18.** $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

**19.** $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**24.** $A$ is a $2 \times 2$ matrix with linearly dependent columns.

**25.** $A$ is a $4 \times 2$ matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

**26.** $A$ is a $4 \times 3$ matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.

**27.** How many pivot columns must a $7 \times 5$ matrix have if its columns are linearly independent? Why?

**28.** How many pivot columns must a $5 \times 7$ matrix have if its columns span $\mathbb{R}^5$? Why?

**29.** Construct $3 \times 2$ matrices $A$ and $B$ such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

**30.** a. Fill in the blank in the following statement: "If $A$ is an $m \times n$ matrix, then the columns of $A$ are linearly independent if and only if $A$ has _____ pivot columns."

b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved *without performing row operations*. [*Hint:* Write $A\mathbf{x} = \mathbf{0}$ as a vector equation.]

**31.** Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

**32.** Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

**21.** a. The columns of a matrix $A$ are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If $S$ is a linearly dependent set, then each vector is a linear combination of the other vectors in $S$.

c. The columns of any $4 \times 5$ matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.

**22.** a. Two vectors are linearly dependent if and only if they lie on a line through the origin.

b. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

c. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

d. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

**23.** $A$ is a $3 \times 3$ matrix with linearly independent columns.

plus twice the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

**33.** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

**34.** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

**35.** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

**36.** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3$ is *not* a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

**37.** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

**38.** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. [*Hint:* Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.]