LINEAR ALGEBRA

week 3

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 – 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 – 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 – 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 – 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different p by writing them as vector equations. In this way, the focus shifts from the solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations.

DEFINITION An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$

Equivalent defn. * $Ax=0$ has only the trivial soln $A=[v_1\, v_2 \dots v_p]$

* A has a pivot in every col. $A=[v_1 \dots v_p]$

FACTS: If A is $m\times n$ $A=[v_1 \dots v_n]$

* If $n>m$ then $\{v_1,\dots,v_n\}$ lin dep
* If $\{v_1,\dots,v_p\}$ are lin ind then $m \geq n$.
* $Ax=0$ has a free var $\Rightarrow \{v_1,\dots,v_n\}$ lin dep.

Itempool

Itempool
later today!

Ex. Which of the following sets of vectors are lin indp/ lin dep

① $\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$

linearly dependent.

$$\left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\1&3&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&1&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

t free

$$\sim \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\end{bmatrix}$$

t

IF $t=1$ $\vec{x} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$

$A \quad \vec{x} \quad = \vec{0}$

$$\begin{bmatrix}1&2&0\\0&1&1\\1&3&1\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + (-1)\begin{bmatrix}2\\1\\3\end{bmatrix} + (1)\begin{bmatrix}0\\1\\1\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$X = t\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

Solns to $A\vec{x}=0$

② $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix} \right\}$

$$\left[\begin{array}{cc|c}1&2&0\\2&4&0\\1&2&0\end{array}\right] \sim$$

DEFINITION An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$\boxed{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}} \quad (2)$$

↑ dependence relation

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Equivalent defn. * $Ax=0$ has only the trivial soln $A=[v_1\, v_2 \dots v_p]$

* A has a pivot in every col. $A=[v_1 \dots v_p]$

FACTS: If A is $m\times n$ $A=[v_1 \dots v_n]$

* If $n>m$ then $\{v_1,\dots,v_n\}$ lin dep
* If $\{v_1,\dots,v_p\}$ are lin ind then $m\geq n$.
* $Ax=0$ has a free var $\Rightarrow \{v_1,\dots,v_n\}$ lin dep.

Ex. Which of the following sets of vectors are lin indp/ lin dep

(1) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

linearly dependent.

$$\left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\1&3&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&1&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

t free

$$\sim \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\end{bmatrix}$$

$$x = t\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$\underbrace{}$ Solns to $A\vec{x}=0$

If $t=5$ $\vec{x}=\begin{bmatrix}10\\-5\\5\end{bmatrix}$

A $\vec{x}$ $=0$

$$\begin{bmatrix}1&2&0\\0&1&1\\1&3&1\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = 10\begin{bmatrix}1\\0\\1\end{bmatrix} + (-5)\begin{bmatrix}2\\1\\3\end{bmatrix} + (5)\begin{bmatrix}0\\1\\1\end{bmatrix} \overset{?}{=} \begin{bmatrix}0\\0\\0\end{bmatrix} ✓$$

(2) $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}\right\}$

$$\left[\begin{array}{cc|c}1&2&0\\2&4&0\\1&2&0\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&0\\0&0&0\\0&0&0\end{array}\right]$$

s

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_2 = s \text{ (free)}\end{cases} \qquad \begin{cases} x_1 = -2s \\ x_2 = s\end{cases}$$

$$x = s\begin{bmatrix}-2\\1\end{bmatrix} \qquad s=1 \quad \vec{x}=\begin{bmatrix}-2\\1\end{bmatrix}$$

dependence relation

$$\boxed{-2\begin{bmatrix}1\\2\\1\end{bmatrix} + (1)\begin{bmatrix}2\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}}$$

lin. independent vectors

A

(2') $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}1&2&0\\0&2&2\\1&0&1\end{bmatrix} \sim \begin{array}{r}\\ \tfrac{1}{2}R_2\to\\ -R_1+R_3\to\end{array}\begin{bmatrix}1&2&0\\0&1&1\\0&-2&1\end{bmatrix}$$

$$0\begin{bmatrix}1\\0\\1\end{bmatrix}+0\begin{bmatrix}2\\2\\0\end{bmatrix}+0\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\sim \begin{array}{r}\\ \\ 2R_2+R_3\to\end{array}\begin{bmatrix}1&2&0\\0&1&1\\0&0&3\end{bmatrix}$$

why b/c A has a pivot in every col

$$\sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{cases}x_1=0\\x_2=0\\x_3=0\end{cases}$$

$$\vec{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Unique soln to $A\vec{x}=\vec{0}$

(3) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\0\end{bmatrix}\right\}$

$$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}$$

in RREF already ☺

lin dep ✓

(4) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}1&-1&2&1\\0&0&1&1\\1&1&0&1\end{bmatrix}$$

has at most 3 pivots

So at least 1 free var!

So lin dependent vectors.

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ lin dep.

$0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ $c \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$??

impossible!

at the end of this section.

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\{v_1, v_2, v_3\}$

lin dep.

set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

THEOREM 8

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

THEOREM 9 If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Solve $A\vec{x} = \vec{0}$

Ex. $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}}$ lin dep? ✓ lin ind?

cols of A.

yes REF? no. RREF?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \sim \begin{bmatrix} 1 & \overset{s}{0} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 0 \\ x_2 = s \text{ (free)} \\ x_3 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = \boxed{s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$s = 1$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Ex.

Itempool

In Exercises 11–14, find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix}1\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\-5\\7\end{bmatrix}, \begin{bmatrix}-1\\5\\h\end{bmatrix}$ **12.** $\begin{bmatrix}2\\-4\\1\end{bmatrix}, \begin{bmatrix}-6\\7\\-3\end{bmatrix}, \begin{bmatrix}8\\h\\4\end{bmatrix}$

you do.

I can do

$\begin{bmatrix}2t\\s\\t\end{bmatrix}$ all in Span$\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

Q #12

$$\begin{bmatrix}2 & -6 & 8\\-4 & 7 & h\\1 & -3 & 4\end{bmatrix} \sim \begin{bmatrix}1 & -3 & 4\\-4 & 7 & h\\2 & -6 & 8\end{bmatrix} \begin{array}{l} \\ \sim 4R_1+R_2 \to \\ -2R_1+R_3 \to \end{array} \begin{bmatrix}1 & -3 & 4\\0 & -5 & 16+h\\0 & 0 & 0\end{bmatrix}$$

MATLAB Exploration #3.

```
+9   scrap89.m   scrap89b.m   scrap90.m
clc
syms h
A=[1 3 -1 ; -1 -5 5 ; 4 7 h]
E1=[1 0 0 ; 1 1 0 ; -4 0 1];
E2=[1 0 0 ; 0 1 0 ; 0 -5/2 1];
reduce=E2*E1*A
```

```
Command Window

A =

[ 1,  3, -1]
[-1, -5,  5]
[ 4,  7,  h]


reduce =

[1,  3,     -1]
[0, -2,      4]
[0,  0,  h - 6]

fx >>
```

1.7 EXERCISES

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1. $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5. $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$

6. $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

In Exercises 9 and 10, (a) for what values of h is $\mathbf{v}_3$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, and (b) for what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*? Justify each answer.

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$

In Exercises 11–14, find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

12. $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

16. $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$

17. $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$

18. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

24. A is a 2×2 matrix with linearly dependent columns.

25. A is a 4×2 matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

26. A is a 4×3 matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.

27. How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

28. How many pivot columns must a 5×7 matrix have if its columns span $\mathbb{R}^5$? Why?

29. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

30. a. Fill in the blank in the following statement: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has ______ pivot columns."

    b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved *without performing row operations*. [*Hint:* Write $A\mathbf{x} = \mathbf{0}$ as a vector equation.]

31. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

32. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column

plus twice the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

21. a. The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

c. The columns of any 4×5 matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.

22. a. Two vectors are linearly dependent if and only if they lie on a line through the origin.

b. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

c. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

d. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. A is a 3×3 matrix with linearly independent columns.

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

34. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

35. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

36. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3$ is *not* a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

37. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

38. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. [*Hint:* Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.]

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Itempool

1.8 : An Introduction to Linear Transforms

Topics

We will cover these topics in this section.

1. The definition of a linear transformation.
2. The interpretation of matrix multiplication as a linear transformation.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear).
2. Characterize linear transforms using the concepts of
   - existence and uniqueness
   - domain, co-domain and range

7:54 AM

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.8 Slide 56

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Section 1.8 Slide 57

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4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

Exam 1 in one week from today @ 6:30 pm rooms listed on Canvas homepage.

Terminology ↓ & definitions

From Matrices to Functions

Let A be an $m \times n$ matrix. We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{v}$$

This is called a **matrix transformation**.

- The **domain** of T is $\mathbb{R}^n$.
- The **co-domain** or **target** of T is $\mathbb{R}^m$.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T
- The set of all possible images $T(\vec{x})$ is the **range**.

This gives us **another** interpretation of $A\vec{x} = \vec{b}$:

- set of equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Section 1.8 Slide 58

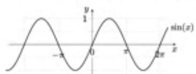
$$T(\vec{x}) = \boxed{A\vec{x} = \vec{b}}$$

Functions from Calculus

Many of the functions we know have **domain** and **codomain** $\mathbb{R}$. We can express the **rule** that defines the function sin this way:

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph, whose horizontal axis is the **domain**, and the vertical axis is the **codomain**.



This is ok when the domain and codomain are $\mathbb{R}$. It's hard to do when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. We would need five dimensions to draw that graph.

Section 1.8 Slide 59

$T(\vec{x}) = A\vec{x}$

$T(\vec{b}) = \begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}7\\5\\7\end{bmatrix} \stackrel{?}{=}$ undefined.

Example 1 $T: \mathbb{R}^2 \to \mathbb{R}^3$

Let $A = \begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}$, $\vec{u} = \begin{bmatrix}3\\4\end{bmatrix}$, $\vec{b} = \begin{bmatrix}7\\5\\7\end{bmatrix}$.

a) Compute $T(\vec{u})$. $= \begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} \stackrel{*}{=} 3\begin{bmatrix}1\\0\\1\end{bmatrix} + 4\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}7\\4\\7\end{bmatrix}$

b) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b}$ $\boxed{\begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}7\\5\\7\end{bmatrix} ?}$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$ $A\vec{x} = \vec{b}$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

Section 1.8 Slide 60

domain of T is $\mathbb{R}^2$

codomain of T is $\mathbb{R}^3$

$\begin{bmatrix}7\\4\\7\end{bmatrix}$ is the image of $\begin{bmatrix}3\\4\end{bmatrix}$.

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$.

So if T is linear, then

$$T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$$

This is called the **principle of superposition**. The idea is that if we know $T(\vec{e}_1), \dots, T(\vec{e}_n)$, then we know every $T(\vec{v})$.

Fact: Every matrix transformation T_A is linear.

Section 1.8 Slide 61

$$\left[\begin{array}{cc|c}1&1&7\\0&1&5\\1&1&7\end{array}\right] \sim \left[\begin{array}{cc|c}1&1&7\\0&1&5\\0&0&0\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&2\\0&1&5\\0&0&0\end{array}\right]$$

$\begin{cases}c_1 = 2\\ c_2 = 5\end{cases}$ $\vec{x} = \begin{bmatrix}2\\5\end{bmatrix}$

$$\left[\begin{array}{cc|c}1&1&7\\0&1&5\\1&1&6\end{array}\right] \sim \left[\begin{array}{cc|c}1&1&7\\0&1&5\\0&0&-1\end{array}\right]$$

$y = x^2$

x	y
0	0
1	1
-1	1
2	4

Example 2

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad T(\vec{x}) = A\vec{x} \quad T: \mathbb{R}^2 \to \mathbb{R}^2$

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$

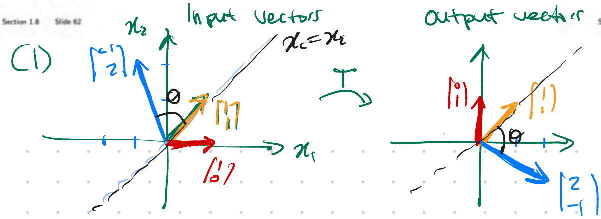
Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(1)

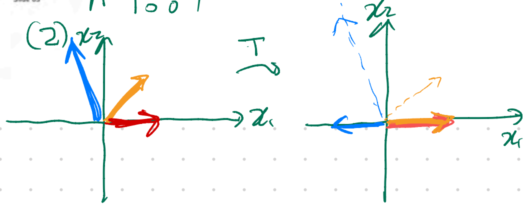


$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

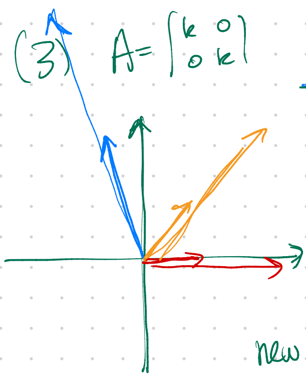


$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ $\quad$ assume $(k > 1)$



$T(\vec{x}) = Ax$

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -k \\ 2k \end{bmatrix} = k\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

new vector is just $k *$ old vector

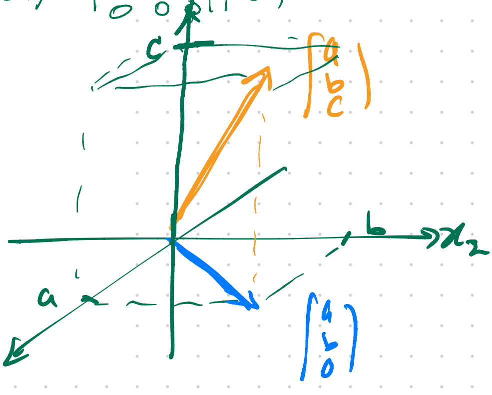
Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

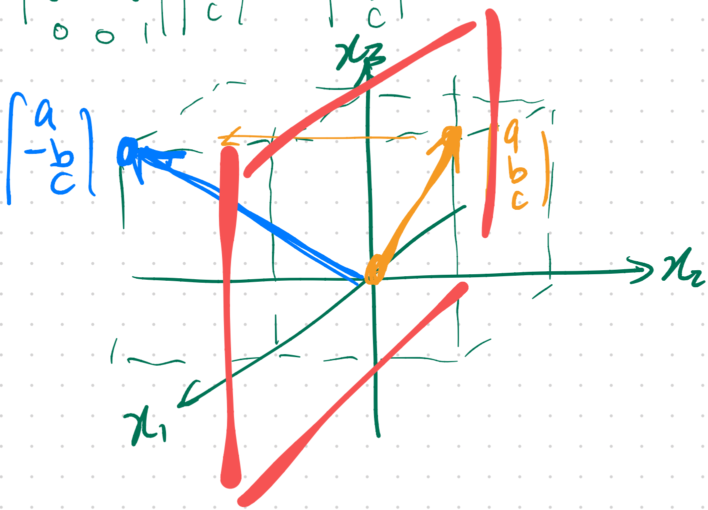
b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$



projection onto x_1x_2-plane

(b) $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -b \\ c \end{bmatrix}$



reflection across $\boxed{x_1x_3\text{-plane}}$

Example 4

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

What is the matrix that represents T?

$$A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\\e\end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}b\\d\\f\end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\bigstar \quad T\left(\begin{bmatrix}c_1\\c_2\end{bmatrix}\right) = c_1 T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + c_2 T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$* \quad T(c\vec{x}) = cT(\vec{x})$$

$$* \quad T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$$

$$f(x) = x^3+2x-1$$

$$f(2) = 8+4-1 = 11$$

$$f(6) \neq 3 \cdot 11$$

$$T\left(\begin{bmatrix}4\\5\end{bmatrix}\right) = T\left(4\begin{bmatrix}1\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 4T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 5T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 4\begin{bmatrix}5\\-7\\2\end{bmatrix} + 5\begin{bmatrix}-3\\8\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\5\end{bmatrix}\right) = \begin{bmatrix}5\\12\\8\end{bmatrix}$$

1.8 EXERCISES

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

7. Let A be a 6×5 matrix. What must a and b be in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^5$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

For Exercises 9 and 10, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

11. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

12. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$.

13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

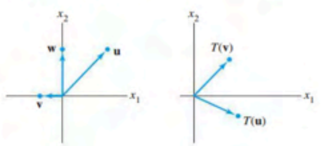
14. $T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u} + 2\mathbf{v}$.

18. The figure shows vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. [*Hint:* First, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.]



19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each $\mathbf{x}$.

In Exercises 21 and 22, mark each statement True or False. Justify each answer

Make two sketches similar to Figure 6 that illustrate properties (i) and (ii) of a linear transformation.

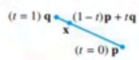
24. Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

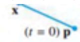
25. Given $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^n$, the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).

26. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, or onto a line through $\mathbf{0}$, or onto just the origin in $\mathbb{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?

27. a. Show that the line through vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ may be written in the parametric form $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$. (Refer to the figure with Exercises 21 and 22 in Section 1.5.)

    b. The line segment from $\mathbf{p}$ to $\mathbf{q}$ is the set of points of the form $(1 - t)\mathbf{p} + t\mathbf{q}$ for $0 \le t \le 1$ (as shown in the figure below). Show that a linear transformation T maps this line segment onto a line segment or onto a single point.





In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation is a special type of function.

b. If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

c. If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

e. A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

22. a. Every matrix transformation is a linear transformation.

b. The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.

c. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if $\mathbf{c}$ is in $\mathbb{R}^m$, then a uniqueness question is "Is $\mathbf{c}$ in the range of T?"

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

e. The superposition principle is a physical description of a linear transformation.

23. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the x_1-axis. (See Practice Problem 2.)

28. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. It can be shown that the set P of all points in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \le a \le 1, 0 \le b \le 1$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain why the image of a point in P under the transformation T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

29. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

a. Show that f is a linear transformation when $b = 0$.

b. Find a property of a linear transformation that is violated when $b \neq 0$.

c. Why is f called a linear function?

30. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(x) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in computer graphics.)

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

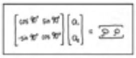
In Exercises 32–36, column vectors are written as rows, such as $\mathbf{x} = (x_1, x_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$.

32. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

1.9 : Matrix of a Linear Transformation

Topics
We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

ILA

CHECK OUT the textbook for Math 1553 which was created by Georgia Tech professors for Intro. Linear Algebra

https://textbooks.math.gatech.edu/ila/

There's a really nice section on linear transformations

Transformations

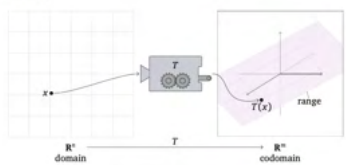
At this point it is convenient to fix our ideas and terminology regarding functions, which we will call *transformations* in this book. This allows us to systematize our discussion of matrices as functions.

Definition. A ***transformation*** from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the ***domain*** of T.
- $\mathbf{R}^m$ is called the ***codomain*** of T.
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the ***image*** of x under T.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the ***range*** of T.

The notation $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means "T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$."

It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.



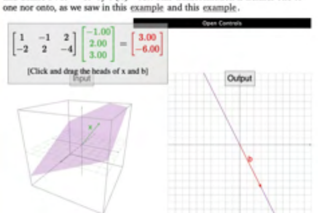
https://textbooks.math.gatech.edu/ila/one-to-one-onto.html

Example (A matrix transformation that is neither one-to-one nor onto).

Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix},$$

and define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x) = Ax$. This transformation is neither one-to-one nor onto, as we saw in this example and this example.



A picture of the matrix transformation T. The violet plane is the solution set of $T(x) = 0$. If you drag x along the violet plane, the output $T(x) = Ax$ does not change. This demonstrates that $T(x) = 0$ has more than one solution, so T is not one-to-one. The range of T is the violet line on the right; this is smaller than the codomain $\mathbf{R}^2$. If you drag b off of the violet line, then the equation $Ax = b$ becomes inconsistent; this means $T(x) = b$ has no solution.

Example (Reflection).

Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

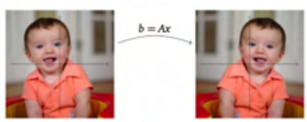
Describe the function $b = Ax$ geometrically.

Solution

In the equation $Ax = b$, the input vector x and the output vector b are both in $\mathbf{R}^2$. First we multiply A by a vector to see what it does:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

Multiplication by A negates the x-coordinate: it *reflects over the y-axis*.



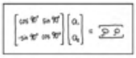
Multiplication by the matrix A reflects over the y-axis. Move the input vector x to see how the output vector b changes.

https://textbooks.math.gatech.edu/ila/matrix-transformations.html

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

1.9 : Matrix of a Linear Transformation

Topics
We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives
For the topics covered in this section, students are expected to be able t do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 - 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 - 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

Definition: The Standard Vectors

The **standard** basis **vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{kth spot}$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

multiplying $A \times e_i$ extracts the ith column of A.

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^m.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n)]$$

The matrix A is the **standard matrix** for a linear transformation.

T rotates vectors in $\mathbb{R}^2$ counter-clockwise by $90°$

$$T(\vec{x}) = A\vec{x}$$

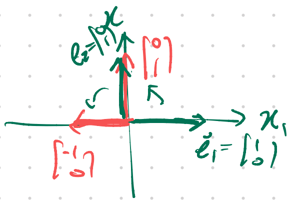
Rotations

Example 1

What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$

Ex. Find A.

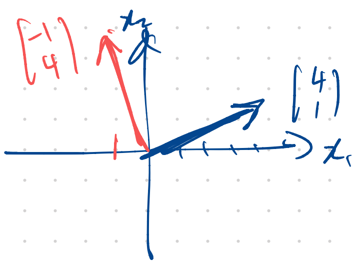


First column of A is $T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

second column of A is $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) \stackrel{?}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$



The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n)]$$

The matrix A is the **standard matrix** for a linear transformation.

Rotations

Example 1

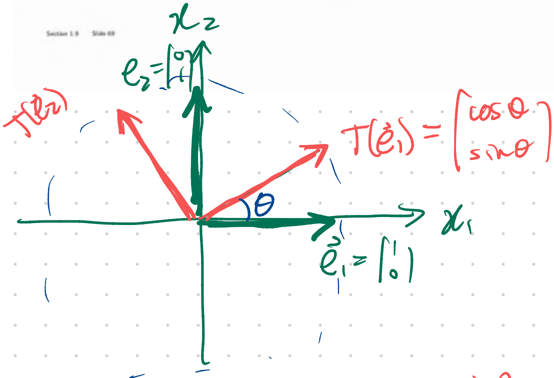
What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$

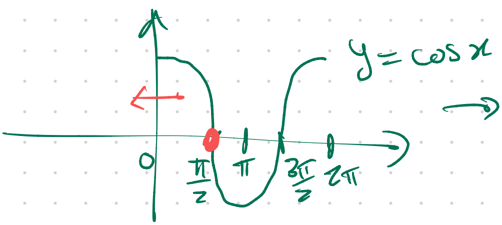
how about arbitrary θ?

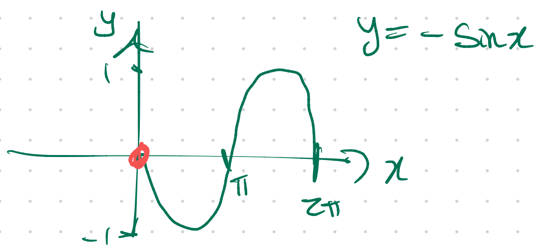
$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

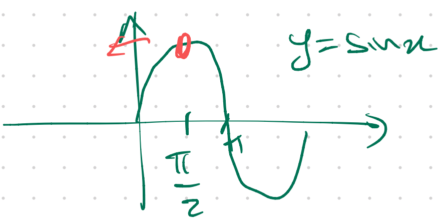
$T(\vec{e}_1) \quad T(\vec{e}_2)$

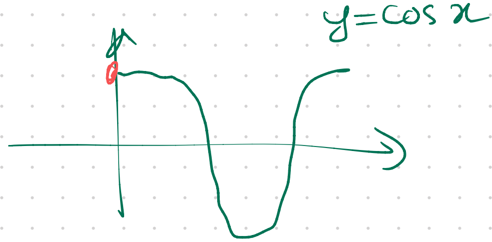


$$T(\vec{e}_2) = \begin{bmatrix} \cos(\theta + 90^\circ) \\ \sin(\theta + 90^\circ) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$









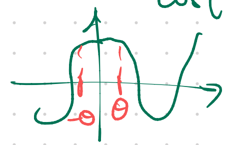
Q: What about Clockwise??

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \leftarrow \text{rotation by } \theta \text{ CCW}$$

$$B = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

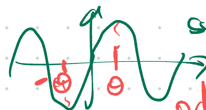
$$\boxed{B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

$\cos(-\theta) = \theta$



even

$\sin(-\theta) = -\sin\theta$



odd

Ex. Let T(x)=Ax be the transformation which first reflects vectors in R^2 across the line y=0, and then projects the resulting vector to the y-axis.

Find the standard matrix of A.

Itempool

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^m.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = [T(\vec{e}_1) \quad T(\vec{e}_3) \quad \cdots \quad T(\vec{e}_n)]$$

The matrix A is the **standard matrix** for a linear transformation.

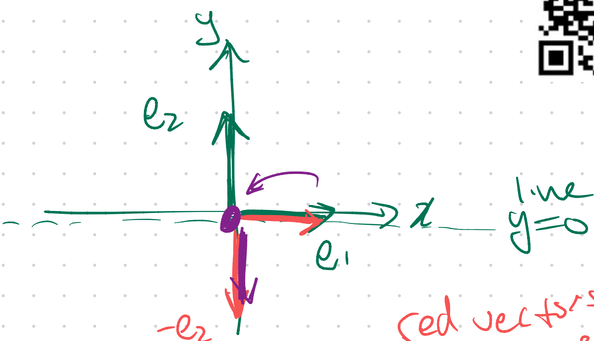
To enter

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In line as text use

[a b ; c d]

ie, MATLAB Syntax.



red vectors are after first transformation.

$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

[0 0 ; 0 -1)

Onto

$T(\vec{x}) = A\vec{x} = \vec{b}$

always consistent.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

Onto is an **existence property:** for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.

Examples

- A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.

Useful Fact

T is onto if and only if its standard matrix A has a pivot in every row.

$\Leftrightarrow$ A has a pivot in every row

$\Leftrightarrow$ RREF of A has no zero rows.

One-to-One

(opposite is "many-to-one")

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.

Examples

- A rotation on the plane is a one-to-one linear transformation.
- A projection in the plane is not one-to-one.

Useful Facts

- T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if the standard matrix A of T has no free variables.

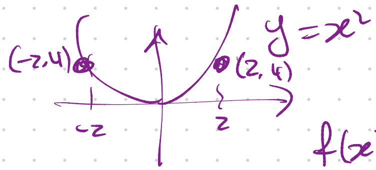
$\Leftrightarrow$ A has a pivot in every column

$\Leftrightarrow$ $Ax=b$ has at most ONE solution

Q: Example of transformation which is

(a) one-to-one but not onto?

(b) onto but not one-to-one?



$f(x)=x^2$

Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, ...)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

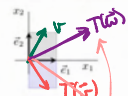
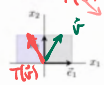
$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = [T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n)]$$

The matrix A is the **standard matrix** for a linear transformation.

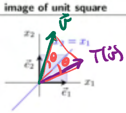
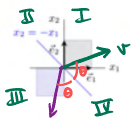
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1−axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection through x_2−axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

onto ? yes

1-1 ? yes.

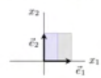
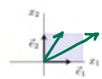
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through $x_2 = x_1$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

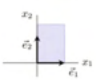
onto ? yes

1-1 ? yes.

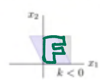
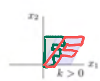
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $\lvert k\rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

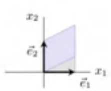
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k\rvert < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

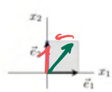
Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
Horizontal Shear(left)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k < 0$
Horizontal Shear(right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$

Two Dimensional Examples: Projections

transformation	image of unit square	standard matrix
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

not onto!

not 1-1!

Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why.**

a) A is a 2×3 standard matrix for a one-to-one linear transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$

b) B is a 3×2 standard matrix for an onto linear transform.

$$B = \begin{pmatrix} 1 & \\ & \\ & \end{pmatrix}$$

c) C is a 3×3 standard matrix of a linear transform that is one-to-one and onto.

$$C = \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}$$

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto.
2. The matrix A has columns which span $\mathbb{R}^m$.
3. The matrix A has m pivotal columns.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

Example 2

Define a linear transformation by
$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Is this one-to-one? Is it onto?

Additional Example (if time permits)

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 8 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Is the transformation onto? Is it one-to-one?

1.9 EXERCISES

In Exercises 1–10, assume that T is a linear transformation. Find the standard matrix of T.

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (counterclockwise).

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $-\pi/4$ radians (clockwise). [*Hint:* $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$.]

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 2\mathbf{e}_2$ but leaves the vector $\mathbf{e}_2$ unchanged.

6. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a horizontal shear transformation that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 3\mathbf{e}_1$.

7. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal x_1-axis. [*Hint:* $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$.]

8. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

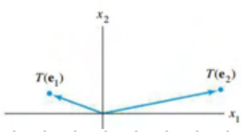
9. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

10. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2-axis and then rotates points $\pi/2$ radians.

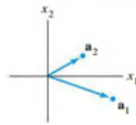
11. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?

12. Show that the transformation in Exercise 8 is merely a rotation about the origin. What is the angle of the rotation?

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2, 1)$.



14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = [\mathbf{a}_1 \ \ \mathbf{a}_2]$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are shown in the figure. Using the figure, draw the image of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation T.



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

15. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

16. $\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$

In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

17. $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

18. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

19. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

20. $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4 \quad (T : \mathbb{R}^4 \to \mathbb{R})$

21. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (3, 8)$.

22. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (-1, 4, 9)$.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.
    b. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle φ, then T is a linear transformation.
    c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.
    d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.
    e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

24. a. Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.
    b. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

c. The standard matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where a and d are ± 1.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17

26. The transformation in Exercise 2

27. The transformation in Exercise 19

28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation T. Use the notation of Example 1 in Section 1.2.

29. $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

30. $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T is one-to-one if and only if A has ____ pivot columns." Explain why the statement is true. [*Hint:* Look in the exercises for Section 1.7.]

32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has ____ pivot columns." Find some theorems that explain why the statement is true.

33. Verify the uniqueness of A in Theorem 10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B. Show that if A is the standard matrix for T, then $A = B$. [*Hint:* Show that A and B have the same columns.]

34. Why is the question "Is the linear transformation T onto?" an existence question?

35. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$, can you give a relation between m and n? If T is one-to-one, what can you say about m and n?

36. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation (from $\mathbb{R}^p$ to $\mathbb{R}^m$). [*Hint:* Compute $T(S(c\mathbf{u} + d\mathbf{v}))$ for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^p$ and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.]

[M] In Exercises 37–40, let T be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if T is a one-to-one mapping. In Exercises 39 and 40, decide if T maps $\mathbb{R}^5$ onto $\mathbb{R}^5$. Justify your answers.

37. $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$

39. $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$