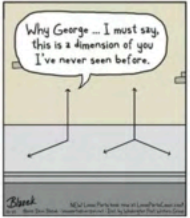
LINEAR ALGEBRA

week 6

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics

We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

Section 2.9 : Dimension and Rank

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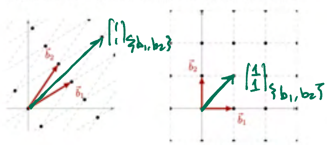
Notes copied from Fall 2023 MediaSpace recording

https://mediaspace.gatech.edu/media/Math%201554%20Section%20G%209_25%20-%20S2.9/1_cqjxbjtw

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.



$1b_1 + 1b_2$

coordinates in $\{b_1, b_2\}$ would be "$\begin{bmatrix}1\\1\end{bmatrix}_{\{b_1,b_2\}}$"

FACT
$\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ for $\mathbb{R}^3$

$e_1, e_2, \dots, e_n$ for $\mathbb{R}^n$

Coordinates

Definition
Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \dots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

And

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\ \vdots \\ c_p\end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinate vector of $\vec{x}$**

the coordinates in the basis $\{e_1, e_2, e_3\}$

$$\begin{bmatrix}-3\\2\\1\end{bmatrix} = -3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Example 1

Let $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{x} = \begin{bmatrix}5\\3\\5\end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

$$\begin{bmatrix}5\\3\\5\end{bmatrix}_{\{\vec{v}_1,\vec{v}_2\}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

?? — the coordinates of $\vec{x}$ in basis $\{b_1, b_2\}$

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\3\\5\end{bmatrix}$$

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Dimension

line dim 1, dim 2, dim 3

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples:

1. $\dim \mathbb{R}^n =$ n
2. $H = \{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0\}$ has dimension $n-1$ — $[1 \; 1 \; \cdots \; 1]$ (n cols.)
3. $\dim(\mathrm{Null}\, A)$ is the number of # free vars
4. $\dim(\mathrm{Col}\, A)$ is the number of # pivot cols.

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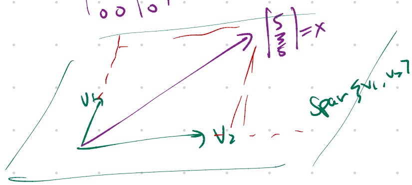
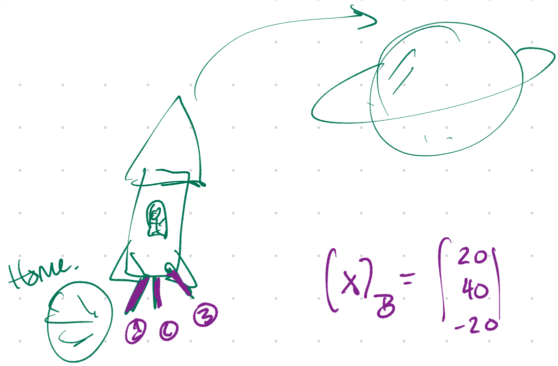
$$\begin{bmatrix}1&1&5\\0&1&3\\1&1&5\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}$$





$$(x)_{\mathcal{B}} = \begin{bmatrix}20\\40\\-20\end{bmatrix}$$

Start here on Wednesday

Rank

Definition

The **rank** of a matrix A is the dimension of its column space. (# pivots of A.)

Example 2: Compute rank(A) and dim(Nul(A)).

$$A = \begin{bmatrix}2&5&-3&-4&8\\4&7&-4&-3&9\\6&9&-5&2&4\\0&-9&6&5&-6\end{bmatrix} \sim \cdots \sim \begin{bmatrix}2&5&-3&-4&8\\0&-3&2&5&-7\\0&0&0&4&-6\\0&0&0&0&0\end{bmatrix}$$

rank(A) = 3

basis for Col A is $\left\{\begin{bmatrix}2\\4\\6\\0\end{bmatrix}, \begin{bmatrix}5\\7\\9\\-9\end{bmatrix}, \begin{bmatrix}-4\\-3\\2\\5\end{bmatrix}\right\}$

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Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

If a matrix A has n columns, then Rank $A + \dim(\mathrm{Nul}\, A) = n$.

Theorem (Basis Theorem)

Any two bases for a subspace have the same dimension.

Theorem (Invertibility Theorem) IMT DLC

Let A be a $n \times n$ matrix. These conditions are equivalent.

a. A is invertible.
m. The columns of A are a basis for $\mathbb{R}^n$.
n. $\mathrm{Col}\, A = \mathbb{R}^n$.
o. $\mathrm{rank}\, A = \dim(\mathrm{Col}\, A) = n$.
p. $\mathrm{Null}\, A = \{\vec{0}\}$.

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dim Nul(A) = 2

Nul(A) is a plane in $\mathbb{R}^5$

$\mathrm{Nul}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$

basis of Nul A is $\left\{\begin{bmatrix}*\\*\\*\\*\\*\end{bmatrix}, \begin{bmatrix}*\\*\\*\\*\\*\end{bmatrix}\right\}$ — from parametric vector form for sol'ns to $A\vec{x} = \vec{0}$.

Example

If possible, give an example of a 2×3 matrix A, in reduced echelon form, with the given properties.

a) $\text{rank}(A) = 3$

$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ w/ 3 pivots. NP

max rank is 2.

b) $\text{rank}(A) = 2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank $A \le \min\{\#\text{rows}, \#\text{cols}\}$

c) $\dim(\text{Null}(A)) = 2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\text{Null} A = \{\vec{0}\}$

NP

(e) $\dim \text{Nul} A = 3$?

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ every vector in $\mathbb{R}^3$ is in Nul A if $A = 0_{2\times 3}$

Q: $\{v_1, v_2\} = \mathcal{B}$ $\quad [x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ Then what is $\vec{x}$?

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

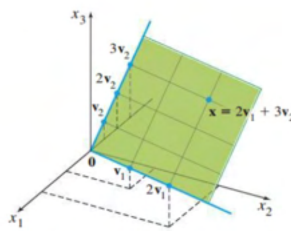


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

Check that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the coordinate vector for $\vec{x}$ in the basis $\{v_1, v_2\}$.

Itempool

$2\vec{v}_1 + 3\vec{v}_2 = \vec{x}$

$2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = \vec{x}$ ✓.

Notes from Dr. Sun's lecture on 2/12 when subbing for Sal

Coordinates

Definition

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \dots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p = \sum_{j=1}^{p} c_j \vec{b}_j = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p]\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$

And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p \qquad \vec{x} = [B]\,[\vec{x}]_B$$

is the **coordinate vector of $\vec{x}$ relative to** $\mathcal{B}$, or the $\mathcal{B}$**-coordinate vector of** $\vec{x}$

$$[B] = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p]$$

- Span $B = H \Rightarrow [\vec{x}]_B$ exists for every $\vec{x} \in H$
- Linear independence of $B \Rightarrow$ the uniqueness of B-coordinate vector of $\vec{x}$.

Example 1

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

Solve a linear system to find $[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$: $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$

$$[\vec{v}_1, \vec{v}_2 \mid \vec{x}] = \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow[①\leftarrow①-②]{③\leftarrow③-①} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array}$$

$$[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

CHECK. $[B]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = \vec{x}$.

Dimension

⚠ "Dimension" is not necessarily defined for every set.

"Dimension" is only defined for vector space, subspace.

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension. Cardinality (number of elements)

$B_1 = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p\}$ and $B_2 = \{\vec{b}_1', \vec{b}_2', \cdots, \vec{b}_q'\}$ are both bases of H, then $p = q$.

Examples:

1. $\dim \mathbb{R}^n = n$
2. $H = \{(x_1, \dots, x_n) : x_1 + \cdots + x_n = 0\}$ has dimension $n-1$
3. $\dim(\text{Null } A)$ is the number of free variables (Non pivotal columns)
4. $\dim(\text{Col } A)$ is the number of pivotal columns
   $\dim(\text{Col } A) = \text{rank}(A)$

Rank

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{2} & 5 & -3 & -4 & 8 \\ 0 & \textcircled{-3} & 2 & 5 & -7 \\ 0 & 0 & 0 & \textcircled{4} & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank (A) = the number of pivotal columns of A = dim Col(A)

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \xrightarrow[③\leftarrow③-3①]{②\leftarrow②-2①} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

$$\xrightarrow[④\leftarrow④-3②]{③\leftarrow③-2②} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -10 & 15 \end{bmatrix} \xrightarrow{④\leftarrow④+\frac{5}{2}③} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

statements in the original theorem in Section 2.3.

THEOREM **The Invertible Matrix Theorem (continued)**

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col } A = \mathbb{R}^n$
o. $\dim \text{Col } A = n$
p. $\text{rank } A = n$
q. $\text{Nul } A = \{\mathbf{0}\}$
r. $\dim \text{Nul } A = 0$

2.9 EXERCISES

In Exercises 1 and 2, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

In Exercises 3–6, the vector $\mathbf{x}$ is in a subspace H with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

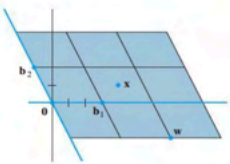
3. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

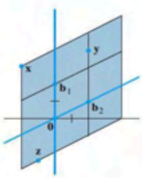
5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$

7. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{w}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}}$. Confirm your estimate of $[\mathbf{x}]_{\mathcal{B}}$ by using it and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{x}$.



8. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_{\mathcal{B}}$, $[\mathbf{y}]_{\mathcal{B}}$, and $[\mathbf{z}]_{\mathcal{B}}$. Confirm your estimates of $[\mathbf{y}]_{\mathcal{B}}$ and $[\mathbf{z}]_{\mathcal{B}}$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{y}$ and $\mathbf{z}$.



Exercises 9–12 display a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

9. $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 13 and 14, find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

13. $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}$

15. Suppose a 3×5 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

16. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here A is an $m \times n$ matrix.

17. a. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then $c_1, \dots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$.
    b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.
    c. The dimension of Col A is the number of pivot columns of A.
    d. The dimensions of Col A and Nul A add up to the number of columns of A.
    e. If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

18. a. If $\mathcal{B}$ is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.
    b. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ makes H look and act the same as $\mathbb{R}^p$.

c. The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.

d. The dimension of the column space of A is rank A.

e. If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.

In Exercises 19–24, justify each answer or construction.

19. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

20. What is the rank of a 4×5 matrix whose null space is three-dimensional?

21. If the rank of a 7×6 matrix A is 4, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

22. Show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\}$ in $\mathbb{R}^n$ is linearly dependent when dim Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\} = 4$.

23. If possible, construct a 3×4 matrix A such that dim Nul $A = 2$ and dim Col $A = 2$.

24. Construct a 4×3 matrix with rank 1.

25. Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.

26. Suppose columns 1, 3, 5, and 6 of a matrix A are linearly independent (but are not necessarily pivot columns) and the rank of A is 4. Explain why the four columns mentioned must be a basis for the column space of A.

Section 3.1 : Introduction to Determinants

Chapter 3 : Determinants

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The definition and computation of a determinant
2. The determinant of triangular matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute determinants of $n \times n$ matrices using a cofactor expansion.
2. Apply theorems to compute determinants of matrices that have particular structures.

Section 3.1 : Introduction to Determinants

Chapter 3 : Determinants

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The definition and computation of a determinant
2. The determinant of triangular matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute determinants of $n \times n$ matrices using a cofactor expansion.
2. Apply theorems to compute determinants of matrices that have particular structures.

5	2/5 - 2/9	2.3,2.4	WS2.2-2.4	2.5	WS2.5	2.8
6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
7	2/19 - 2/23	3.3	WS3.2	4.9	WS3.3,4.9	5.1
8	2/26 - 3/1	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

A Definition of the Determinant

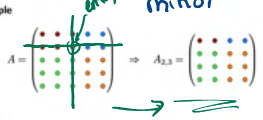
Suppose A is $n \times n$ and has elements a_{ij}.

1. If $n = 1$, $A = [a_{11}]$, and has determinant $\det A = a_{11}$.
2. Inductive case: for $n > 1$,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

where A_{ij} is the submatrix obtained by eliminating row i and column j of A.

Example



Example 1

Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $= ad - bc$

Ex. $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1\cdot 4 - 2\cdot 3 = -2$ (rank $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 2$)

$\det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1\cdot 4 - 2\cdot 2 = 0$ (rank $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 1$.)

Example 2

Compute $\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{vmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix}.$

A

pick row 1 (can be any row or any column)

$$\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = 1\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix} - (-5)\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= 1(0-(-2)) - (-5)(0-0) + 0(4-0)$$

$$= 1\cdot 2 - 0 + 0 = \boxed{2}$$

↖ the determinant of A.

Cofactors

The ij-cofactor is the signed determinant of the minor A_{ij}

Cofactors give us a more convenient notation for determinants.

Definition: Cofactor

The (i, j) cofactor of an $n \times n$ matrix A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

The pattern for the negative signs is

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

a_{11}, a_{22}, a_{33}, a_{44}

↙ a_{ij} do $(-1)^{i+j}$

if $i+j$ is even $+$

$i+j$ is odd $-$

$$\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = 0\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -5 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & -5 \\ 2 & 4 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= 0 + (2-0) + 0 = \boxed{2}$$

↗ same answer using any row/column of A for the expansion.

a_{ij} ↙ entry in row i & col j

Theorem

The determinant of a matrix A can be computed down any row or column of the matrix. For instance, down the j^{th} column, the determinant is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This gives us a way to calculate determinants more efficiently.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Example 3

Compute the determinant of $\begin{bmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$. $= A$

$$\det A = 5\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 4 & 3 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

0 0 0

$$= 5\left(0\cdot\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}\right)$$

$$= 5\cdot 3(1-(-2)) = 5\cdot 3\cdot 3 = 3^2\cdot 5 = \boxed{45}$$

Triangular Matrices

> **Theorem**
> If A is a triangular matrix then
> $$\det A = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Example 4
Compute the determinant of the matrix. Empty elements are zero.

$$\begin{bmatrix} 2 & 1 & & & & & \\ & 2 & 1 & & & & \\ & & 2 & 1 & & & \\ & & & 2 & 1 & & \\ & & & & 2 & 1 & \\ & & & & & 2 & 1 \\ & & & & & & 2 \end{bmatrix}$$

Computational Efficiency

Note that computation of a co-factor expansion for an $N \times N$ matrix requires roughly $N!$ multiplications.

- A 10×10 matrix requires roughly $10! = 3.6$ million multiplications
- A 20×20 matrix requires $20! \approx 2.4 \times 10^{18}$ multiplications

This doesn't mean that determinants are not useful.

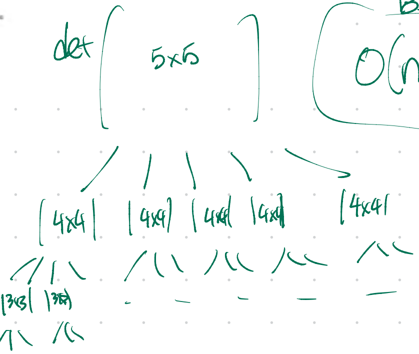
- We will explore other methods that further the efficiency of their calculation.
- Determinants are very useful in multivariable calculus for solving certain integration problems.

either upper/lower (same trick)

$$\det = \begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= 2(2)\begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2\cdot 2\cdot 2\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = (2\cdot 2\cdot 2\cdot 2\cdot 2) = 2^5$$



So So Sooooo BAD!!

$O(n!)$

$$\det\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 2\cdot 3\cdot -1 = -6$$ ✓

$$\det\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{pmatrix} = -0\begin{vmatrix} b & c \\ e & f \end{vmatrix} + 0\begin{vmatrix} a & c \\ d & f \end{vmatrix} - 0\begin{vmatrix} a & b \\ d & f \end{vmatrix}$$

$$= -0 + 0 - 0 = 0$$ ✓

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix}$ **6.** $\begin{vmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$ **8.** $\begin{vmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

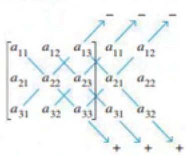
9. $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$ **10.** $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 5 & -6 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{vmatrix}$ **12.** $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:



Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4×4 *or larger matrices.*

15. $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$ **16.** $\begin{vmatrix} 0 & 3 & 1 \\ 4 & -5 & 0 \\ 3 & 4 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -3 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix}$ **18.** $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

21. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

22. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$

23. $\begin{bmatrix} a & b & c \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ a & b & c \\ 4 & 5 & 6 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$ **28.** $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **30.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **36.** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39. a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

b. The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

"A problem isn't finished just because you've found the right answer."
- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

Topics and Objectives

Topics
We will cover these topics in this section.

- The relationships between row reductions, the invertibility of a matrix, and determinants.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.
2. Use determinants to determine whether a square matrix is invertible.

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5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

Row Operations

- We saw how determinants are difficult or impossible to compute with a cofactor expansion for large N.
- Row operations give us a more efficient way to compute determinants.

Theorem: Row Operations and the Determinant

Let A be a square matrix.

1. If a multiple of a row of A is added to another row to produce B, then $\det B = \det A$.
2. If two rows are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by a scalar k to produce B, then $\det B = k \det A$.

Example 1 Compute $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

Invertibility

Important practical implication: If A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times \text{(product of pivots)}, & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular.} \end{cases}$$

Example 2 Compute the determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{vmatrix}$$

THEOREM 3 **Row Operations**

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

FIGURE 1
Typical echelon forms of square matrices.

THEOREM 4 A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM 6 **Multiplicative Property**

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Properties of the Determinant

For any square matrices A and B, we can show the following.

1. $\det A = \det A^T$.
2. A is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$.

Additional Example (if time permits)

Use a determinant to find all values of λ such that matrix C is not invertible.

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda I_3$$

Additional Example (if time permits)

Determine the value of

$$\det A = \det\left(\begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^8\right).$$

3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 3 & -5 & 2 \end{vmatrix}$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

11. $\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

14. $\begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix}$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$

16. $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$

17. $\begin{bmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{bmatrix}$

18. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

21. $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 27 and 28, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.
    b. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.
    c. If the columns of A are linearly dependent, then $\det A = 0$.
    d. $\det(A+B) = \det A + \det B$.

28. a. If three row interchanges are made in succession, then the new determinant equals the old determinant.
    b. The determinant of A is the product of the diagonal entries in A.
    c. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.
    d. $\det A^{-1} = (-1)\det A$.

29. Compute $\det B^4$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

32. Suppose that A is a square matrix such that $\det A^3 = 0$. Explain why A cannot be invertible.

33. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.

34. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

35. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

36. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

39. Let A and B be 3×3 matrices, with $\det A = -3$ and $\det B = 4$. Use properties of determinants (in the text and in the exercises above) to compute:
    a. $\det AB$ b. $\det 5A$ c. $\det B^T$
    d. $\det A^{-1}$ e. $\det A^3$

40. Let A and B be 4×4 matrices, with $\det A = -3$ and $\det B = -1$. Compute:
    a. $\det AB$ b. $\det B^5$ c. $\det 2A$
    d. $\det A^T BA$ e. $\det B^{-1}AB$

41. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$

42. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A+B) = \det A + \det B$ if and only if $a + d = 0$.

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

NOTE: Cramers rule and Adjoint of a matrix are NOT covered in Math 1554

Topics and Objectives

Topics

We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

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5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	Exam 2, Review	Cancelled	5.3

Supplementary FREE textbook

https://textbooks.math.gatech.edu/ila/

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Interactive Linear Algebra
Dan Margalit, Joseph Rabinoff

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Interactive Linear Algebra

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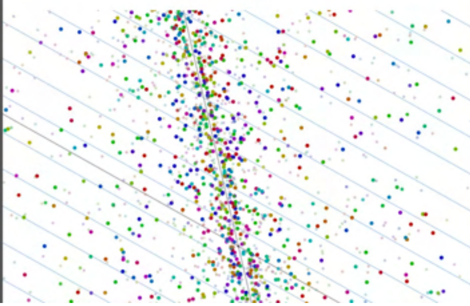
June 3, 2019



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4.3 Determinants and Volumes

Objectives

1. Understand the relationship between the determinant of a matrix and the volume of a parallelepiped.
2. Learn to use determinants to compute volumes of parallelograms and triangles.
3. Learn to use determinants to compute the volume of some curvy shapes like ellipses.
4. *Pictures:* parallelepiped, the image of a curvy shape under a linear transformation.
5. *Theorem:* determinants and volumes.
6. *Vocabulary word:* ***parallelepiped***.

In this section we give a geometric interpretation of determinants, in terms of *volumes*. This will shed light on the reason behind three of the four defining properties of the determinant. It is also a crucial ingredient in the change-of-variables formula in multivariable calculus.

Parallelograms and Paralellepipeds

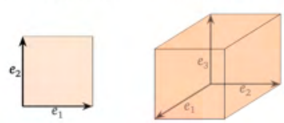
The determinant computes the volume of the following kind of geometric object.

Definition. The ***paralellepiped*** determined by n vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$ is the subset

$$P = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n \mid 0 \le a_1, a_2, \ldots, a_n \le 1\}.$$

In other words, a parallelepiped is the set of all linear combinations of n vectors with coefficients in $[0,1]$. We can draw parallelepipeds using the parallelogram law for vector addition.

Example (The unit cube). The parallelepiped determined by the standard coordinate vectors $e_1, e_2, \ldots, e_n$ is the unit n-dimensional cube.



Topics and Objectives

Topics

We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

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For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

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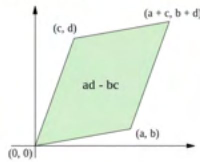
Determinants, Area and Volume

In $\mathbb{R}^2$, determinants give us the area of a parallelogram.



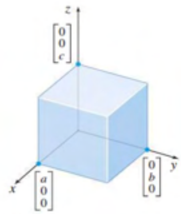
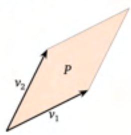
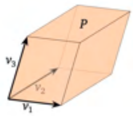


FIGURE 3
Volume $= |abc|$.

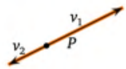
Example (Parallelograms). When $n = 2$, a paralellepiped is just a paralellogram in $\mathbf{R}^2$. Note that the edges come in parallel pairs.

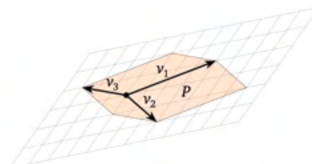


Example. When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.



When does a parallelepiped have zero volume? This can happen only if the parallelepiped is flat, i.e., it is squashed into a lower dimension.



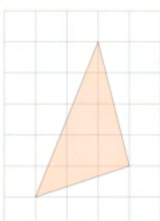


This means exactly that $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*, which by this corollary in Section 4.1 means that the matrix with rows $v_1, v_2, \ldots, v_n$ has determinant zero. To summarize:

Key Observation. The parallelepiped defined by $v_1, v_2, \ldots, v_n$ has zero volume if and only if the matrix with rows $v_1, v_2, \ldots, v_n$ has zero determinant.

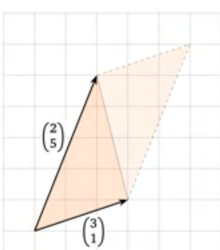
Example (Area of a triangle).

Find the area of the triangle with vertices $(-1,-2)$, $(2,-1)$, $(1,3)$.



Solution

Doubling a triangle makes a paralellogram. We choose two of its sides to be the rows of a matrix.



Determinants as Area, or Volume

Theorem

The volume of the parallelpiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar c.

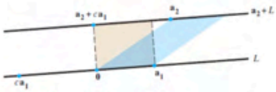


FIGURE 2 Two parallelograms of equal area.

Any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det(A)|$.

FIGURE 4 Two parallelepipeds of equal volume.

Example 1

Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1), (6,4)$

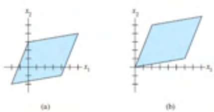


FIGURE 5 Translating a parallelogram does not change its area.

Linear Transformations

Theorem

If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then

$$\text{volume}\,(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$$

An example that applies this theorem is given in this week's worksheets.

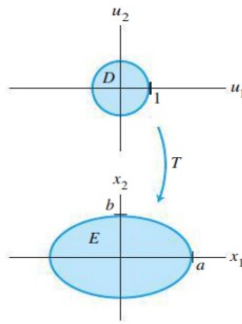
THEOREM 10

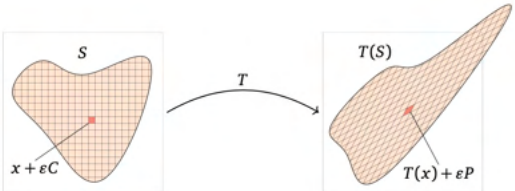
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \qquad (5)$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \qquad (6)$$



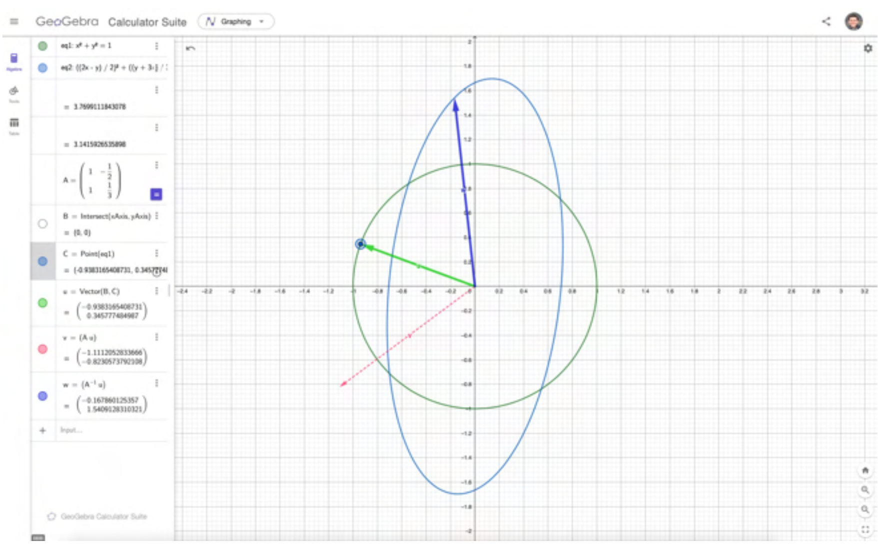


Example (Area of an ellipse).

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1.$$

https://www.geogebra.org/calculator/mkxeqfjy



Example (Area of an ellipse). ^

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2+\left(\frac{y+3x}{3}\right)^2=1.$$

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0), (5, 2), (6, 4), (11, 6)$

20. $(0, 0), (-2, 4), (4, -5), (2, -1)$

21. $(-2, 0), (0, 3), (1, 3), (-1, 0)$

22. $(0, -2), (5, -2), (-3, 1), (2, 1)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$, and $(-1, 3, -1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [*Hint:* Think about the columns of A.]

26. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^n$.

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

30. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

[*Hint:* Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are