

LINEAR

ALGEBRA

Week

7

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

NOTE: Cramer's rule and Adjoint of a matrix are NOT covered in Math 1554

Topics and Objectives

Topics

We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

Section 3.3 : Volume, Linear Transformations

Topics and Objectives

5	2/5 - 2/9	2.3.2.4	WS2.2-2.4	2.5	WS2.5	2.8
6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
7	2/19 - 2/23	3.3	WS3.2	4.9	WS3.3,4.9	5.1
8	2/26 - 3/1	5.2	WS5.1,5.2	Exam 2, Review	Cancelled	5.3

Chapter 3 : Determinants
Math 1554 Linear Algebra

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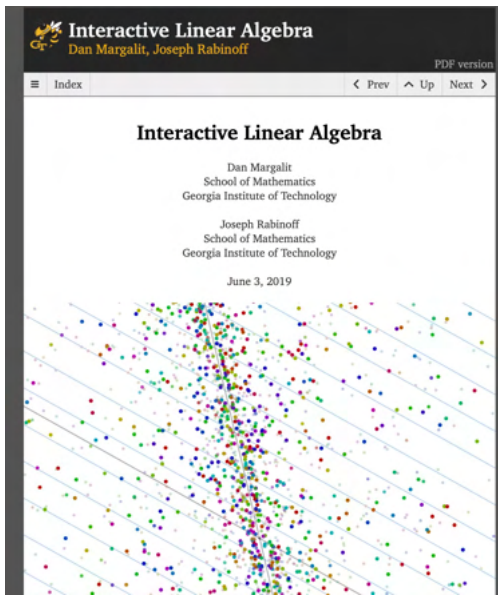
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Supplementary FREE textbook

<https://textbooks.math.gatech.edu/ila/>

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Interactive Linear Algebra
Dan Margalit, Joseph Rabinoff

PDF version

Index < Prev ^ Up Next >

4.3 Determinants and Volumes

Objectives

1. Understand the relationship between the determinant of a matrix and the volume of a parallelepiped.
2. Learn to use determinants to compute volumes of parallelograms and triangles.
3. Learn to use determinants to compute the volume of some curvy shapes like ellipses.
4. *Pictures:* parallelepiped, the image of a curvy shape under a linear transformation.
5. *Theorem:* determinants and volumes.
6. *Vocabulary word:* **parallelepiped**.

In this section we give a geometric interpretation of determinants, in terms of *volumes*. This will shed light on the reason behind three of the four defining properties of the determinant. It is also a crucial ingredient in the change-of-variables formula in multivariable calculus.

Parallelograms and Parallelepipeds

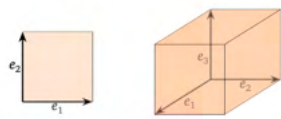
The determinant computes the volume of the following kind of geometric object.

Definition. The *parallelepiped* determined by n vectors v_1, v_2, \dots, v_n in \mathbb{R}^n is the subset

$$P = \{a_1x_1 + a_2x_2 + \dots + a_nx_n \mid 0 \leq a_1, a_2, \dots, a_n \leq 1\}.$$

In other words, a parallelepiped is the set of all linear combinations of n vectors with coefficients in $[0, 1]$. We can draw parallelepipeds using the parallelogram law for vector addition.

Example (The unit cube). The parallelepiped determined by the standard coordinate vectors e_1, e_2, \dots, e_n is the unit n -dimensional cube.



Topics and Objectives

Determinants, Area and Volume

$A = [\vec{v}_1, \vec{v}_2]$ $\det A = \text{Vol}(S)$

Topics

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- Relationships between area, volume, determinants, and linear transformations.

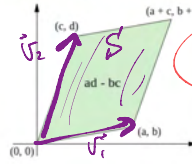
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In \mathbb{R}^2 , determinants give us the area of a parallelogram.



~~$\int_0^b \tau d\tau = \frac{1}{2} b^2$~~

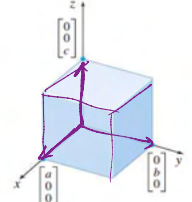


FIGURE 3

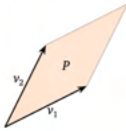
Volume = $|abc|$.

$\det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = a \cdot b \cdot c$

Q: What does $\det A$ mean geometrically?

$\det A = 0$
 $\iff A$ not invertible

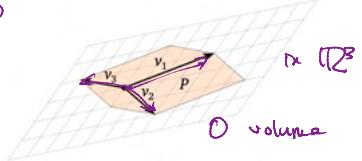
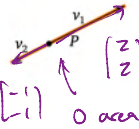
Example (Parallelograms). When $n = 2$, a parallelepiped is just a parallelogram in \mathbb{R}^2 . Note that the edges come in parallel pairs.



$\text{Vol } P = |\det[\vec{v}_1, \vec{v}_2]|$

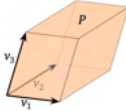
When does a parallelepiped have zero volume? This can happen only if the parallelepiped is flat, i.e., it is squashed into a lower dimension.

$\det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = 0$



$\det A = 0??$

Example. When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.



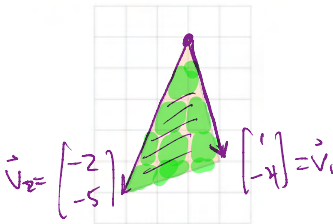
$\text{Vol } P = |\det[\vec{v}_1, \vec{v}_2, \vec{v}_3]|$

This means exactly that $\{v_1, v_2, \dots, v_n\}$ is linearly dependent, which by this corollary in Section 4.1 means that the matrix with rows v_1, v_2, \dots, v_n has determinant zero. To summarize:

Key Observation. The parallelepiped defined by v_1, v_2, \dots, v_n has zero volume if and only if the matrix with rows v_1, v_2, \dots, v_n has zero determinant.

Example (Area of a triangle). I LA. (§ 4.3)

Find the area of the triangle with vertices $(-1, -2)$, $(2, -1)$, $(1, 3)$.

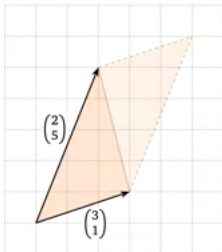


$\det \begin{bmatrix} 1 & -2 \\ -4 & -5 \end{bmatrix} = -5 - (-8) = -13$

$|\det[\vec{v}_1, \vec{v}_2]| = 13$ (crossed out) $\frac{13}{2}$ (circled)

Solution

Doubling a triangle makes a parallelogram. We choose two of its sides to be the rows of a matrix.



Volume of triangle is $\frac{1}{2}$ * volume of parallelepiped

Determinants as Area, or Volume

Theorem

The volume of the parallelepiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors \vec{a}, \vec{b} is equal to the area spanned by $\vec{a}, \vec{a} + \vec{b}$, for any scalar c .

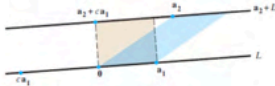
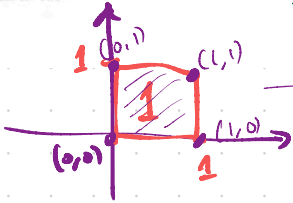


FIGURE 2 Two parallelograms of equal area.

Section 1.1 Slide 108



$\det A = 1$

$T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} = \vec{x}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det(A)|$.

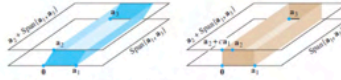


FIGURE 4 Two parallelepipeds of equal volume.



Example 1

Calculate the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1), (6, 4)$

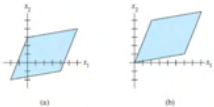


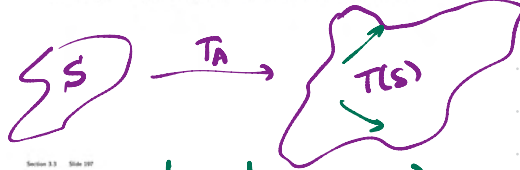
FIGURE 6 Translating a parallelogram does not change its area.

Linear Transformations

Theorem

If $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and S is some parallelogram in \mathbb{R}^n , then $\text{volume}(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$

An example that applies this theorem is given in this week's worksheets.



$\text{Vol}(S) * |\det A| = \text{Vol}(T(S))$

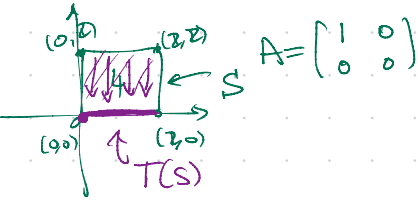
$-\int_{\partial S} f(\vec{c}) d\vec{c}$

Low Priority

Section 1.1 Slide 108

Section 1.1 Slide 107

③ project onto $x_2=0$ (x_1 -axis)

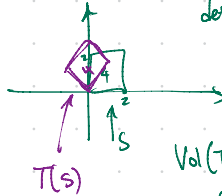


$\text{Vol}(T(S)) = 0$

$\checkmark 4 * |\det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}|$

$= 4 * 0 = 0$

Ex. ① rotation by θ



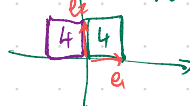
$\det \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \cos^2 \theta - (-\sin^2 \theta)$

$= \cos^2 \theta + \sin^2 \theta = 1$

$\text{Vol}(T(S)) = |\det A| * \text{Vol}(S)$

\uparrow new vol $\uparrow 1 * \uparrow$ old vol.

② reflect across x_2 -axis



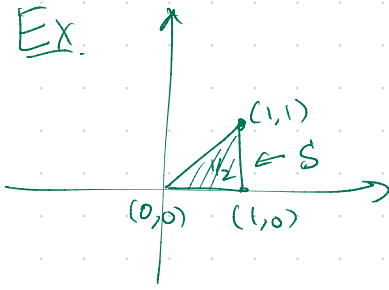
$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = (-1)(1)$

$= -1$

Ex:



do ① rotate by $\pi/2$ CCW

② horizontal scaling by 3

③ vertical scaling by 2

④ reflect about $x_1=0$. (x_2 -axis)

Qs What is $\text{Vol}(T(S)) = ?$

$$\text{Vol}(S) = 1/2$$

$$\text{Vol}(T(S)) = \frac{1}{2} * |-6| = 3$$

$$\textcircled{1} A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\textcircled{2} A_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\textcircled{3} A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\textcircled{4} A_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = A_4 A_3 A_2 A_1$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \vec{x} = (A_4 A_3 A_2) \underbrace{A_1}_{\substack{\text{A}_1 \text{ goes} \\ \text{first}}} \vec{x}$$

↑ what in what order.

$$\begin{aligned} \det A &= \det A_4 \det A_3 \det A_2 \det A_1 \\ &= -1 * 2 * 3 * 1 \\ &= -6 \end{aligned}$$

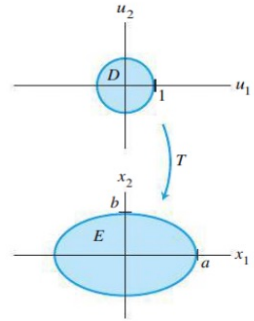
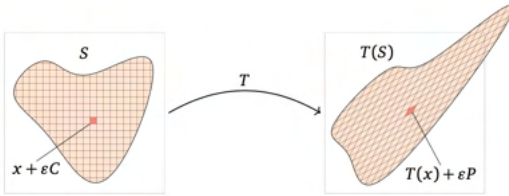
THEOREM 10

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A . If S is a parallelogram in \mathbb{R}^2 , then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \quad (5)$$

If T is determined by a 3×3 matrix A , and if S is a parallelepiped in \mathbb{R}^3 , then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \quad (6)$$

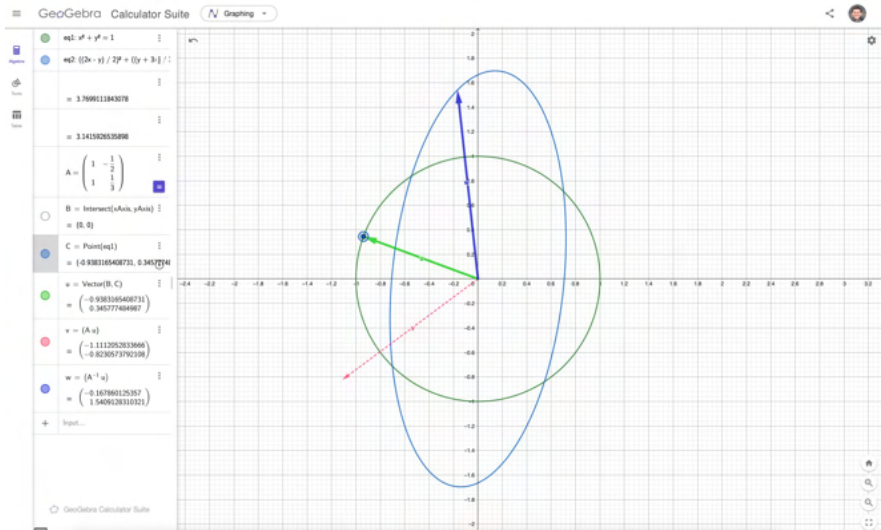


Example (Area of an ellipse). ^

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1.$$

<https://www.geogebra.org/calculator/mkxseqfjy>



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$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1.$$

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0), (5, 2), (6, 4), (11, 6)$
20. $(0, 0), (-2, 4), (4, -5), (2, -1)$
21. $(-2, 0), (0, 3), (1, 3), (-1, 0)$
22. $(0, -2), (5, -2), (-3, 1), (2, 1)$
23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.
24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$, and $(-1, 3, -1)$.
25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [Hint: Think about the columns of A .]
26. Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation, and let \mathbf{p} be a vector and S a set in \mathbb{R}^m . Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in \mathbb{R}^n .
27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.
28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$.
29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, \mathbf{v}_1 , and \mathbf{v}_2 in \mathbb{R}^2 .
30. Let R be the triangle with vertices at (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

[Hint: Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a , b , and c are

Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

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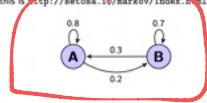
Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces
Math 1554 Linear Algebra

Example 1

- A small town has two libraries, A and B.
- After 1 month, among the books checked out of A,
 - 80% returned to A
 - 20% returned to B
- After 1 month, among the books checked out of B,
 - 30% returned to A
 - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is <http://www.math.uwaterloo.ca/markov/index.html>



$$P \cdot \vec{x}_0 = \vec{x}_1$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1100 \\ 900 \end{bmatrix}$$

$$1000 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} + 1000 \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1100 \\ 900 \end{bmatrix}$$

@ A Started @ A
 ← come back to A
 ← Started @ B
 ← never to A

$$1000(0.8) + 1000(0.3) = 1100$$

@ B started @ A
 ← started @ B

$$1000(0.2) + 1000(0.7) = 900$$

$$P \cdot \vec{x}_1 = \vec{x}_2$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1100 \\ 900 \end{bmatrix} = \begin{bmatrix} 1150 \\ 850 \end{bmatrix}$$

$$1100 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} + 900 \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1150 \\ 850 \end{bmatrix}$$

@ A
 $1100(0.8) + 900(0.3) = 1150$ ← @ A after 2 months

@ B
 $1100(0.2) + 900(0.7) = 850$ ← @ B after 2 months

$$P \vec{x}_k = \vec{x}_{k+1}$$

$$P \vec{x}_0 = \vec{x}_1$$

$$P \vec{x}_1 = \vec{x}_2 = P P \vec{x}_0 = P^2 \vec{x}_0$$

$$P^k \vec{x}_0 = \vec{x}_k$$

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Section 4.9 : Applications to Markov Chains

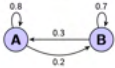
Chapter 4 : Vector Spaces
Math 1554 Linear Algebra

Section 4.9	Topic 2.2	5	2/5 - 2/9	2.3,2.4	WS2.2-2.4	2.5	WS2.5	2.8
		6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
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		8	2/26 - 3/1	5.2	WS5.1.5.2	Exam 2. Review	Cancelled	5.3

Example 1

- A small town has two libraries, A and B.
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- After 1 month, among the books checked out of B,
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 - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is <http://setosa.io/markov/index.html>



Example 1 Continued

The books are equally divided by between the two branches, denoted by $x_0 = \begin{bmatrix} 500 \\ 500 \end{bmatrix}$. What is the distribution after 1 month, call it x_1 ? After two months?

After k months, the distribution is x_k , which is what in terms of x_0 ?

clc

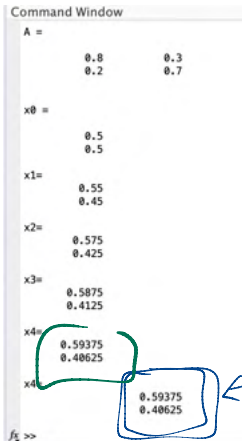
```
format shortG
% library example
A=[.8 .3 ; .2 .7]
x0=[.5;.5]
```

```
% car rental example
%A=[.8 .1 .2 ; .2 .8 .3 ; 0 .3 .5]
%x0=[.2; .2; .6]
```

```
% election example
%A=[.7 .1 .3 ; .2 .8 .3 ; .1 .1 .4]
%x0=[.55;.4;.05]
```

```
% set value of n (largest index to compute)
%n=10;
for i=1:n
    % convert current index to string and create xi string
    index=string(i);
    s=strcat('x',index, '=');
    % compute xi value
    xi=A^i*x0;
    % display each xi=A^i*x0
    disp(s)
    disp(xi)
end
```

```
% display final xn=A^n*x0
format longG
n=4;
index=string(n);
s=strcat('x',index, '=');
xi=A^n*x0;
disp(s)
disp(xi)
```



$0.59999 \dots 9$
 $0.40000 \dots 1$
 $\approx \begin{pmatrix} .60 \\ .40 \end{pmatrix}$

Q1: What is a Markov chain?
a sequence of probability vectors
s.t. $x_{k+1} = P x_k$, P stochastic matrix.

Q2: What questions might one ask about these?
What is the long term trend?
 $\lim_{k \rightarrow \infty} P^k x_0 = ??$

Q3: How do you specify a particular Markov chain?

$x_0 \in P$
Then I can generate
 x_0, x_1, x_2, \dots
the whole chain.

Markov Chains

A few definitions:

- A **probability vector** is a vector, \vec{x} , with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, P , whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors \vec{x}_k , and a stochastic matrix P , such that:

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

- A **steady-state vector** for P is a vector \vec{q} such that $P\vec{q} = \vec{q}$.

a probability vector that stays the same after multiplying by P .

$$P\vec{x} - \vec{x} = \vec{0}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(P - I)\vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{x} = r \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

Convergence

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

as $k \rightarrow \infty$.

Definition: a stochastic matrix P is **regular** if there is some k such that P^k only contains strictly positive entries.

Theorem

If P is a regular stochastic matrix, then P has a unique steady-state vector \vec{q} , and $\vec{x}_{k+1} = P\vec{x}_k$ converges to \vec{q} as $k \rightarrow \infty$.

Example 2

Determine a steady-state vector for the stochastic matrix

$$\text{Want } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ s.t. } \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = P$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$P\vec{x} = \vec{x}$$

$$\text{Solve } \begin{cases} 0.8x_1 + 0.3x_2 = x_1 \\ 0.2x_1 + 0.7x_2 = x_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.8x_1 - x_1 + 0.3x_2 = 0 \\ 0.2x_1 + 0.7x_2 - x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (0.8 - 1)x_1 + 0.3x_2 = 0 \\ 0.2x_1 + (0.7 - 1)x_2 = 0 \end{cases}$$

$$\begin{bmatrix} 0.8 - 1 & 0.3 \\ 0.2 & 0.7 - 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$$

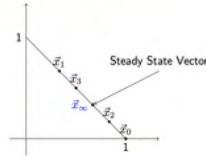
60% @ A
40% @ B

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.4 + 0.6 \\ 0.6 + 1.4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.48 + 0.12 \\ 0.12 + 0.28 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \checkmark$$

Stochastic Vectors in the Plane

The stochastic vectors in the plane are the line segment below, and a stochastic matrix maps stochastic vectors to themselves. Iterates $P^k \vec{x}_0$ converge to the steady state.



$$P^k \rightarrow [\vec{x}_\infty \quad \vec{x}_\infty]$$

How many books should be at each library so that same # of books next month @ each?

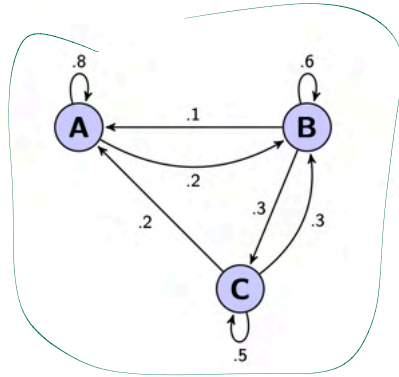
Example 3

A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

	rented from		
	A	B	C
returned to A	.8	.1	.2
returned to B	.2	.6	.3
returned to C	.0	.3	.5

There are 10 cars at each location today.

- Construct a stochastic matrix, P , for this problem.
- What happens to the distribution of cars after a long time? You may assume that P is regular.



$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

Section 4.9 Slide 206

put at

$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix}$$

← A
← B
← C

gongto ✓

Section 4.9 Slide 207

What does this mean?!

↓ \vec{x}_0

$$P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix} = \begin{bmatrix} .66 \\ .28 \\ .06 \end{bmatrix}$$

$$P^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .66 \\ .28 \\ .06 \end{bmatrix} = \vec{x}_2$$

$$P \left(\begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix} \right) = P \left(P \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right) = P^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{x}_2$$

$$P(\vec{x}_1) = P(P\vec{x}_0) = P^2\vec{x}_0 = \vec{x}_2$$

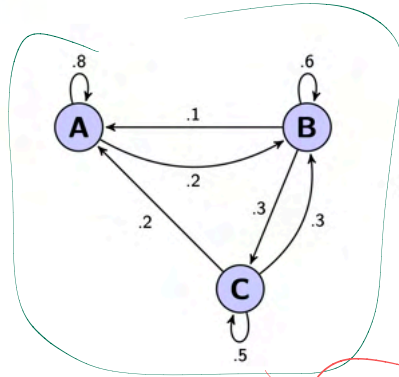
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There are 10 cars at each location today.

- Construct a stochastic matrix, P , for this problem.
- What happens to the distribution of cars after a long time? You may assume that P is regular.



$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

Section 4.9 Slide 206

put at

$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix}$$

← A
← B
← C

going to

ANS

~~$$\vec{q} = \begin{bmatrix} 11/6 \\ 5/3 \\ 1 \end{bmatrix}$$~~

Section 4.9 Slide 207

Q: Find steady state \vec{x}_0 probability vector st. $P\vec{x} = \vec{x}$

Idea solve $P\vec{x} = \vec{x} \Rightarrow P\vec{x} - \vec{x} = \vec{0} \Rightarrow (P - I)\vec{x} = \vec{0}$

opt 2
($b=6$)

$$\vec{x} = \begin{bmatrix} 11 \\ 10 \\ 6 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} 11/6 \\ 10/6 \\ 6/6 \end{bmatrix}$$

Sum is 27 $\div 27$

$$\begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.2 & .1 & .2 \\ .2 & -.4 & .3 \\ 0 & .3 & -.5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 2 & -4 & 3 \\ 0 & 3 & -5 \end{bmatrix}$$

$\vec{q} = \begin{pmatrix} 2 \\ 11 \\ 6 \end{pmatrix}$

$$\vec{q} = \frac{2}{9} \begin{bmatrix} 11/6 \\ 5/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/27 \\ 10/27 \\ 6/27 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 5 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -11/6 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 11/6 \\ 5/3 \\ 1 \end{bmatrix}$$

Current Sum $\frac{11}{6} + \frac{5}{3} + 1 = \frac{11+10+6}{6} = \frac{27}{6} = \frac{9}{2}$

THEOREM 18

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector q . Further, if x_0 is any initial state and $x_{k+1} = Px_k$ for $k = 0, 1, 2, \dots$, then the Markov chain $\{x_k\}$ converges to q as $k \rightarrow \infty$.

WARNING!!

Defn. P stochastic matrix is regular

if P^k has no zero entries for some $k \geq 1$.

Ex.
 $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ not regular.
 has zeros

$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ (also has 0 entries!)
 $P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 = P \cdot P^2 = P \cdot I = P$

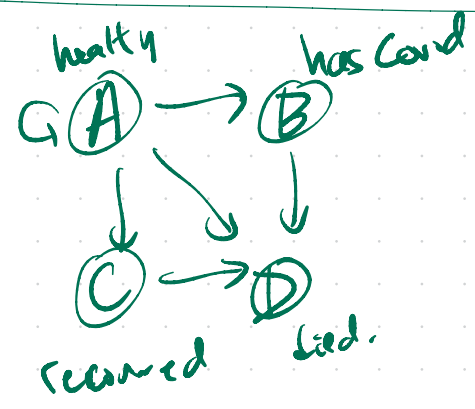
From last lecture

$P = \begin{bmatrix} .8 & .3 \\ -.2 & .7 \end{bmatrix}$

$\textcircled{A} \xrightarrow{.2} \textcircled{B}$
 $\textcircled{A} \xrightarrow{.3} \textcircled{B}$
 $\textcircled{A} \xrightarrow{.8} \textcircled{A}$
 $\textcircled{B} \xrightarrow{.7} \textcircled{B}$
 $\textcircled{B} \xrightarrow{.3} \textcircled{A}$

$P^k \begin{bmatrix} .5 \\ .5 \end{bmatrix} \approx \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has a unique steady state!
 $\hat{q} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$



THEOREM 18

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector q . Further, if x_0 is any initial state and $x_{k+1} = Px_k$ for $k = 0, 1, 2, \dots$, then the Markov chain $\{x_k\}$ converges to q as $k \rightarrow \infty$.

WARNING!!

Defn. P stochastic matrix is regular if P^k has no zero entries for some $k \geq 1$.

Ex.
 $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ not regular. $P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ (also has 0 entries!)
 $P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 = P \cdot P^2 = P \cdot I = P$

From last lecture
 $P = \begin{bmatrix} .8 & .3 \\ -.2 & -.7 \end{bmatrix}$ (A) $\xrightarrow{.2}$ (B) $P \cdot 7$
 $P^k \begin{bmatrix} .5 \\ .5 \end{bmatrix} \approx \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

4.9 EXERCISES

- A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.
 - Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.
 - Give the initial state vector.
 - What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?
- A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food

- on one trial, it will choose the same food on the next trial with a probability of 50%, and it will choose the other foods on the next trial with equal probabilities of 25%.
- What is the stochastic matrix for this situation?
 - If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?



- On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy

- tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.
- What is the stochastic matrix for this situation?
 - Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?
 - If a student is well today, what is the probability that he or she will be well two days from now?
- The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 60% chance the weather will be good tomorrow, a 30% chance the weather will be indifferent, and a 10% chance the weather will be bad. If the weather is indifferent today, it will be good tomorrow with probability .40 and indifferent with probability .30. Finally, if the weather is bad today, it will be good tomorrow with probability .40 and indifferent with probability .50.
 - What is the stochastic matrix for this situation?
 - Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?
 - Suppose the predicted weather for Monday is 40% indifferent weather and 60% bad weather. What are the chances for good weather on Wednesday?

In Exercises 5–8, find the steady-state vector.

- $\begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$
 - $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$
 - Determine if $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ is a regular stochastic matrix.
 - Determine if $P = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ is a regular stochastic matrix.
- Find the steady-state vector for the Markov chain in Exercise 1.
 - At some time late in the day, what fraction of the listeners will be listening to the news?
 - Refer to Exercise 2. Which food will the animal prefer after many trials?
 - Find the steady-state vector for the Markov chain in Exercise 3.
 - What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?
 - Refer to Exercise 4. In the long run, how likely is it for the weather in Columbus to be good on a given day?
 - [M] The Demographic Research Unit of the California State Department of Finance supplied data for the following migration matrix, which describes the movement of the United

EXAMPLE 1 Section 1.10 examined a model for population movement between a city and its suburbs. See Figure 1. The annual migration between these two parts of the metropolitan region was governed by the *migration matrix* M :

$$M = \begin{array}{cc} \text{From:} & \\ \begin{array}{c} \text{City} \quad \text{Suburbs} \end{array} & \begin{array}{c} \text{To:} \\ \text{City} \\ \text{Suburbs} \end{array} \\ \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} & \end{array}$$

That is, each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. The columns of M are probability vectors, so M is a stochastic matrix. Suppose the 2014 population of the region is 600,000 in the city and 400,000 in the suburbs. Then the initial distribution of the population in the region is given by \mathbf{x}_0 in (1) above. What is the distribution of the population in 2015? In 2016?



FIGURE 1 Annual percentage migration between city and suburbs.

EXAMPLE 4 The probability vector $\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ is a steady-state vector for the population migration matrix M in Example 1, because

$$M\mathbf{q} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .35625 + .01875 \\ .01875 + .60625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \mathbf{q} \quad \blacksquare$$

More
practice

EXAMPLE 2 Suppose the voting results of a congressional election at a certain voting precinct are represented by a vector \mathbf{x} in \mathbb{R}^3 :

$$\mathbf{x} = \begin{bmatrix} \% \text{ voting Democratic (D)} \\ \% \text{ voting Republican (R)} \\ \% \text{ voting Libertarian (L)} \end{bmatrix}$$

Suppose we record the outcome of the congressional election every two years by a vector of this type and the outcome of one election depends only on the results of the preceding election. Then the sequence of vectors that describe the votes every two years may be a Markov chain. As an example of a stochastic matrix P for this chain, we take

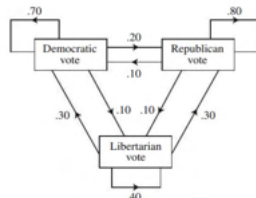


FIGURE 2 Voting changes from one election to the next.

More practice

	From:			
	D	R	L	To:
$P =$.70	.10	.30	D
	.20	.80	.30	R
	.10	.10	.40	L

SOLUTION The outcome of the next election is described by the state vector \mathbf{x}_1 and that of the election after that by \mathbf{x}_2 , where

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} \begin{array}{l} 44\% \text{ will vote D.} \\ 44.5\% \text{ will vote R.} \\ 11.5\% \text{ will vote L.} \end{array}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix} \begin{array}{l} 38.7\% \text{ will vote D.} \\ 47.8\% \text{ will vote R.} \\ 13.5\% \text{ will vote L.} \end{array}$$

To understand why \mathbf{x}_1 does indeed give the outcome of the next election, suppose 1000 persons voted in the "first" election, with 550 voting D, 400 voting R, and 50 voting L. (See the percentages in \mathbf{x}_0 .) In the next election, 70% of the 550 will vote D again, 10% of the 400 will switch from R to D, and 30% of the 50 will switch from L to D. Thus the total D vote will be

$$.70(550) + .10(400) + .30(50) = 385 + 40 + 15 = 440 \quad (2)$$

Thus 44% of the vote next time will be for the D candidate. The calculation in (2) is essentially the same as that used to compute the first entry in \mathbf{x}_1 . Analogous calculations could be made for the other entries in \mathbf{x}_1 , for the entries in \mathbf{x}_2 , and so on. ■

EXAMPLE 3 Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consider a system whose state is described by the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$, for $k = 0, 1, \dots$. What happens to the system as time passes? Compute the state vectors $\mathbf{x}_1, \dots, \mathbf{x}_{15}$ to find out.

more practice

SOLUTION

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

The results of further calculations are shown below, with entries rounded to four or five significant figures.

$$\begin{aligned} \mathbf{x}_4 &= \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, & \mathbf{x}_5 &= \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, & \mathbf{x}_6 &= \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, & \mathbf{x}_7 &= \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix} \\ \mathbf{x}_8 &= \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, & \mathbf{x}_9 &= \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, & \mathbf{x}_{10} &= \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, & \mathbf{x}_{11} &= \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix} \\ \mathbf{x}_{12} &= \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, & \mathbf{x}_{13} &= \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, & \mathbf{x}_{14} &= \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, & \mathbf{x}_{15} &= \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix} \end{aligned}$$

These vectors seem to be approaching $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$. The probabilities are hardly changing from one value of k to the next. Observe that the following calculation is exact (with no rounding error):

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

When the system is in state \mathbf{q} , there is no change in the system from one measurement to the next. ■

EXAMPLE 5 Let $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Find a steady-state vector for P .

more practice

SOLUTION First, solve the equation $P\mathbf{x} = \mathbf{x}$.

$$\begin{aligned} P\mathbf{x} - \mathbf{x} &= \mathbf{0} \\ P\mathbf{x} - I\mathbf{x} &= \mathbf{0} && \text{Recall from Section 1.4 that } I\mathbf{x} = \mathbf{x}. \\ (P - I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

For P as above,

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

To find all solutions of $(P - I)\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $x_1 = \frac{3}{4}x_2$ and x_2 is free. The general solution is $x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

Next, choose a simple basis for the solution space. One obvious choice is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

but a better choice with no fractions is $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (corresponding to $x_2 = 4$).

Finally, find a probability vector in the set of all solutions of $P\mathbf{x} = \mathbf{x}$. This process is easy, since every solution is a multiple of the solution \mathbf{w} above. Divide \mathbf{w} by the sum of its entries and obtain

$$\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

As a check, compute

$$P\mathbf{q} = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix} \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \mathbf{q} \quad \blacksquare$$

$$\begin{cases} x_1 + 3x_2 - 4x_3 = 2 \\ -x_1 - x_2 + 5x_3 = 3 \\ x_2 + 2x_3 = 1 \end{cases}$$

Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

$$x = c \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

$$Ax = b$$

$$AB = I_n$$

$$A^{-1}b = \vec{x}$$

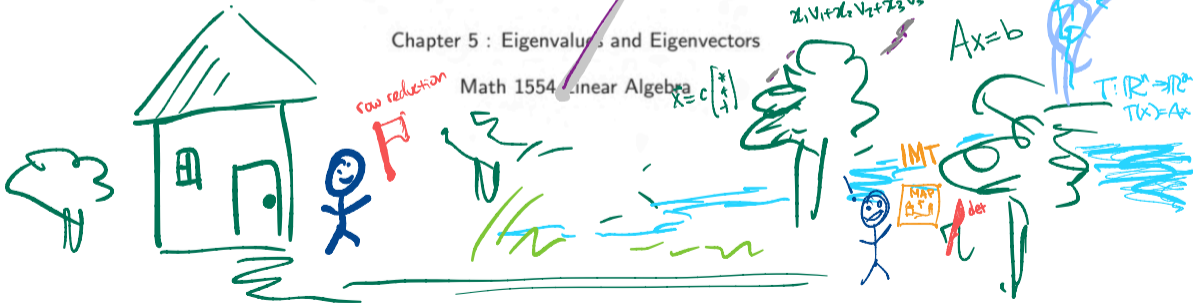
$$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

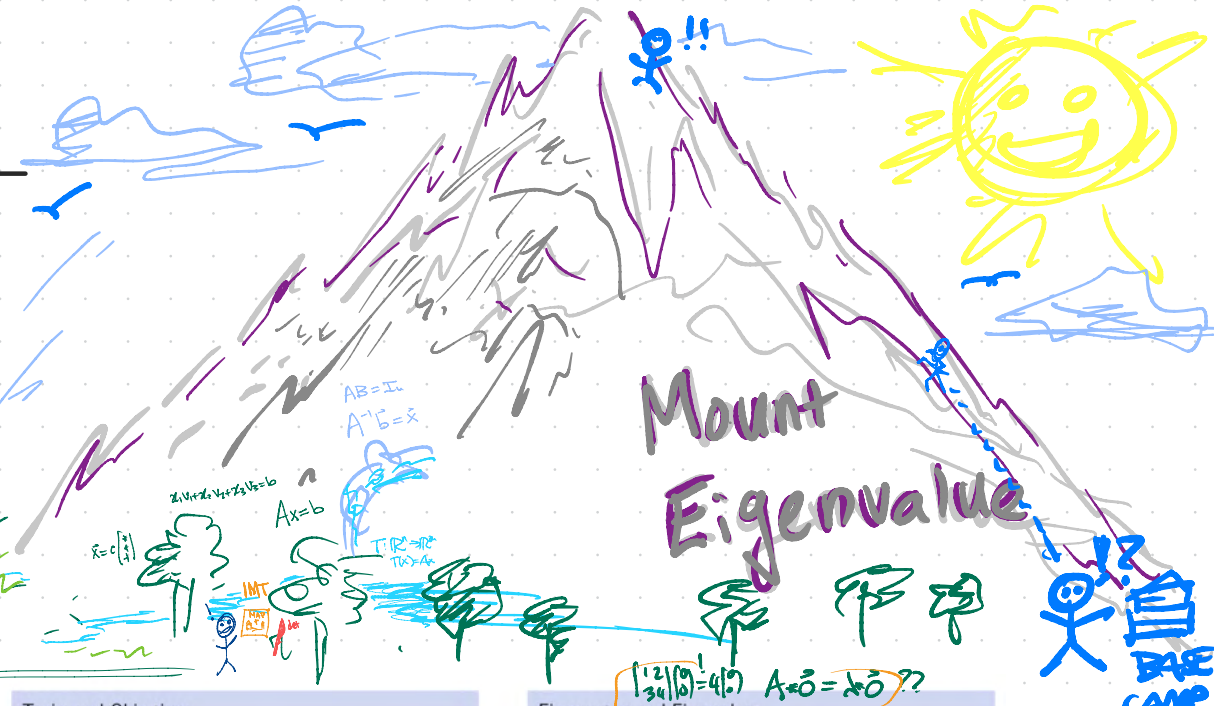
$$T(x) = Ax$$

IMT

MAP

det





Mount Eigenvalue

Topics and Objectives

Topics
We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

Eigenvectors and Eigenvalues

If $A \in \mathbb{R}^{n \times n}$, and there is a $\vec{v} \neq \vec{0}$ in \mathbb{R}^n , and

$$A\vec{v} = \lambda\vec{v}$$

then \vec{v} is an **eigenvector** for A , and $\lambda \in \mathbb{R}$ the corresponding **eigenvalue**.

- Note that
- We will only consider square matrices.
 - If $\lambda \in \mathbb{R}$, then
 - when $\lambda > 0$, $A\vec{v}$ and \vec{v} point in the same direction
 - when $\lambda < 0$, $A\vec{v}$ and \vec{v} point in opposite directions
 - Even when all entries of A and \vec{v} are real, λ can be complex (a rotation of the plane has no real eigenvalues.)
 - We explore complex eigenvalues in Section 5.5.

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v} \neq \vec{0}$$

Says \vec{v} is an eigenvector of A w/ eigenvalue.

5.1 EIGENVECTORS AND EIGENVALUES

Although a transformation $x \mapsto Ax$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of \vec{u} and \vec{v} under multiplication by A are shown in Figure 1. In fact, $A\vec{v}$ is just $2\vec{v}$. So A only "stretches," or dilates, \vec{v} .



FIGURE 1 Effects of multiplication by A .

As another example, readers of Section 4.9 will recall that if A is a stochastic matrix, then the steady-state vector \vec{q} for A satisfies the equation $A\vec{q} = \vec{q}$. That is, $A\vec{q} = 1 \cdot \vec{q}$.

5	9/18 - 9/22	2.3.2.4	WS2.2.2.3	2.5	WS2.4.2.5	2.8
6	9/25 - 9/29	2.9	WS2.8.2.9	3.1.3.2	WS3.1.3.2	3.3
7	10/2 - 10/6	4.9	WS3.3.4.9	5.1.5.2	WS5.1.5.2	5.2
8	10/9 - 10/13	Break	Break	Exam 2, Review	Cancelled	5.3

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

← Not an eigenvector

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A w/ eigenvalue $\lambda=2$.

Example 1

you try it!!

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

yes $\lambda = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$$

b) $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

yes $\lambda = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 0$

c) $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Section 5.1 Slide 212

Section 5.1 Slide 213

no
not an eigenvector
we exclude
it (v3 defn)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} ??$$

↑ true for any λ !!

$$3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

← linear comb of eigenvectors

Idea



$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left(3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = 3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 3 \cdot 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \cdot 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$T(\vec{x}) = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$$

Eigenspace

Definition

Suppose $A \in \mathbb{R}^{n \times n}$. The eigenvectors for a given λ span a subspace of \mathbb{R}^n called the λ -eigenspace of A .

Note: the λ -eigenspace for matrix A is $\text{Nul}(A - \lambda I)$.

Example 3

Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \lambda = -1, 2$$

Section 5.1 Slide 214

Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

Theorems

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.
2. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A .
3. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are linearly independent.

Section 5.1 Slide 215

Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$

Cols lin ind/dep?

Count pivots!

$\text{Nul}(A-3I) = \text{span}\{(-4, 1)\}$

Looking for $\vec{x} \neq \vec{0}$ s.t.

$A\vec{x} = 3\vec{x} \quad ??$

$A - 3I$

$= \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$

$\vec{x} = s \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\Rightarrow A\vec{x} - 3\vec{x} = \vec{0}$

$\Rightarrow (A - 3I)\vec{x} = \vec{0} \quad ??$

$\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is an eigenvector of A w/ $\lambda = 3$.

CHECK!

$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$= \begin{pmatrix} -12 \\ 3 \end{pmatrix}$

$= 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

non-zero $P\vec{q} = \vec{q} \Rightarrow P\vec{q} = 1 \cdot \vec{q}$
 Steady state vector
 ARE EIGENVECTORS w/ $\lambda = 1$

Eigenspace

Subspaces

Col A
Nul A

Theorems

Definition
 Suppose $A \in \mathbb{R}^{n \times n}$. The eigenvectors for a given λ span a subspace of \mathbb{R}^n called the λ -eigenspace of A .

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

Note: the λ -eigenspace for matrix A is $\text{Nul}(A - \lambda I)$.

- The diagonal elements of a triangular matrix are its eigenvalues.
- A invertible $\Leftrightarrow 0$ is not an eigenvalue of A .
- If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are linearly independent.

Example 3
 Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \lambda = -1.2$

$A\vec{x} = \lambda\vec{x} \Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$
 $\vec{x} \in \text{Nul}(A - \lambda I)$

eigenspace for $\lambda = -1$

$\text{Nul}(A + I) = \text{span}\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$

basis is $\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$

$\lambda = -1$

Find a basis for $\text{Nul}(A - \lambda I)$

$A - \lambda I = A - (-1)I = A + I$

$= A + I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \vec{x} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

check $(s=1)$ $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(s=3)$ $A \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = (-1) \begin{pmatrix} 3 \\ 3 \end{pmatrix} \checkmark$

Eigenspace

pages Null A

Definition

Suppose $A \in \mathbb{R}^{n \times n}$. The eigenvectors for a given λ span a subspace of \mathbb{R}^n called the λ -eigenspace of A .

Note: the λ -eigenspace for matrix A is $\text{Null}(A - \lambda I)$.

$A\vec{x} = \lambda\vec{x}$
 $\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$
 $\Rightarrow \vec{x} \in \text{Null}(A - \lambda I)$

Example 3

Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \lambda = -1, 2$

you find $A = 2I$ for $\lambda = 2$

$\lambda = -1$

Find a basis for $\text{Null}(A - \lambda I)$

$A - \lambda I = A - (-1)I = A + I$
 $= A + I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check $(s=1)$ $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 2$ $A \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = (-1) \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ ✓

Theorems

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.

2. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A .

3. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are linearly independent.

e.g. $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 9$

$A\vec{x} = 0 \Rightarrow \vec{x} = \vec{0}$
 $\vec{x} \neq \vec{0}$

$\text{Null}(A - \lambda I)$
 $\lambda = 0$ Eigen space

eigenspace for $\lambda = -1$

$\text{Null}(A + I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

T/F If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ lin ind eigenvectors of A
 $\Rightarrow \lambda_1, \lambda_2, \lambda_3$ all different.
FALSE

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$
 $\vec{e}_1, \vec{e}_2, \vec{e}_3$
 all e-vectors w/ $\lambda = 1$.

Warning!

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

Example: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- But the reduced echelon form of A is:
- The reduced echelon form is triangular, and its eigenvalues are:

Additional Resource

3Blue1Brown

A beautiful, animated, and visual explanation of eigenvalues and eigenvectors.

<http://bit.ly/2lKjJPg>

Section 5.1 Slide 224

Section 5.1 Slide 227

DEFINITION

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution \mathbf{x} of $A\mathbf{x} = \lambda\mathbf{x}$; such an \mathbf{x} is called an *eigenvector corresponding to λ* .¹

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are \mathbf{u} and \mathbf{v} eigenvectors of A ?

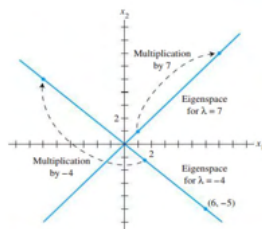


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

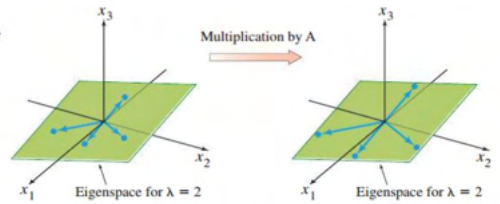


FIGURE 3 A acts as a dilation on the eigenspace.

THEOREM 2

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

5.1 EXERCISES

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?
2. Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not?
3. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If so, find the eigenvalue.
4. Is $\begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$? If so, find the eigenvalue.
5. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.
6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.
7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.
8. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

274 CHAPTER 5 Eigenvalues and Eigenvectors

9. $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1, 5$
10. $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$, $\lambda = 4$
11. $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$, $\lambda = 10$
12. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$, $\lambda = 1, 5$
13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\lambda = 1, 2, 3$
14. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$, $\lambda = -2$
15. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$, $\lambda = 3$
16. $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $\lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ 18. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

19. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify your answer.

In Exercises 21 and 22, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

21. a. If $Ax = \lambda x$ for some vector x , then λ is an eigenvalue of A .
 b. A matrix A is not invertible if and only if 0 is an eigenvalue of A .
 c. A number c is an eigenvalue of A if and only if the equation $(A - cI)x = 0$ has a nontrivial solution.

- d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.
 e. To find the eigenvalues of A , reduce A to echelon form.

22. a. If $Ax = \lambda x$ for some scalar λ , then x is an eigenvector of A .
 b. If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
 c. A steady-state vector for a stochastic matrix is actually an eigenvector.
 d. The eigenvalues of a matrix are on its main diagonal.
 e. An eigenspace of A is a null space of a certain matrix.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

24. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

25. Let λ be an eigenvalue of an invertible matrix A . Show that λ^{-1} is an eigenvalue of A^{-1} . [Hint: Suppose a nonzero x satisfies $Ax = \lambda x$.]

26. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T . [Hint: Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

28. Use Exercise 27 to complete the proof of Theorem 1 for the case when A is lower triangular.

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s . Show that s is an eigenvalue of A . [Hint: Use an eigenvector.]

30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s . Show that s is an eigenvalue of A . [Hint: Use Exercises 27 and 29.]

In Exercises 31 and 32, let A be the matrix of the linear transformation T . Without writing A , find an eigenvalue of A and describe the eigenspace.

31. T is the transformation on \mathbb{R}^2 that reflects points across some line through the origin.

32. T is the transformation on \mathbb{R}^3 that rotates points about some line through the origin.

33. Let u and v be eigenvectors of a matrix A , with corresponding eigenvalues λ and μ , and let c_1 and c_2 be scalars. Define $x_k = c_1 \lambda^k u + c_2 \mu^k v$ ($k = 0, 1, 2, \dots$)

- a. What is x_{k+1} , by definition?
 b. Compute Ax_k from the formula for x_k , and show that $Ax_k = x_{k+1}$. This calculation will prove that the sequence $\{x_k\}$ defined above satisfies the difference equation $x_{k+1} = Ax_k$ ($k = 0, 1, 2, \dots$).

34. Describe how you might try to build a solution of a difference equation $x_{k+1} = Ax_k$ ($k = 0, 1, 2, \dots$) if you were given the initial x_0 and this vector did not happen to be an eigenvector of A . [Hint: How might you relate x_0 to eigenvectors of A ?]

35. Let u and v be the vectors shown in the figure, and suppose u and v are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by $T(x) = Ax$ for each x in \mathbb{R}^2 , and let $w = u + v$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(u)$, $T(v)$, and $T(w)$.



36. Repeat Exercise 35, assuming u and v are eigenvectors of A that correspond to eigenvalues -1 and 3 , respectively.

[M] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

37. $\begin{bmatrix} 8 & -10 & -5 \\ 2 & 17 & 2 \\ -9 & -18 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ -42 & -33 & 21 & -45 \end{bmatrix}$

39. $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$

40. $\begin{bmatrix} -4 & -4 & 20 & -8 & -1 \\ 14 & 12 & 46 & 18 & 2 \\ 6 & 4 & -18 & 8 & 1 \\ 11 & 7 & -37 & 17 & 2 \\ 18 & 12 & -60 & 24 & 5 \end{bmatrix}$