Math 2602 **Exam 1** Spring 2015

Instructor: Sal Barone

Name: KEY.

GT username: ____________________

Circle your TA/section:

(F1) Qiqin (F2) Changong (F3) George (L1) Scott (L2) Vaidehi

1. No books or notes are allowed.
2. You will not need a calculator for this exam. All electronic devices are not allowed.
3. Show all work and fully justify your answer to receive full credit.
4. Good luck!

Page	Max. Possible	Points
1	24	
2	24	
3	24	
4	28	
Total	100	

1. Let $A = \{1, 2, 3\}$.

(a) Give an example of a relation $\mathcal{R}$ on A which is reflexive, not symmetric, and not transitive. (8 pts.)

For example,

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$$

(b) Explain why your relation $\mathcal{R}$ is not transitive. (4 pts.)

Not transitive since

$(1,2) \in \mathcal{R} \wedge (2,3) \in \mathcal{R}$ but $(1,3) \notin \mathcal{R}$.

2. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if $g \circ f$ is one-to-one, then f is one-to-one. (8 pts.)

Proof: ~~Suppose $g \circ f$ is one-to-one.~~

~~(Proof by contradiction) Suppose, seeking~~

(Prove the contrapositive)

Suppose f is not one-to-one.

Then $\exists x_1, x_2 \in A$ s.t. $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Thus $g(f(x_1)) = g(f(x_2))$, but $x_1 \neq x_2$. So $g \circ f(x_1) = g \circ f(x_2)$ but $x_1 \neq x_2$, so $g \circ f$ is not one-to-one. ∎

(b) Write down the converse of the statement you proved above and give an example of functions f, g where it is false. (4 pts.)

Converse: If f is one-to-one then so is $g \circ f$.

False consider $f : \{1,2,3\} \to \{a,b,c\}$ and $g : \{a,b,c\} \to \{z\}$

$1 \mapsto a$ $\quad a \mapsto z$

$2 \mapsto b$ $\quad b \mapsto z$

$3 \mapsto c$ $\quad c \mapsto z$.

Then f is one-to-one but $g \circ f(x) = z$ $\forall x \in \{1,2,3\}$

so $g \circ f$ is not 1-1. ∎

3. Prove that the arguments below are valid. (12 pts. each)

(a)

$$\begin{array}{cc} ① & p \vee q \\ ② & \neg p \\ ③ & q \to r \\ ④ & s \to \neg r \\ \hline ⑤ & \neg s \end{array}$$

Proof. Suppose all premises are true.

① is true so either p or q is true.

② is true so p is false, hence by ① q is true.

③ is true and q is true, so r is true.

④: If s were true then since ④ is true r would be false, but r is true so s must be false.

Hence ⑤ is true: s is false. ∎

4.

$$\begin{array}{c} p \vee (q \wedge \neg r) \\ \neg q \to \neg r \\ \hline p \end{array}$$

Proof. Suppose the conclusion is false.

Then p is false. If the first premise is true then $(q \wedge \neg r)$ is true since p is false.

But $(q \wedge \neg r) \Leftrightarrow \neg(r \to q)$,

and $(\neg q \to \neg r) \Leftrightarrow r \to q$.

Therefore, if the conclusion is false and premise 1 is true, then premise 2 is false.

In particular, the argument is valid. ∎

5. Short answer section: put a number, statement, or argument in each box. Please show your work for potential partial credit. (6 pts. each)

(i) Give an example of an invalid argument with at least two premises using only one atomic variable p.

For example:

$$\begin{array}{c} p \\ p \vee \neg p \\ \hline \neg p \end{array}$$

(ii) Give an example of an implication which is always true using the atomic variables p, q.

for example:

$$(p \wedge \neg p) \rightarrow q$$

(iii) Simplify the negation of the statement $p \vee (q \rightarrow r)$.

$$\boxed{\neg p \wedge q \wedge \neg r}$$

$$\neg(p \vee (q \rightarrow r)) \Leftrightarrow \neg p \wedge \neg(q \rightarrow r)$$

$$\Leftrightarrow \neg p \wedge q \wedge \neg r.$$

(iv) Let $A = \{a, b\}$. Then, the number of subsets of A equals $\boxed{4}$

Subsets of A are

$\phi, \{a\}, \{b\}, A.$

5. True and false questions. Instructions: for each statement below, circle TRUE if the statement is always true and circle FALSE otherwise. Your work will NOT be graded. (4 pts. each)

(i) The relation $\mathcal{R} = \{(a, b) \in \mathbb{Z}^2 \mid a - b \geq 0\}$ is reflexive, anti-symmetric, and transitive. TRUE FALSE

(ii) The statement $\overset{F}{q} \overset{T}{\rightarrow} (\neg p \rightarrow q) \Leftrightarrow \mathbf{T}$ is a neither a contradiction nor a tautology. TRUE FALSE

(iii) The statement $\neg p \wedge (q \rightarrow r)$ is true when p, q, and r are all true. TRUE FALSE

false? $\neg p$ is false so the "and" is false.

(iv) For any $a \in \mathbb{Z}$, the number a is odd if and only if the number a^2 is odd. TRUE FALSE

(v) $\forall r \in \mathbb{R}(\exists n \in \mathbb{Z}(n \leq r < n + 1))$. TRUE FALSE

(vi) There are n equivalence classes of integers under the equivalence relation remainder modulo n. TRUE FALSE

(vii) If A, B are sets and both $A \subseteq B$ and $A \cap B = \emptyset$, then $B = \emptyset$. TRUE FALSE

$A = \emptyset$ $B = \{1, 2\}$

Then $A \subseteq B$ and $A \cap B = \emptyset$ but $B \neq \emptyset$.