Math 1552, Integral Calculus

Section 8.5: Partial Fractions

Evaluate the following integrals using any method we have learned.

1. $\int \frac{x+3}{(x-1)(x^2-4x+4)}dx$

The denominator factors into $(x-1)(x-2)^2$, so the partial fraction decomposition is:

$$\frac{x+3}{(x-1)(x^2-4x+4)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Solving for A, B, C yields $A = 4$, $B = -4$, and $C = 5$, so:

$$\begin{aligned}\int \left[\frac{4}{x-1} - \frac{4}{x-2} + \frac{5}{(x-2)^2}\right] dx &= 4\ln|x-1| - 4\ln|x-2| - \frac{5}{x-2} + C \\ &= 4\ln\left|\frac{x-1}{x-2}\right| - \frac{5}{x-2} + C.\end{aligned}$$

2. $\int \frac{x+4}{x^3+x}dx$

The denominator factors into $x(x^2+1)$, so the partial fraction decomposition is:

$$\frac{x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Solving yields $A = 4$, $B = -4$, and $C = 1$, so the integral becomes:

$$\begin{aligned}\int \left[\frac{4}{x} + \frac{-4x+1}{x^2+1}\right] dx &= \int \left[\frac{4}{x} - \frac{4x}{x^2+1} + \frac{1}{x^2+1}\right] dx \\ &= 4\ln|x| - 2\ln(x^2+1) + \tan^{-1}(x) + C.\end{aligned}$$

3. $\int x^5 \ln(x)dx$

Integration by parts: let $u = \ln x$ and $dv = x^5 dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^6}{6}$, so:

$$\begin{aligned}\int x^5 \ln(x)dx &= \frac{x^6 \ln x}{6} - \frac{1}{6}\int x^5 dx \\ &= \frac{x^6 \ln x}{6} - \frac{x^6}{36} + C.\end{aligned}$$

4. $\int \tan(x)\ln[\cos(x)]dx$

u-substitution: let $u = \ln[\cos(x)]$, then $du = -\tan x dx$:

$$\int \tan(x) \ln[\cos(x)] dx = -\int u du = -\frac{1}{2}(\ln[\cos(x)])^2 + C.$$

5. $\int \frac{x+2}{x+1} dx$

Carrying out the long division, we have:

$$\int \frac{x+2}{x+1} dx = \int \left(1 + \frac{1}{x+1}\right) dx = x + \ln|x+1| + C.$$

6. $\int \sqrt{25 - x^2} dx$

Trig substitution: let $x = 5\sin\theta$, then $dx = 5\cos\theta d\theta$. Then

$$\begin{aligned}
\int \sqrt{25 - x^2} dx &= \int (\sqrt{25 - 25\sin^2\theta})(5\cos\theta) d\theta \\
&= \int (\sqrt{25\cos^2\theta})(5\cos\theta) d\theta \\
&= \int (5\cos\theta)(5\cos\theta) d\theta \\
&= 25 \int \cos^2\theta d\theta \\
&= \frac{25}{2} \int [1 + \cos(2\theta)] d\theta \\
&= \frac{25}{2} \left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{25}{2} \left[\sin^{-1}\left(\frac{x}{5}\right) + \frac{x}{5} \cdot \frac{\sqrt{25 - x^2}}{5}\right] + C \\
&= \frac{25}{2} \sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25 - x^2}}{2} + C.
\end{aligned}$$

7. $\int \tan^3(x) \sec^4(x) dx$

$$\begin{aligned}
\int \tan^3(x) \sec^4(x) dx &= \int \tan^3(x) \sec^2(x) \sec^2(x) dx \\
&= \int \tan^3(x)[1 + \tan^2(x)] \sec^2(x) dx \\
&= \int [\tan^3(x) + \tan^5(x)] \sec^2(x) dx \\
&= \int (u^3 + u^5) du \quad [u = \tan(x)] \\
&= \frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C.
\end{aligned}$$

8. $\int x \tan^{-1}(x) dx$

Integration by parts: let $u = \tan^{-1}(x)$ and $dv = xdx$. Then $du = \frac{1}{1+x^2}dx$ and $v = \frac{x^2}{2}$, so

$$\begin{aligned}
\int x \tan^{-1}(x) dx &= \frac{x^2}{2}\tan^{-1}(x) - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{1+x^2} dx \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\left[x - \tan^{-1}(x)\right] + C \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C.
\end{aligned}$$

9. $\int \frac{dx}{x\sqrt{1+x^2}}$

Trig substitution: let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$. So

$$\begin{aligned}
\int \frac{dx}{x\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}} d\theta \\
&= \int \frac{\sec^2\theta}{\tan\theta\sqrt{\sec^2\theta}} d\theta \\
&= \int \frac{\sec^2\theta}{\tan\theta\sec\theta} d\theta \\
&= \int \frac{\sec\theta}{\tan\theta} d\theta \\
&= \int \frac{1/\cos\theta}{\sin\theta/\cos\theta} d\theta \\
&= \int \frac{1}{\sin\theta} d\theta \\
&= \int \csc(\theta) d\theta \\
&= -\ln|\csc\theta + \cot\theta| + C \\
&= -\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C.
\end{aligned}$$

10. $\int \frac{x+1}{x^2(x-1)} dx$

Using partial fractions:

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiply both sides by the common denominator, and combine like terms to get

$$x + 1 = (A + C)x^2 + (-A + B)x - B.$$

Solving for A, B, and C yields $A = -2$, $B = -1$, and $C = 2$. Then

$$\begin{aligned}\int \frac{x+1}{x^2(x-1)}dx &= \int \left(\frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\right)dx \\ &= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.\end{aligned}$$

11. $\int \frac{x+1}{x^2-4x+8}dx$

Since we cannot factor the denominator any further, we use the method for rational functions. First, we try to obtain the derivative of the denominator in the numerator, then we break up our problem into two separate integrals:

$$\begin{aligned}\int \frac{x+1}{x^2-4x+8}dx &= \frac{1}{2}\int \frac{2x+2}{x^2-4x+8}dx \\ &= \frac{1}{2}\int \frac{2x-4+4+2}{x^2-4x+8}dx \\ &= \frac{1}{2}\int \frac{2x-4}{x^2-4x+8}dx + 3\int \frac{dx}{x^2-4x+8} \\ &= \frac{1}{2}\ln|x^2-4x+8| + 3\int \frac{dx}{x^2-4x+4-4+8} \\ &= \frac{1}{2}\ln|x^2-4x+8| + 3\int \frac{dx}{(x-2)^2+4} \\ &= \frac{1}{2}\ln|x^2-4x+8| + \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C.\end{aligned}$$