Review for Test 3

Math 1552, Integral Calculus

Sections 8.8, 10.1-10.5

1. Terminology review: complete the following statements.

(a) A geometric series has the general form $\sum_{k=0}^{\infty} r^k$. The series converges when
$\underline{|r| \text{ is less than one}}$ and diverges when
$\underline{|r| \text{ is greater than or equal to one}}$.

(b) A p-series has the general form $\sum_{k=1}^{\infty} \frac{1}{k^p}$. The series converges when
$\underline{p \text{ is greater than one}}$
and diverges when $\underline{p \text{ is less than or equal to one}}$. To show these results, we can use the <u>integral</u> test.

(c) The harmonic series <u>diverges</u> and telescoping series <u>converge</u>.

(d) If you want to show a series converges, compare it to a <u>larger</u> series that also converges. If you want to show a series diverges, compare it to a <u>smaller</u> series that also diverges.

(e) If the direct comparison test does not have the correct inequality, you can instead use the <u>limit comparison</u> test. In this test, if the limit is a <u>finite, positive</u> number (not equal to 0) then both series converge or both series diverge.

(f) In the ratio and root tests, the series will <u>converge</u> if the limit is less than 1 and <u>diverge</u> if the limit is greater than 1. If the limit equals 1, then the test is <u>INCONCLUSIVE</u>.

(g) If $\lim_{n\to\infty} a_n = 0$, then what do we know about the series $\sum_k a_k$?
<u>absolutely NOTHING!</u>

(h) An integral is improper if either one or both limits of integration are <u>infinite</u>, or the function has a <u>vertical asymptote</u> on the interval $[a, b]$.

(i) A **sequence** is an infinite <u>list</u> of terms.
A sequence $\{a_n\}$ converges if: $\underline{\text{the limit of the terms exists and is finite as } n \to \infty}$.

(j) The smallest value that is greater than or equal to every term in a sequence is called the least upper bound (l.u.b.). The largest value that is less than or equal to every term in the sequence is called the greatest lower bound (g.l.b). If both of these values are finite, then we say the sequence is bounded.

(k) A sequence is called monotonic if the terms are increasing, monotonically increasing, decreasing, or monotonically decreasing. If a sequence is both monotonic and bounded, then we know it must converge.

2. Sum the series

$$\sum_{k=2}^{\infty} \frac{4^{2k}-1}{17^{k-1}}.$$

Solution:

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{4^{2k}-1}{17^{k-1}} &= \sum_{k=2}^{\infty} \left(\frac{4^{2k}}{17^{k-1}}\right) - \sum_{k=2}^{\infty} \frac{1}{17^{k-1}} \\
&= \sum_{k=2}^{\infty} \frac{16^k}{17^k \cdot 17^{-1}} - \sum_{k=2}^{\infty} \frac{1}{17^k \cdot 17^{-1}} \\
&= 17\sum_{k=2}^{\infty} \left(\frac{16}{17}\right)^k - 17\sum_{k=2}^{\infty} \left(\frac{1}{17}\right)^k \\
&= 17\left[\frac{1}{1-\frac{16}{17}} - 1 - \frac{16}{17}\right] - 17\left[\frac{1}{1-\frac{1}{17}} - 1 - \frac{1}{17}\right] = 255\frac{15}{16}.
\end{aligned}$$

3. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)}.$$

Solution: Using partial fractions on the telescoping series, we see that:

$$\frac{1}{(2k-1)(2k+3)} = \frac{1/4}{2k-1} - \frac{1/4}{2k+3},$$

so the series becomes:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)} &= \frac{1}{4}\sum_{k=1}^{\infty} \left[\frac{1}{2k-1} - \frac{1}{2k+3}\right] \\
&= \frac{1}{4}\left[(1-\frac{1}{5}) + (\frac{1}{3}-\frac{1}{7}) + (\frac{1}{5}-\frac{1}{9}) + (\frac{1}{7}-\frac{1}{11}) + ...\right] \\
&= \frac{1}{4}\left[1+\frac{1}{3}\right] = \frac{1}{3}.
\end{aligned}$$

4. Determine whether the following series converge or diverge. Justify your answers using the tests we discussed in class.

(a) $\sum_{k=1}^{\infty} \frac{e^k}{(1+4e^k)^{3.2}}$

Solution: Use the Integral Test. We first check the conditions to apply this test. Since $k \geq 1$, $e^k > 0$ so the function is positive and continuous. Letting $f(x) = \frac{e^x}{(1+4e^x)^{3.2}}$, we can find the derivative:

$$f'(x) = \frac{e^x(1 - 8.8e^x)}{(1 + 4e^x)^{4.2}}.$$

Note that $f'(x) < 0$ when $x \geq 1$, so the function is decreasing. Now evaluate the integral:

$$\begin{aligned}
\int_1^{\infty} \frac{e^x}{(1+4e^x)^{3.2}} dx &= \lim_{b\to\infty} \int_1^b \frac{e^x}{(1+4e^x)^{3.2}} \\
&= \frac{1}{4} \lim_{b\to\infty} \int_{x=1}^{x=b} \frac{1}{u^{3.2}} du \quad (u = 1 + 4e^x) \\
&= \frac{1}{4} \lim_{b\to\infty} \left[-\frac{1}{2.2(1+4e^x)^{2.2}} \right]_1^b \\
&= -\frac{1}{8.8} \lim_{b\to\infty} \left[\frac{1}{(1+4e^b)^{2.2}} - \frac{1}{(1+4e)^{2.2}} \right] \\
&= -\frac{1}{8.8} (0 - \frac{1}{(1+4e)^{2.2}}) = \frac{1}{8.8(1+4e)^{2.2}}.
\end{aligned}$$

Since the integral converges, the series also **converges**.

(b) $\sum_{k=2}^{\infty} \left(\frac{k-5}{k}\right)^{k^2}$

Solution: Use the Root Test:

$$\lim_{n\to\infty} \left[\left(\frac{n-5}{n} \right)^{n^2} \right]^{1/n} = \lim_{n\to\infty} \left(1 - \frac{5}{n}\right)^n = e^{-5}.$$

Since the limit is $e^{-5} < 1$, the series **converges** by the Root Test.

(c) $\sum_{k=1}^{\infty} \frac{k^2 \cdot 2^{k+1}}{k!}$

Solution: Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+2}}{(n+1)!} \cdot \frac{n!}{n^2 \cdot 2^{n+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^n \cdot 2^2 \cdot n!}{(n+1) \cdot n! \cdot n^2 \cdot 2^n \cdot 2} \\
&= \lim_{n\to\infty} \frac{2(n+1)}{n^2} = 0.
\end{aligned}$$

Since the limit is 0, which is less than 1, the series **converges** by the ratio test.

(d) $\sum_{k=1}^{\infty} \frac{1}{1+2+3+...+k}$

Solution: Recall the formula: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, so we can rewrite this series as:

$$\sum_{k=1}^{\infty} \frac{1}{1+2+3+...+k} = \sum_{k=1}^{\infty} \frac{2}{k(k+1)}.$$

The series is telescoping, so it converges and we can find its sum using partial fractions:

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)} = 2\sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 2.$$

Alternately, we can compare to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Since $k+1 > k$, we see that $\frac{1}{k+1} < \frac{1}{k}$ and thus $\frac{2}{k(k+1)} < \frac{2}{k^2}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-series with $p = 2 > 1$), $\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so the series $\sum_{k=1}^{\infty} \frac{1}{1+2+3+...+k} = \sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ also converges by the Basic Comparison Test.

5. For each sequence, determine: (i) the l.u.b. and g.l.b.; (ii) whether the sequence is monotonic; (iii) whether the sequence converges or diverges, and the limit if it is convergent.

(a) $\left\{\left(\frac{n}{n+2}\right)^{3n}\right\}$

Solution: First, let's find the limit:

$$\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^{3n} = \left[\lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{2}{n}\right)^n}\right)\right]^3 = \left(\frac{1}{e^2}\right)^3 = \frac{1}{e^6},$$

so the sequence converges.

Since the sequence is convergent, it is bounded. Writing out a few terms, we can see the sequence is decreasing (take the derivative to confirm). Therefore, it is monotonic, and l.u.b.=$\frac{1}{27}$ (the first term), g.l.b.=$\frac{1}{e^6}$ (limit).

(b) $\left\{\frac{\cos(n\pi)}{4^n}\right\}$

Solution: Note that since $-1 \le \cos(n\pi) \le 1$, we see that:

$$-\frac{1}{4^n} \le \frac{\cos(n\pi)}{4^n} \le \frac{1}{4^n}.$$

Since $\lim_{n\to\infty} -\frac{1}{4^n} = \lim_{n\to\infty}\frac{1}{4^n} = 0$, by the Sandwich Theorem, we also have $\lim_{n\to\infty}\frac{\cos(n\pi)}{4^n} = 0$, so the sequence converges.

Since the terms alternate, the sequence is bounded, but it is **not** monotonic. We find that l.u.b.=$\frac{1}{16}$ (first positive term) and g.l.b.=$-\frac{1}{4}$ (first negative term).

(c) $\left\{(-1)^n\frac{n+2}{n+4}\right\}$

Solution: Note that $\lim_{n\to\infty}\frac{n+2}{n+4} = 1$, so when n is odd, $(-1)^n = -1$ and the terms approach -1, but when n is even, $(-1)^n = +1$ and the terms approach 1. As the limits for different values of n are not equal, there is no limit, and thus the sequence **diverges**.

Since we have limits for values where n is positive and negative, the sequence is bounded, and those limits are the bounds. Thus, l.u.b.=1 and g.l.b=-1. The sequence alternates signs, so it is **not** monotonic.

6. Determine if the improper intergral converges or diverges. If it converges, evaluate the integral.

(a)

$$\int_2^\infty \frac{x}{(x^2-1)^{3/2}}dx.$$

Solution: Let $u = x^2 - 1$, then $du = 2xdx$, so we have:

$$\begin{aligned}
\lim_{b\to\infty} \int_{x=2}^{x=b} \frac{1}{2}u^{-3/2}du &= \lim_{b\to\infty} -\frac{1}{\sqrt{x^2-1}}|_2^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{\sqrt{b^2-1}} + \frac{1}{\sqrt{3}}\right] = \frac{1}{\sqrt{3}}.
\end{aligned}$$

Thus, the integral converges.

(b)

$$\int_0^2 \frac{dx}{x^2-5x+6}.$$

Solution: Using partial fractions:

$$\begin{aligned}
\int_0^2 \frac{dx}{x^2-5x+6} &= \lim_{c\to 2^-} \int_0^c \left[\frac{1}{x-3} - \frac{1}{x-2}\right] dx \\
&= \lim_{c\to 2^-} \ln\left|\frac{x-3}{x-2}\right| |_0^c \\
&= \lim_{c\to 2^-} \left[\ln\left|\frac{c-3}{c-2}\right| - \ln\frac{3}{2}\right] = \infty,
\end{aligned}$$

so the integral diverges.