Math 1552, Integral Calculus

Section 5.6: Area Between Curves

1. Suppose that $y = f(x)$ and $y = g(x)$ are both continuous functions on the interval $[a, b]$. Determine if each statement below is always true or sometimes false.

(a) Suppose that $f(c) > g(c)$ for some number $c \in (a, b)$. Then the area bounded by f, g, $x = a$, and $x = b$ can be found by evaluating the integral $\int_a^b (f(x) - g(x))\ dx$. [FALSE; THERE MAY BE A POINT WHERE THE FUNCTIONS SWAP FROM "TOP" TO "BOTTOM"]

(b) If $\int_a^b (f(x) - g(x))\ dx$ evaluates to -5, then the area bounded by f, g, $x = a$, and $x = b$ is 5. [FALSE; THE AREA MAY BE NEGATIVE ON ONE PIECE AND POSITIVE ON ANOTHER–THIS IS ABSOLUTE NET AREA, NOT TOTAL AREA]

(c) If $f(x) > g(x)$ for every $x \in [a, b]$, then $\int_a^b |f(x) - g(x)|\ dx = \int_a^b (f(x) - g(x))\ dx$. [TRUE]

2. Find the area bounded by the region between the curves $f(x) = x^3 + 2x^2$ and $g(x) = x^2 + 2x$.

Solution: The curves intersect when $x = 0, 1, -2$. Breaking the interval into pieces, we see that f is larger on $[-2, 0]$ and g is larger on $[0, 1]$, so:

$$A = \int_{-2}^{0} [f(x) - g(x)]dx + \int_0^1 [g(x) - f(x)]dx.$$

Plugging in the functions and evaluating the integral yields a total area of $\frac{37}{12}$ square units.

3. Find the area bounded by the region enclosed by the three curves $y = x^3$, $y = -x$, and $y = -1$.

Solution: DRAW A PICTURE. Points of intersection are (0,0) (when $x^3 = -x$), (-1,-1) (when $x^3 = -1$), and (1,-1) (when $-x = -1$).

$$A = \int_{-1}^{0} (x^3 - (-1))dx + \int_0^1 (-x - (-1))dx.$$

Evaluating gives an answer of $\frac{5}{4}$ square units.

4. Find the area bounded by the curves $y = \cos x$ and $y = \sin(2x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution: Setting the curves equal, and using the trig identity $\sin(2x) = 2\sin x \cos x$, we see they intersect at $x = \frac{\pi}{2}$ $x = \frac{\pi}{6}$. On the interval $\left[0, \frac{\pi}{2}\right]$, the $\cos x$ is greater on $\left[0, \frac{\pi}{6}\right]$ and $\sin(2x)$ is greater on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, so:

$$A = \int_0^{\pi/6} (\cos x - \sin(2x))dx + \int_{\pi/6}^{\pi/2} (\sin(2x) - \cos x)dx.$$

Evaluating the integral gives an answer of $\frac{1}{2}$ square unit.

5. Find the area of the triangle with vertices at the points (0,1), (3,4), and (4,2). USE CALCULUS.

Solution: DRAW A PICTURE. Let L_1 be the line between (0,1) and (3,4), L_2 the line between (3,4) ad (4,2), and L_3 the line between (0,1) and (4,2). Using the point-slope form of a line, we see that:

$$L_1 : y = x + 1, \quad L_2 : y = -2x + 10, \quad L_3 : y = \frac{1}{4}x + 1.$$

L_3 is always the "bottom" function; the top function changes from L_1 to L_2 at the point (3,4). Thus:

$$A = \int_0^3 \left[(x + 1) - (\frac{1}{4}x + 1)\right] dx + \int_3^4 \left[(-2x + 10) - (\frac{1}{4}x + 1)\right] dx.$$

Evaluating this integral yields an area of 4.5 square units.