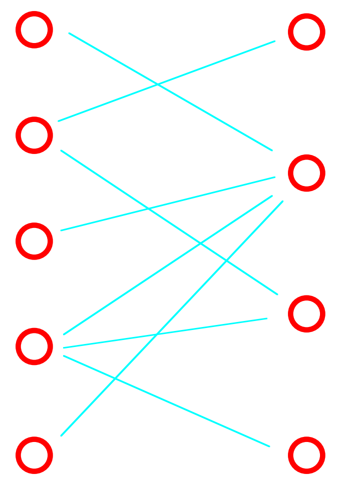
The König-Egerváry Theorem

<u>König-Egerváry Theorem</u>: In any bipartite graph, the maximum size of a matching equals the minimum size of a cover.

For any matching M and any cover C, it is easy to see that |C| ≥ |M|. The theorem will therefore be proved if we can demonstrate that every bipartite graph always has one matching and one cover <u>having the same size</u>.

We will demonstrate this for the bipartite graph at right. The argument will be sufficiently general to apply to an arbitrary bipartite graph.

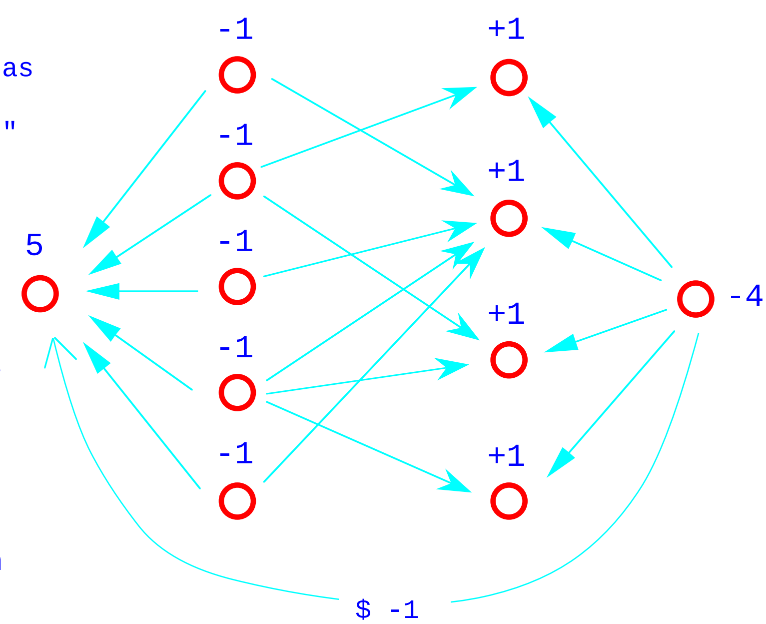


original bipartite graph

Construct a network with demands as indicated. All arcs are assigned zero cost, except for the "return" arc which has cost -1.

In this network, every integer feasible flow corresponds to a matching in the orginal bipartite graph.

The flow in the return arc equals the size of the corresponding matching. Therefore, any minimum cost flow corresponds to a largest possible matching.



The network simplex algorithm solves the minimum cost flow problem for this network. Since all demands and costs are integer, the algorithm finds an *integer* flow x of minimum cost. This flow will correspond to a matching M of largest possible size (M will be the set of original arcs having flow = 1).

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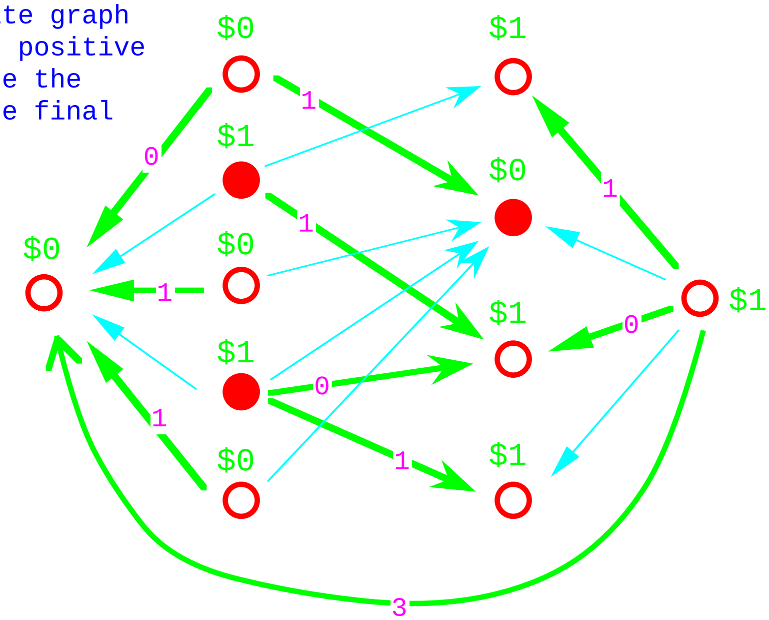
The flow in the return arc equals the maximum size of a matching, which must be greater than zero (except in the trivial case where the original bipartite graph has no arcs). But any arc with positive flow belongs to the tree. Hence the return arc always belongs to the final tree. In our example, |M| = 3.



The algorithm also terminates with node prices p solving the dual LP. Since the min cost flow problem is neither infeasible nor unbounded, its minimum value equals the maximum in the dual LP -- that is p'b = c'x.

Let $0 be the final price of the left-source node (one price may always be set arbitrarily). Since the return arc is in the final tree, $1 is the final price at the right-source node. It follows that all final prices are zeros and ones.

At termination there are no "bad" arcs, so every original arc either
   has price $1 at the left node, or
   has price $0 at the right node.
Hence, the union of original left $1 nodes with the original right $0 nodes (the solid nodes in the figure) forms a cover C for the bipartite graph.

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We will now give two demonstrations that $|C| = |M|$.

first demo:

-size of the matching
= cx = pb
= 0 - (#nodes with price $1 on left)
+ (#nodes with price $1 on right) - (#nodes on right)
= -(#nodes with price $1 on left)
- (#nodes with price $0 on right)
= -size of the covering

second demo:

To show $|C| \leq |M|$ (we already know that $|C| \geq |M|$), we define a 1-1 mapping from C into M. Note

i. every node in C belongs to an arc in M (a $1 left node has no tree arcs branching leftwards, so one of the rightward ones must have flow 1; a $0 right node has no tree arcs branching rightwards, so one of the leftward ones must have flow 1.)

ii. every arc in M has the same price at both nodes, so no arc in M contains two nodes in C.

The first observation allows us to define a function associating each node of C to an arc of M. The second observation ensures that this function is 1-1. Thus $|C| \leq |M|$.