EXISTENCE OF A SOLUTION TO AN EQUATION ARISING FROM THE THEORY OF MEAN FIELD GAMES

WILFRID GANGBO AND ANDRZEJ ŚWIĘCH

Abstract. We construct a small time strong solution to a nonlocal Hamilton–Jacobi equation (1.1) introduced in [48], the so-called master equation, originating from the theory of Mean Field Games. We discover a link between metric viscosity solutions to local Hamilton–Jacobi equations studied in [2, 19, 20] and solutions to (1.1). As a consequence we recover the existence of solutions to the First Order Mean Field Games equations (1.2), first proved in [48], and make a more rigorous connection between the master equation (1.1) and the Mean Field Games equations (1.2).

Contents

1991 *Mathematics Subject Classification.* 35R15, 45K05, 49L25, 49N70, 91A13.

1. Introduction.

The theory of Mean Field Games (MFG) analyzes differential games with a large number of players, each player having a very little influence on the overall system. This theory, which encompasses games with a continuum of players, was developed by Lasry–Lions [44, 45, 46, 47]. Similar ideas were independently introduced at the same time and studied in the engineering literature by Huang–Caines–Malhamé [36, 37, 38, 40]. Games with a continuum of players or traders, first appeared in economics, starting with the seminal work of Aumann [5]. Later a theory of non-atomic games was presented in a book by Aumann–Shapley [6]. In this pioneering work, Aumann–Shapley proposed a profound mathematical theory for economics, the potential of which has not yet been fully exploited. The term "Mean Field Games" was introduced by analogy with the mean field models in mathematical physics where the behaviors of many identical particles are analyzed. We refer the readers to [9, 12, 22, 29, 32] for several excellent surveys on the theory of MFG and its extensions. In particular, the notes [12] from the lectures of P.-L. Lions [48] have been a great contribution to the field, and have clarified the current state of the theory of MFG. This was the starting point of our study.

The theory of MFG has attracted significant attention. In the past five years alone, a large number of manuscripts have been devoted to it, revealing its importance, impact, and possible applications (see e.g. [7, 8, 15, 23, 24, 25, 26, 27, 28, 30, 31, 33, 34, 35, 39, 41, 42, 43, 49, 50, 51, 52]). In light of the publications [44, 45, 46, 47], we restrict our study to the simplest framework of games: those with identical players. Our effort will be devoted mainly to the study of the master equation of MFG (1.1); only a small part of the manuscript deals with the MFG equations (1.2) which were studied in [46, 48, 12]. Our main result establishes the short time existence of a regular solution to (1.1).

Let us denote by $\mathcal{P}(\mathbb{T}^d)$ the set of probability measures on the d–dimensional torus $\mathbb{T}^d$, let $T > 0$ be a real number, and let

$$F,\ u_* : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

be Lipschitz functions. The objective is to find a continuous function

$$u : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

such that

$$\begin{cases} \partial_s u(s,q,\mu) + \big\langle \nabla_\mu u(s,q,\mu), \nabla_q u(s,\cdot,\mu) \big\rangle_\mu + \frac{|\nabla_q u(s,q,\mu)|^2}{2} + F(q,\mu) = 0, \\ u(0,\cdot,\cdot) = u_*(\cdot,\cdot) \end{cases} \tag{1.1}$$

is satisfied in some sense. Here, $\nabla_\mu u$ stands for the Wasserstein gradient of u and we have set

$$\langle \nabla_\mu u(s,q,\mu), \nabla_q u(s,\cdot,\mu) \rangle_\mu = \int_{\mathbb{T}^d} \nabla_\mu u(s,q,\mu)(z) \cdot \nabla_q u(s,z,\mu) \mu(dz).$$

We will call (1.1) the master equation of the theory of MFG.

A heuristic derivation of (1.1) as the limit of a large system of Hamilton–Jacobi equations arising from Nash equilibria in feedback form for many players, can be found in [48] (see also [12]). Furthermore, [48] describes the connection between (1.1) and the first order MFG equations

$$\begin{cases} \partial_t U(t,q) + \frac{|\nabla U(t,q)|^2}{2} + F(q,\sigma_t) = 0 \\ \partial_t \sigma_t + \nabla \cdot (\sigma_t \nabla U(t,q)) = 0 \quad \text{in} \quad \mathcal{D}'\big((0,T)\big) \times \mathbb{T}^d\big) \\ U_0 = u_*(q,\sigma_0), \quad \sigma_T = \mu. \end{cases} \tag{1.2}$$

In (1.2), the first equation is supposed to be satisfied in the viscosity sense and U represents the value function of a typical player. The second equation is supposed to be satisfied in the distribution sense and σ_t, represents the probability distribution of all the players at time t. The measures μ and σ_t in (1.2) are also supposed to be absolutely continuous with respect to the Lebesgue measure for every t.

The main difficulty in dealing with (1.1) is the following. Observe that for each (s,q,μ) fixed, the knowledge of $\partial_s u$, $\nabla_\mu u$ and $\nabla_q u$ at (s,q,μ) is not sufficient to verify that the equation is satisfied since we need the knowledge of $\nabla_q u(s,z,\mu)$ for all $z \in \mathbb{T}^d$ to fully describe (1.1). In other words, (1.1) is non–local in $\nabla_q u$. This difficulty is coupled with the infinite dimensional character of the equation. Interpreting in what weak sense (1.1) may be satisfied has remained a puzzle so far. We try to unravel it by providing a possible definition in the current manuscript (see Definition 7.3). More importantly, we prove the existence of a strong solution to (1.1) for a short time, assuming that the data are sufficiently smooth. We hope this work will help uncover some groundbreaking facts and improve our understanding of the theory of MFG.

Not to overshadow the main ideas with technical details, we have opted in this manuscript to restrict the study of (1.1) to a particular – nevertheless important – class of F's. More precisely, we choose $\phi \in C^3(\mathbb{T}^d)$ and consider

$$F(q,\mu) = \phi * \mu(q). \tag{1.3}$$

However we stress that the approach developed in this paper can be carried out for a wider and quite general class of functionals.

The starting point of our work is the value function $\mathcal{U}$ which is the unique metric viscosity solution, in the sense of [19] and [20] (see also [2] and [18]), to the Hamilton–Jacobi equation

$$\begin{cases} \partial_t \mathcal{U} + \frac{1}{2}||\nabla_\mu \mathcal{U}||^2_\mu + \frac{1}{2}\int_{\mathbb{T}^d} \phi * \mu d\mu = 0 \quad \text{in} \quad (0,T)\times \mathcal{P}(\mathbb{T}^d) \\ \mathcal{U}(0,\cdot) = \mathcal{U}_* \quad \text{on} \quad \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{1.4}$$

We draw the attention of the reader to the fact that having the coefficient 1/2 in front of ϕ in (1.4) and not in (1.3) is not a typo.

According to the well-established theory of endowing the set of probability measures $\mathcal{P}(\mathbb{T}^d)$ with a weak Riemannian structure (see e.g. [4]), the Wasserstein gradient $\nabla_\mu \mathcal{U}$ of $\mathcal{U}$ at $\mu \in \mathcal{P}(\mathbb{T}^d)$ is an element of

$$\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d) := \overline{\nabla C^\infty(\mathbb{T}^d)}^{L^2(\mu)}, \tag{1.5}$$

the tangent space to $\mathcal{P}(\mathbb{T}^d)$ at μ. Hence

$$\nabla_\mu \mathcal{U}(t,\mu) : \mathbb{T}^d \to \mathbb{T}^d$$

is a map which, formally at least, is the gradient of a function $u(t,\cdot,\mu)$:

$$\nabla_q u(t,q,\mu) = \nabla_\mu \mathcal{U}(t,\mu)(q). \tag{1.6}$$

One of the tasks of the current manuscript will include finding a function u satisfying (1.6) which will also satisfy (1.1). The identity (1.6) linking (1.1) to (1.4), appears to be an unexpected connection between two different directions of research which, over the past several years, have been pursued by different research groups using different methods. Indeed, so far the study of (1.4) was primarily motivated by aspects of fluids mechanics (see e.g. [17, 18, 19, 20]). The lectures of P.-L. Lions [48] presented in the notes by Cardialaguet [12] seem to be the first to imply a connection between these two directions. We stress here that the readers should not be misled to think that they need a prior knowledge of the various viscosity solutions concepts introduced in [2, 18, 19, 20] to grasp the content of this manuscript. We have mentioned the works on metric viscosity solutions just to emphasize that there is connection between (1.1) and (1.4) via the identity (1.6), which could be explored in future studies.

The cornerstone of our work, besides establishing identity (1.6), is a good understanding of the regularity properties with respect to the μ variable, of the inverse

$X_s^t[\mu]$ of the map $\Sigma_s^t[\mu]$. The latter map is defined uniquely for small enough T and $s \in [0,T]$ by the system of differential equations

$$\left\{ \begin{array}{ll} \partial_{tt}\Sigma_s^t[\mu](q) = -\nabla_q F\left(\Sigma_s^t[\mu](q), \Sigma_s^t[\mu]_\#\mu\right), & \text{on } (0,T)\times\mathbb{T}^d \\ \Sigma_s^s[\mu](q) = q & \text{on } \mathbb{T}^d \\ \partial_t\Sigma_s^0[\mu](q) = \nabla_q u_*\big(\Sigma_s^0[\mu](q), \Sigma_s^0[\mu]_\#\mu\big) & \text{on } \mathbb{T}^d. \end{array} \right. \tag{1.7}$$

We will often write $\Sigma(t,s,q,\mu)$ for $\Sigma_s^t[\mu](q)$. The regularity property of Σ in the variables (t,s,q) and the invertibility property of $\Sigma(t,s,\cdot,\mu)$ are obtained by standard methods. However, the regularity property with respect to μ of the inverse of $\Sigma(t,s,\cdot,\mu)$ is subtle. We overcome this obstacle by first discretizing $\Sigma_s^t[\mu](q)$ in its μ–variable and then studying the maps

$$(t,q,\mathbf{x}) \in (0,T)\times\mathbb{T}^d\times(\mathbb{T}^d)^n \;\rightarrow\; (t, \Sigma_s^t[\mu^{\mathbf{x}}](q), \mathbf{x}),$$

where we have set

$$\mu^{\mathbf{x}} := \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \quad \mathbf{x} = (x_1,...,x_n).$$

The determinant of the Jacobian of $\nabla_{t,q,\mathbf{x}}\Sigma_s^t[\mu^{\mathbf{x}}](q)$ is shown to be controlled in terms of the finite dimensional determinant $\det \nabla_q \Sigma_s^t[\mu^{\mathbf{x}}](q)$. This allows us to apply the Inverse Function Theorem and then obtain bounds on partial derivatives of the inverse of this map using the bounds on the partial derivatives of $S(t,s,q,\mathbf{x}) := \Sigma_s^t[\mu^{\mathbf{x}}](q)$. This task is completed in Section 8.

In Sections 5 and 6 we show that, if $\mu \in \mathcal{P}(\mathbb{T}^d)$ and if $s>0$ is small enough, the infimum in the variational problem (6.2) related to the Hamilton–Jacobi equation (1.1) is attained by a path $(\sigma, \mathbf{v})$, where

$$\sigma_t = \Sigma_s^t[\mu]_\#\mu, \quad \mathbf{v}_s = \nabla_\mu \mathcal{U}(s,\sigma_t).$$

In Section 7 we construct a function $u(t,q,\mu)$ such that the pair $U(t,q) = u(t,q,\sigma_t)$ and σ satisfy the First Order Mean Field Games equations (1.2). We also show that u is a solution to (1.1) in some weak sense (see Lemma 7.1 and Definition 7.3). The statement (1.6) is one of the things we prove at this stage of the analysis. Then, in Section 9 we prove regularity properties of the function u for small times t that allow us to differentiate u with respect to each variable and show that u satisfies (1.1) pointwise. We call such a function a strong solution of (1.1). Uniqueness of strong solutions remains open. Finally, in Subsection 9.3 we make a rigorous link between strong solutions to the master equation (1.1) and the Mean Field Games equations (1.2) by showing in Lemma 9.9 that any strong solution u to (1.1) allows to construct a pair (U,σ) which solves (1.2), and argue that u also allows to construct an analogue of a Nash equilibrium for a game with a continuum of players.

After the manuscript was completed we learned about the papers [10, 13] which deal with formal derivation of the Master Equations in both deterministic and stochastic cases and their analysis. Also during the second submission of the paper a referee pointed out preprints [11, 14] which deal with classical solutions of Master Equations for stochastic Mean Field Games.

2. Preliminaries.

2.1. **Notation and definitions.**

Throughout this manuscript, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus. When there is no possible confusion we identify an element of the quotient space $\mathbb{T}^d$ with the unique $q \in [0,1)^d$. We denote by $|q^* - q|_{\mathbb{T}^d}$ the distance on $\mathbb{T}^d$ between $q^*, q \in \mathbb{T}^d$. The Euclidean distance between $q^*, q \in \mathbb{R}^d$ is denoted by $|q^* - q|$. If $\xi \in \mathbb{R}^{d\times m}$ we denote

$$|\xi|^2 = \sum_{i=1}^{d}\sum_{j=1}^{m} \xi_{ij}^2.$$

We denote by $Id : \mathbb{T}^d \to \mathbb{T}^d$ the identity map and by I_d the $d \times d$ identity matrix.

Definition 2.1. Let $f : \mathbb{R}^d \to \mathbb{R}$.

(i) By $f : \mathbb{T}^d \to \mathbb{R}^k$ we mean that $f : \mathbb{R}^d \to \mathbb{R}$ and if $q, q^* \in \mathbb{R}^d$ are such that $q - q^* \in \mathbb{Z}^d$, then $f(q^*) = f(q)$.

(ii) By $f : \mathbb{T}^d \to \mathbb{T}^d$ we mean that if $q, q^* \in \mathbb{R}^d$ are such that $q - q^* \in \mathbb{Z}^d$ then $f(q^*) - f(q) \in \mathbb{Z}^d$.

(ii) By $X \in C(\mathbb{T}^d; \mathbb{T}^d)$ we mean that $X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $X : \mathbb{T}^d \to \mathbb{T}^d$.

If $T > 0$ and $S \in W^{2,\infty}\big((0,T) \times \mathbb{T}^d; \mathbb{T}^d\big)$, unless explicitly stated otherwise, $\nabla_{tq} S := \partial_t \nabla_q S$, $\nabla_{qt} S := \nabla_q \partial_t S$, etc..., denote the distributional derivatives of S. Since for instance, the distributional derivatives $\partial_t \nabla_q S$ and $\nabla_q \partial_t S$ coincide, we denote them by $\nabla_{tq} S = \nabla_{qt} S$. Since the distributional derivatives coincide almost everywhere with the pointwise derivatives, expressions such as $||\nabla_q S||_\infty$ will be used to denote the essential supremum of the function $|\nabla_q S|$.

Given two metric spaces $\mathcal{S}_1$, $\mathcal{S}_2$ and a map

$$S : [0,T] \times [0,T] \times \mathcal{S}_1 \to \mathcal{S}_2$$

we use the notation

$$S(t,s,\xi) = S_s^t(\xi), \quad (t,s,\xi) \in [0,T] \times [0,T] \times \mathcal{S}_1.$$

If $S : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is a bounded Borel function, the smallest number A such that

$$|S(t,q,\mu)| \le A$$

for almost every $(t,q) \in [0,T] \times \mathbb{T}^d$ and all $\mu \in \mathcal{P}(\mathbb{T}^d)$ is denoted by $||S||_\infty$.

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ with finite second moments. On $\mathcal{P}_2(\mathbb{R}^d)$ we can define a class of equivalence (cf. e.g. [21]): we say that $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are equivalent if for all $f \in C(\mathbb{T}^d)$ we have

$$\int_{\mathbb{R}^d} f(q)\mu(dq) = \int_{\mathbb{R}^d} f(q)\nu(dq).$$

We use the notation

$$\int_{\mathbb{T}^d} f(q)\mu(dq) := \int_{\mathbb{R}^d} f(q)\mu(dq).$$

The quotient of $\mathcal{P}_2(\mathbb{R}^d)$ by the equivalence relation is $\mathcal{P}(\mathbb{T}^d)$, the set of Borel probability measures on $\mathbb{T}^d$. The set $\mathcal{P}(\mathbb{T}^d)$ has been amply studied in [21], as the quotient space of $\mathcal{P}_2(\mathbb{R}^d)$, and so, we refer to that manuscript for more details. We just recall that any measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ yields a measure $\bar{\mu}$ on $[0,1)^d$ which is defined by

$$\bar{\mu}(B) = \sum_{k \in \mathbb{Z}^d} \mu(B+k)$$

for a Borel $B \subset [0,1)^d$.

Given $\mu \in \mathcal{P}(\mathbb{T}^d)$, we denote by $L^2(\mu)$ the set of Borel maps $\xi : \mathbb{T}^d \to \mathbb{R}^d$ which are square integrable and we set

$$||\xi||_\mu^2 = \int_{\mathbb{T}^d} |\xi|^2 \mu(dq).$$

Given a Borel map $X : \mathbb{T}^d \to \mathbb{T}^d$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$, we denote by $X_\# \mu$ the push forward of μ by X.

Definition 2.2 (cf. [4]). Let $\sigma \in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$. We say that a Borel vector field $\mathbf{v} : (0,T) \times \mathbb{T}^d \to \mathbb{R}^d$ is a velocity for σ if $t \to ||\mathbf{v}_t||_{\sigma_t}$ is in $L^2(0,T)$ and

$$\partial_t \sigma + \nabla \cdot (\sigma \mathbf{v}) = 0$$

in the sense of distributions on $(0,T) \times \mathbb{T}^d$. The latter statement means that for every $f \in C_c^1\big((0,T) \times \mathbb{T}^d\big)$

$$\int_0^T \left(\int_{\mathbb{T}^d} \big(\partial_t f(t,q) + \mathbf{v}_t(q) \nabla f(t,q)\big) \sigma_t(dq) \right) dt = 0.$$

When $x_1, \cdots, x_n \in \mathbb{T}^d$ we set $\mathbf{x} = (x_1, \cdots, x_n)$ and

$$\mu^{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}.$$

Definition 2.3. Given $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, we define $\Gamma(\mu, \nu)$ to be the set of measures $\gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d)$ which have μ as the first marginal, and ν as the second marginal. We denote by $\Gamma_0(\mu, \nu)$ the set of $\gamma \in \Gamma(\mu, \nu)$ such that

$$W_2^2(\mu, \nu) := \min_{\bar{\gamma} \in \Gamma(\mu,\nu)} \int_{\mathbb{T}^d \times \mathbb{T}^d} |r - q|^2_{\mathbb{T}^d} \bar{\gamma}(dq, dr) = \int_{\mathbb{T}^d \times \mathbb{T}^d} |r - q|^2_{\mathbb{T}^d} \gamma(dq, dr).$$

Recall that $\mathcal{P}(\mathbb{T}^d)$ endowed with the Wasserstein distance W_2 is a compact metric space and a sequence $\{\mu_k\}_k \subset \mathcal{P}(\mathbb{T}^d)$ converges to μ in the Wasserstein metric if and only if its converges narrowly.

Definition 2.4. If $\mu \in \mathcal{P}(\mathbb{T}^d)$, we define $\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d)$ to be the closure in $L^2(\mu)$ of the set $\nabla C^\infty(\mathbb{T}^d) := \{\nabla f \ : \ f \in C^\infty(\mathbb{T}^d)\}$.

Let $\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. We define the Lagrangian $\mathcal{L}$ and the Hamiltonian $\mathcal{H}$ by

$$\mathcal{L}(\mu, \xi) := \frac{1}{2}||\xi||^2_\mu - \mathcal{F}(\mu), \quad \mathcal{H}(\mu, \xi) := \frac{1}{2}||\xi||^2_\mu + \mathcal{F}(\mu) \tag{2.1}$$

for $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $\xi \in L^2(\mu)$. The assumptions on $\mathcal{F}$ will be given in Subsection 2.2.

Recall that a function $\psi : \mathbb{R}^m \to \mathbb{R}$ is λ–convex (respectively, λ–concave) if $\psi(x) - \lambda/2|x|^2$ is convex (respectively, concave). Such functions are called semiconvex (respectively, semiconcave). By analogy, the concept of λ–convex functions on $\mathcal{P}(\mathbb{T}^d)$ was introduced in [4]. We refer the reader to the same book for more on the Wasserstein space, absolutely continuous curves in metric spaces, etc.

Following [18] we give a definition of the sub–differential which in general does not coincide with that of [4] except for λ–convex functions.

Definition 2.5. Let $\mathcal{G} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and let $\mu \in \mathcal{P}(\mathbb{T}^d)$.

(i) We say that ξ belongs to the subdifferential of $\mathcal{G}$ at μ and we write $\xi \in \partial_\cdot \mathcal{G}(\mu)$ if $\xi \in L^2(\mu)$ and

$$\mathcal{G}(\nu) - \mathcal{G}(\mu) \geq \sup_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{T}^d \times \mathbb{T}^d} \xi(q) \cdot (r - q) \gamma(dq, dr) + o\big(W_2(\mu, \nu)\big) \qquad \forall \nu \in \mathcal{P}(\mathbb{T}^d). \tag{2.2}$$

(ii) We say that ξ belongs to the superdifferential of $\mathcal{G}$ at μ and we write $\xi \in \partial^\cdot \mathcal{G}(\mu)$ if $-\xi \in \partial_\cdot(-\mathcal{G})(\mu)$. The unique element of minimal norm in $\partial^\cdot \mathcal{G}(\mu)$ belongs to $\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d)$ and is called the gradient of $\mathcal{G}$ at μ and is denoted by $\nabla_\mu \mathcal{G}(\mu)$.

(iii) We say that $\mathcal{G}$ is differentiable at μ if both $\partial_\cdot \mathcal{G}(\mu)$ and $\partial^\cdot \mathcal{G}(\mu)$ are non empty. In that case (see e.g. [18]) both sets coincide and

$$\partial_\cdot \mathcal{G}(\mu) \cap \mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d) = \partial^\cdot \mathcal{G}(\mu) \cap \mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d) = \{\nabla_\mu \mathcal{G}(\mu)\}.$$

Remark 2.6. Here are few remarks.

(i) We refer the reader to Remark 3.2 of [18] for property (iii) in Definition 2.5.
(ii) Thanks to Proposition 8.5.4 of [4], note that (2.2) holds for ξ if and only if it holds for any $\xi_0 \in L^2(\mu)$ such that $\xi_0 - \xi$ belongs to the orthogonal complement of $\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d)$ in $L^2(\mu)$. Rephrasing, if (2.2) holds for $\xi_0 \in L^2(\mu)$ then it holds for ξ defined as the orthogonal projection of ξ_0 onto $\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d)$.

Remark 2.7 (Basic properties of the determinant). Let $\xi = (\xi_{ij}) \in \mathbb{R}^{d\times d}$ and denote by $\bar\xi = (\bar\xi_{ij})$ the matrix of its cofactors.

(i) We can write $\xi = LQ$ where L is lower triangular and Q is orthogonal. Thus, $\xi\xi^T = LL^T$ and so,

$$|\det \xi|^{\frac{2}{d}} = |\det L|^{\frac{2}{d}} \leq \frac{l_{11}^2 + \cdots + l_{dd}^2}{d} \leq \frac{|L|^2}{d} = \frac{|\xi|^2}{d}.$$

(ii) We have $\partial_{\xi_{ij}} \det \xi = \bar\xi_{ij}$ and so, by (i), if $d > 1$ then

$$|\partial_{\xi_{ij}} \det \xi| \leq \frac{|\xi|^{d-1}}{\sqrt{d-1}^{\,d-1}} \tag{2.3}$$

and, using the fact that $d^2 \leq 4(d-1)^{d-1}$ we conclude that

$$|\nabla_\xi \det \xi| \leq 2|\xi|^{d-1} \tag{2.4}$$

If $d = 1$ then $\det \xi = \xi$. In that case (2.4) continues to hold.

2.2. **Assumptions.** We state here general assumptions that will be used in the manuscript. In the second part of the paper (from Section 6 on) we will further assume that the functions $F, \mathcal{F}, u_*, \mathcal{U}_*$ have particular forms.

Let

$$\kappa \geq 1 \tag{2.5}$$

be a given constant. We assume we have a differentiable function

$$F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

and a differentiable κ–Lipschitz function

$$\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

such that for any $q \in \mathbb{T}^d$ and any $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\nabla_q F(q, \mu) = \nabla_\mu \mathcal{F}(\mu)(q), \tag{2.6}$$

and

(2.7)
$$\left| \mathcal{F}(\nu) - \mathcal{F}(\mu) - \int_{\mathbb{T}^d\times\mathbb{T}^d} \nabla_\mu \mathcal{F}(\mu)(q) \cdot (y - q)\gamma(dq, dy) \right| \leq \kappa \int_{\mathbb{T}^d\times\mathbb{T}^d} |q - y|^2_{\mathbb{T}^d} \gamma(dq, dy),$$

for all $\nu \in \mathcal{P}(\mathbb{T}^d)$ and all $\gamma \in \Gamma(\mu, \nu)$

We further assume that

$$\nabla_q F(q,\mu), \ \ \nabla_{qq} F(q,\mu), \ \ \nabla_{qqq} F(q,\mu) \quad \text{exist and are continuous,} \tag{2.8}$$

$$\|\nabla_q F\|_\infty, \ \ \|\nabla_{qq} F\|_\infty, \ \ \|\nabla_{qqq} F(q,\mu)\|_\infty \leq \kappa, \tag{2.9}$$

and

$$\nabla_q F \quad \text{is } \kappa\text{-Lipschitz.} \tag{2.10}$$

We assume to be given a κ–Lipschitz function

$$u_* : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

such that

$$|u_*| \leq \kappa. \tag{2.11}$$

We assume there is a differentiable κ–Lipschitz function

$$\mathcal{U}_* : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

such that for any $q \in \mathbb{T}^d$ and any $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\nabla_q u_*(q,\mu) = \nabla_\mu \mathcal{U}_*(\mu)(q), \tag{2.12}$$

and

(2.13)

$$\left| \mathcal{U}_*(\nu) - \mathcal{U}_*(\mu) - \int_{\mathbb{T}^d\times\mathbb{T}^d} \nabla_\mu \mathcal{U}_*(\mu)(q)\cdot(y-q)\gamma(dq,dy) \right| \leq \kappa \int_{\mathbb{T}^d\times\mathbb{T}^d} |q-y|^2_{\mathbb{T}^d}\gamma(dq,dy),$$

for all $\nu \in \mathcal{P}(\mathbb{T}^d)$ and all $\gamma \in \Gamma(\mu,\nu)$.

We further assume that

$$\nabla_q u_*, \ \ \nabla_{qq} u_*, \ \ \nabla_{qqq} u_* \quad \text{exist and are continuous,} \tag{2.14}$$

$$||\nabla_q u_*||_\infty, \ \ ||\nabla_{qq} u_*||_\infty, \ \ ||\nabla_{qqq} u_*||_\infty \leq \kappa, \tag{2.15}$$

and

$$\nabla_q u_* \quad \text{is } \kappa\text{-Lipschitz.} \tag{2.16}$$

Remark 2.8. Observe that the requirement on $\mathcal{F}$ in (2.7) is more restrictive than 2κ–geodesic convexity and 2κ–geodesic concavity (see Proposition 4.2 of [3]) since we do not require that $\gamma \in \Gamma_0(\mu,\nu)$. A similar remark applies to (2.13).

If $s \in [0,T]$ and $\sigma \in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$ has $\mathbf{v}$ as a velocity, we define the augmented action

$$\mathcal{A}(s;\sigma,\mathbf{v}) := \int_0^s \mathcal{L}(\sigma_l,\mathbf{v}_l)dl + \mathcal{U}_*(\sigma_0).$$

For $s \in [0,T]$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$, we define the map

$$M_s[\mu] : C\big([0,T] \times \mathbb{T}^d; \mathbb{T}^d\big) \to C\big([0,T] \times \mathbb{T}^d; \mathbb{T}^d\big)$$

by

$$\begin{aligned} M_s[\mu](S)(t,q) &= q + (t-s)\nabla_q u_*\big(S^0(q), S^0_{\#}\mu\big) \\ &+ \int_t^s dl \int_0^l \nabla_q F\big(S^\tau(q), S^\tau_{\#}\mu\big) d\tau. \end{aligned} \tag{2.17}$$

where we used the notation S^τ for $S(\tau,\cdot)$.

Example 2.9. Let ϕ, U^0, $U^1 \in C^3(\mathbb{T}^d)$ be such that ϕ and U^1 are even and (6.1) holds. For any $q \in \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathbb{T}^d)$ we set

$$u_*(q,\mu) = U^0(q) + U^1 * \mu(q), \qquad F(q,\mu) = \phi * \mu(q),$$

$$\mathcal{U}_*(\mu) = \int_{\mathbb{T}^d} \big(U^0 + \frac{1}{2} U^1 * \mu\big)(y)\mu(dy), \tag{2.18}$$

$$\mathcal{F}(\mu) = \frac{1}{2}\int_{\mathbb{T}^d} \phi * \mu(y)\mu(dy). \tag{2.19}$$

We have

$$\nabla_q u_*(q,\mu) = \nabla U^0(q) + \nabla U^1 * \mu(q), \qquad \nabla_q F(q,\mu) = \nabla\phi * \mu(q),$$

and it can be shown, using techniques of [4], that F, $\mathcal{F}$, u_* and $\mathcal{U}_*$ satisfy all the assumptions of this section.

3. Uniqueness of a fix point of $M_s[\mu]$.

Throughout this section, $T > 0$ is a prescribed number. Further restrictions on T will be placed later. We denote $C_T := T(1+T)$.

3.1. **Elementary properties of $M_s[\mu]$.** Let

$$S \in W^{1,\infty}\big([0,T] \times \mathbb{T}^d; \mathbb{T}^d\big).$$

Using the notation $S^t = S(t,\cdot)$ we have

$$\partial_t\big(M_s[\mu](S)\big)(t,q) = \nabla_q u_*\big(S^0(q), S^0_{\#}\mu\big) - \int_0^t \nabla_q F(S^l(q), S^l_{\#}\mu) dl, \tag{3.1}$$

$$\partial_{tt}\big(M[\mu](S)\big)(t,q) = -\nabla_q F(S^t(q), S^t_{\#}\mu) \tag{3.2}$$

and

$$\begin{aligned}\nabla_q M_s[\mu](S)(t,q) &= I_d + (t-s)\nabla_{qq}u_*\big(S^0(q), S^0_{\#}\mu\big)\nabla_q S^0(q)\\ &+ \int_t^s dl \int_0^l \nabla_{qq}F(S^\tau(q), S^\tau_{\#}\mu)\nabla_q S^\tau(q)d\tau.\end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned}\nabla_{tq} M_s[\mu](S)(t,q) &= \nabla_{qq}u_*\big(S^0(q), S^0_{\#}\mu\big)\nabla_q S^0(q)\\ &- \int_0^t \nabla_{qq}F\big(S^\tau(q), S^\tau_{\#}\mu\big)\nabla_q S^\tau(q)d\tau.\end{aligned} \tag{3.4}$$

Lemma 3.1. *Let $S \in W^{2,\infty}\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ and let $A > 0$ be such that*

$$||\nabla_q S||_\infty,\ ||\nabla_{qq}S||_\infty \le A.$$

Then for any $\mu \in \mathcal{P}(\mathbb{T}^d)$ and any $s \in [0,T]$ we have:

(i)

$$||M_s[\mu](S)||_\infty \le \frac{\sqrt{d}}{2} + \kappa C_T,\ ||\partial_t M_s[\mu](S)||_\infty \le \kappa(1+T),\ ||\partial_{tt} M_s[\mu](S)||_\infty \le \kappa. \tag{3.5}$$

(ii)

$$||\nabla_q M_s[\mu](S)||_\infty \le \sqrt{d} + \kappa A C_T,\ \ ||\nabla_{tq} M_s[\mu](S)||_\infty \le \kappa A(1+T). \tag{3.6}$$

(iii)

$$||\nabla_{qq} M_s[\mu](S)||_\infty \le \kappa A(1+A)C_T.$$

Proof. To show (i) we use (2.9) and (2.15) to obtain the first inequality in (i). We use the formulas for first and second derivatives of $M_s[\mu](S)$ with respect to t, given by (3.1) and (3.2) and then use (2.9) and (2.15) to obtain the second and third inequalities in (i). Similarly, the inequalities in (3.6) are obtained from (3.3) and (3.4), using (2.9) and (2.15). To get the inequality in (iii) we differentiate (3.3) with respect to q and again use (2.9) and (2.15) and the assumptions on S. ∎

We also suppose that $A > 0$ and $T > 0$ are such that

$$3\kappa\sqrt{d} \le A, \quad 2T,\ \ 3\kappa C_T < 1, \quad 4\kappa T A(\sqrt{d}+1)^{d-1} \le 1. \tag{3.7}$$

We observe that the second and third inequalities above give

$$\kappa(1+A)C_T \le 1. \tag{3.8}$$

Lemma 3.2. *Let $\mu \in \mathcal{P}(\mathbb{T}^d)$ and let*

$$\Sigma \in W^{1,\infty}\big((0,T)\times(0,T)\times\mathbb{T}^d;\mathbb{T}^d\big), \qquad \bar\Sigma(\cdot,s,\cdot) := M_s[\mu]\big(\Sigma(s,\cdot,\cdot)\big).$$

Then

$$\left\|\partial_s\bar\Sigma\right\|_\infty \le \kappa\Big(1+T+\sqrt{2}T\Big(1+\frac{T}{2}\Big)\|\partial_s\Sigma\|_\infty\Big)$$

and

$$\left\|\partial_s\partial_t\bar\Sigma\right\|_\infty \le \sqrt{2}\kappa(1+T)\|\partial_s\Sigma\|_\infty.$$

Proof. Since $Z\to Z_\#\mu$ is a 1–Lipschitz map of $C(\mathbb{T}^d;\mathbb{T}^d)$ into $\mathcal{P}(\mathbb{T}^d)$ and $\nabla_q u_*, \nabla_q F$ are κ–Lipschitz, we conclude that maps

$$Z\to\nabla_q u_*\big(Z(q),Z_\#\mu\big)\quad\text{and}\quad Z\to\nabla_q F\big(Zq,Z_\#\mu\big)$$

are $\sqrt{2}\kappa$–Lipschitz for $q\in\mathbb{T}^d$ fixed. We use this to obtain the first inequality. The second inequality is obtained in a similar manner applying the above arguments to the formula for $\partial_t\bar\Sigma$. ∎

Remark 3.3. The following hold:

(i) The map $Z\to Z_\#\mu$ is 1–Lipschitz of $C(\mathbb{T}^d;\mathbb{T}^d)$ into $\mathcal{P}(\mathbb{T}^d)$.

(ii) If $Z:\mathbb{T}^d\to\mathbb{T}^d$ is l–Lipschitz, so is $\mu\to Z_\#\mu$. As a consequence if $Z\in C(\mathbb{T}^d;\mathbb{T}^d)$ then $\zeta_Z:\mu\to Z_\#\mu$ is a continuous map of $\mathcal{P}(\mathbb{T}^d)$ into itself $\mathcal{P}(\mathbb{T}^d)$. Therefore, if $S\in C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$, then $\zeta_{S(0,\cdot)}$ is a continuous map of $\mathcal{P}(\mathbb{T}^d)$ into itself.

Proof. (i) If $S_1,S_2\in C(\mathbb{T}^d;\mathbb{T}^d)$ then

$$\gamma:=(S_1\times S_2)_\#\mu\in\Gamma\big(S_{1\,\#}\mu,S_{2\,\#}\mu\big).$$

and so,

$$W_2^2\big(S_{1\,\#}\mu,S_{2\,\#}\mu\big)\le\|S_1-S_2\|_\mu^2\le\|S_1-S_2\|_\infty^2.$$

(ii) Let $Z:\mathbb{T}^d\to\mathbb{T}^d$ be l–Lipschitz and let $\mu_1,\mu_2\in\mathcal{P}(\mathbb{T}^d)$. If $\gamma\in\Gamma_0(\mu_1,\mu_2)$ then

$$\bar\gamma:=(Z\times Z)_\#\gamma\in\Gamma(Z_\#\mu_1,Z_\#\mu_2).$$

Hence,

$$\begin{aligned}W_2^2\big(Z_\#\mu_1,Z_\#\mu_2\big) &\le \int_{\mathbb{T}^d\times\mathbb{T}^d}|q-r|_{\mathbb{T}^d}^2\bar\gamma(dq,dr)\\ &= \int_{\mathbb{T}^d\times\mathbb{T}^d}|Z(x)-Z(y)|_{\mathbb{T}^d}^2\gamma(dx,dy)\\ &\le l^2W_2^2(\mu_1,\mu_2).\end{aligned}\tag{3.9}$$

This proves that ζ_Z is l–Lipschitz. Let now $Z\in C(\mathbb{T}^d;\mathbb{T}^d)$ and let $\{Z^k\}_k$ be a sequence of Lipschitz functions that converges uniformly to Z on $\mathbb{T}^d$. By (i)

$$W_2(\zeta_Z\mu,\zeta_{Z_k}\mu)\le\|Z-Z^k\|_\infty$$

and so, ζ_Z is continuous as a uniform limit of Lipschitz maps. ∎

Remark 3.4. The following hold:

(i) By assumption $\nabla_q F$ and $\nabla_q u_*$ are κ–Lipschitz. Since, by Remark 3.3 (i), $Z \to Z_{\#}\mu$ is a 1–Lipschitz map of $C(\mathbb{T}^d;\mathbb{T}^d)$ into $\mathcal{P}(\mathbb{T}^d)$, we conclude that if $s \in (0,T]$ then $M_s[\mu] : C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big) \to C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ is Lipschitz continuous with the Lipschitz constant which is less than or equal to $\sqrt{2}\kappa C_T < 1$. Thus $M_s[\mu]$ is a contraction.

(ii) If $S \in C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ then, by Remark 3.3 (ii), $\mu \to S^0_{\#}\mu$ is a continuous map of $\mathcal{P}(\mathbb{T}^d)$ into itself. Since $\nabla_q u_*$ is κ–Lipschitz, we obtain that $\mu \to \nabla_q u_*\big(S^0(q), S^0_{\#}\mu\big)$ is continuous. We use (i) and the fact that $\nabla_q F$ is κ–Lipschitz to conclude that the map $(s,S,\mu) \to M_s[\mu](S)$ is a continuous map of $[0,T]\times C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)\times\mathcal{P}(\mathbb{T}^d)$ into $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$.

Definition 3.5. We define

(i) $\mathcal{C}_A$ to be the set of $S \in C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ such that

$$\|S\|_\infty \le \frac{\sqrt{d}+1}{2},\ \|\partial_t S\|_\infty \le 2\kappa,$$

and

$$\|\partial_{tt} S\|_\infty \le \kappa,\ \|\nabla_q S\|_\infty,\ \|\nabla_{qq} S\|_\infty \le A,\ \|\nabla_{tq} S\|_\infty \le \frac{3}{2}\kappa A.$$

(ii) We define $\mathcal{C}^*_A$ to be the set of $\Sigma \in C\big([0,T]\times[0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ such that for every $s \in [0,T]$, $\Sigma(\cdot,s,\cdot) \in \mathcal{C}_A$ and

$$\|\partial_s\Sigma\|_\infty \le A, \quad \|\partial_{ts}\Sigma\|_\infty \le \sqrt{2}\kappa(1+T)A.$$

Lemma 3.6. *The following hold:*

(i) *$\mathcal{C}_A$ is a compact set in $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$.*

(ii) *If $s \in [0,T]$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$ then $M_s[\mu]$ maps $\mathcal{C}_A$ into itself.*

Proof. (i) We omit the proof of (i) because it is elementary.

(ii) Let $S \in \mathcal{C}_A$. Since $3\kappa C_T \le 1$, we use Lemma 3.1 (i) and the fact that $T \le 1$ to obtain

$$\|M_s[\mu](S)\|_\infty \le \frac{\sqrt{d}+1}{2},\ \|\partial_t M_s[\mu](S)\|_\infty \le 2\kappa,\ \|\partial_{tt} M_s[\mu](S)\|_\infty \le \kappa. \tag{3.10}$$

By (3.7) since $\kappa \ge 1$ we have $3\sqrt{d} \le A$. We use the latter inequality in Lemma 3.1 (ii) and use the fact that $3\kappa C_T \le 1$ to obtain

$$\|\nabla_q M_s[\mu](S)\|_\infty \le A. \tag{3.11}$$

The inequality $\kappa(1+A)C_T \le 1$ implies $\kappa A(1+A)C_T \le A$. This, together with Lemma 3.1 (iii) gives

$$\|\nabla_{qq} M_s[\mu](S)\|_\infty \le A. \tag{3.12}$$

Since $T \le 1$, we obtain

$$\|\nabla_{qt} M_s[\mu](S)\|_\infty \le \kappa A(1+T) \le \frac{3}{2}\kappa A.$$

This, together with (3.10), (3.11) and (3.12), yields $M_s[\mu](S) \in \mathcal{C}_A$. ∎

Lemma 3.7. *Let* $\mu \in \mathcal{P}(\mathbb{T}^d)$ *and let* $\Sigma \in \mathcal{C}_A^*$. *Define*

$$\bar\Sigma(t,s,q) = M_s[\mu]\big(\Sigma(\cdot,s,\cdot)\big)(t,q) \quad \forall (t,s,q) \in [0,T]\times[0,T]\times\mathbb{T}^d.$$

Then $\bar\Sigma \in \mathcal{C}_A^*$.

Proof. Since for any $s \in [0,T]$ we have $\Sigma(\cdot,s,\cdot) \in \mathcal{C}_A$, Lemma 3.6 yields $\bar\Sigma(\cdot,s,\cdot) \in \mathcal{C}_A$. By Lemma 3.2, since $\Sigma \in \mathcal{C}_A^*$ and $T < 1/2$, we have

$$\big\|\partial_s \bar\Sigma\big\|_\infty \le \kappa\Big(\frac{3}{2} + \frac{3\sqrt{2}}{2}TA\Big).$$

The first inequality in (3.7) ensures that $A \ge 3$ and then the third inequality there gives $12\kappa T \le 1$. Therefore, using again the first inequality in (3.7),

$$\kappa\Big(\frac{3}{2} + \frac{3\sqrt{2}}{2}TA\Big) \le \frac{A}{2} + \frac{3\sqrt{2}}{2}\cdot\frac{1}{12}A < A.$$

We use the second inequality in Lemma 3.2 and the fact that $\big\|\partial_s\bar\Sigma\big\|_\infty \le A$ to complete the proof. ∎

Theorem 3.8. *Let* $\mu \in \mathcal{P}(\mathbb{T}^d)$ *and let* $0 \le s \le T$. *Then* $M_s[\mu]$ *admits a unique fixed point* $\Sigma_s[\mu]$ *in* $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$. *Furthermore,* $\Sigma_s[\mu]$ *belongs to every closed subset of* $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ *which is invariant under* $M_s[\mu]$. *As a consequence we have:*

(i) *For any* $k \in \mathbb{Z}^d$,

$$\Sigma_s[\mu](t,q+k) = \Sigma_s[\mu](t,q) + k.$$

(ii) $\Sigma_s[\mu] \in \mathcal{C}_A$.

Proof. Since $M_s[\mu]$ is a contraction in $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$, it has a unique fixed point $\Sigma_s[\mu]$. Any closed subset $\mathcal{C}$ of $C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ is also a complete metric space. Hence, if $\mathcal{C}$ is invariant under $M_s[\mu]$, there must be a unique fixed point of $M_s[\mu]$ in $\mathcal{C}$ which, by uniqueness, must be equal to $\Sigma_s[\mu]$. Since, by Lemma 3.6, $\mathcal{C}_A$ is a compact set invariant under $M_s[\mu]$, we thus obtain (ii).

(i) If $k \in \mathbb{Z}^d$, the set $\mathcal{C}$ which consists of $S \in C\big([0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ such that $S^t(q+k) = S^t(q)$ for all $t \in [0,T]$ and all $q \in \mathbb{T}^d$, is closed. To show that $\Sigma_s[\mu] \in \mathcal{C}$, its remains to show that $\mathcal{C}$ is invariant under $M_s[\mu]$. If $S \in \mathcal{C}$, using the facts that

$$\nabla_q u_*(\cdot,\nu),\ \ \nabla_q F(\cdot, S^\tau_\# \mu) : \mathbb{T}^d \to \mathbb{R}^d,$$

we obtain $M_s[\mu](S) \in \mathcal{C}$. ∎

Lemma 3.9. *Let $\mu \in \mathcal{P}(\mathbb{T}^d)$, and let $\Sigma^0 \in \mathcal{C}_A^*$. Define inductively $\Sigma_s^k = M_s[\mu](\Sigma_s^{k-1})$ for $k \geq 1$. Then the sequence $\{\Sigma^k\}$ converges uniformly to $\Sigma[\mu]$, where $\Sigma[\mu](t,s,q) = \Sigma_s[\mu](t,q)$, and $\Sigma[\mu] \in \mathcal{C}_A^*$.*

Proof. By Lemma 3.7, an induction argument shows that $\Sigma^k \in \mathcal{C}_A^*$. In particular for each $s \in [0,T]$, $\Sigma^k(\cdot, s, \cdot) \in \mathcal{C}_A$. Recall that, by Lemma 3.6, $\mathcal{C}_A$ is a compact set in $C\big([0,T] \times \mathbb{T}^d; \mathbb{T}^d\big)$ and $M_s[\mu]$ maps $\mathcal{C}_A$ into $\mathcal{C}_A$. Since $M_s[\mu]$ is a contraction we conclude that $\{\Sigma^k(\cdot, s, \cdot)\}_k$ converges uniformly on $[0,T] \times \mathbb{T}^d$ to $\Sigma_s[\mu]$. We use the equicontinuity of $\{\Sigma^k\}_k$ to infer its uniform convergence on $[0,T] \times [0,T] \times \mathbb{T}^d$ to $\Sigma[\mu]$. Since $\mathcal{C}_A^*$ is closed for the uniform convergence, $\Sigma[\mu] \in \mathcal{C}_A^*$. ∎

Definition 3.10. Under the assumptions of Theorem 3.8, we define $\Sigma_s[\mu]$ to be the unique fixed point of $M_s[\mu]$ and write $\Sigma_s^t[\mu]$ in place of $\Sigma[\mu](t,s,\cdot)$. We will sometimes also use the notation $\Sigma(t,s,q,\mu)$.

Lemma 3.9 ensures that we can always assume $\Sigma[\mu] \in \mathcal{C}_A^*$.

3.2. Differentiability properties of $\Sigma_s[\mu]$ on $[0,T] \times \mathbb{T}^d$.

In the sequel, we assume that $T > 0, A > 0$ and (3.7) holds.

Remark 3.11. Let $\xi \in \mathbb{R}^{d \times d}$ be such that $|\xi| \leq 3/2\kappa A$. For any $s \in [0,T]$, we have:

(i)
$$|\det(I + s\xi) - 1| \leq \frac{3}{4}.$$

(ii) $I + \tau s\xi$ is invertible and
$$\left|(I + \tau s\xi)^{-1}\right| \leq c_d := 8(\sqrt{d} + 1)^{d-1}.$$

Proof. (i) We use (3.7) to obtain that $3/2\kappa TA < 1$ and so, for any $\tau \in [0,1]$,

$$|I + \tau s\xi| \leq |I| + \frac{3}{2}\kappa TA \leq \sqrt{d} + 1. \tag{3.13}$$

We use write the first order Taylor expansion of $\det(I + s\xi)$ to obtain $\tau \in [0,1]$ such that
$$|\det(I + s\xi) - \det I| = s|\nabla_\xi \det(I + \tau s\xi) \cdot \xi|.$$
We then apply (2.4) and (3.7) to conclude that
$$|\det(I + s\xi) - \det I| \leq T|\xi| 2|I + \tau s\xi|^{d-1} \leq 3\kappa TA(\sqrt{d} + 1)^{d-1} \leq \frac{3}{4}.$$

(ii) By (i), $I + s\xi$ is invertible. Since
$$(I + s\xi)^{-1} = \frac{\big(\nabla_\xi \det(I + s\xi)\big)^T}{\det(I + s\xi)},$$

we use (i), (2.4) and (3.13) to conclude that

$$\left|(I+s\xi)^{-1}\right| \leq 8|I+s\xi|^{d-1} \leq 8(\sqrt{d}+1)^{d-1}.$$

∎

Remark 3.12. If $s \in [0,T]$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$ then $\Sigma_s[\mu]$ is the unique solution to the system of differential equations

$$\begin{cases} \partial_{tt}\Sigma_s^t[\mu](q) & = -\nabla_q F\Big(\Sigma_s^t[\mu](q),\, \Sigma_s^t[\mu]_{\#}\mu\Big) & \text{on } (0,T)\times\mathbb{T}^d \\ \Sigma_s^s[\mu](q) & = q & \text{on } \mathbb{T}^d \\ \partial_t\Sigma_s^0[\mu](q) & = \nabla_q u_*\Big(\Sigma_s^0[\mu](q),\, \Sigma_s^0[\mu]_{\#}\mu\Big) & \text{on } \mathbb{T}^d. \end{cases} \tag{3.14}$$

Lemma 3.13. *Let $\mu \in \mathcal{P}(\mathbb{T}^d)$.*

(i) *For $s \in [0,T]$ and $t \in [0,T]$ we have $4\det\nabla_q\Sigma_s^t[\mu] \geq 1$.*

(ii) *For $s \in [0,T]$ and $t \in [0,T]$, $\Sigma_s^t[\mu] : \mathbb{T}^d \to \mathbb{T}^d$ is a diffeomorphism whose inverse is denoted by $X_s^t[\mu]$.*

(iii) *There exists a constant C_A independent of s and μ such that*

$$\|\partial_t\Sigma_s[\mu]\|_{W^{2,\infty}\big((0,T)\times\mathbb{T}^d\big)},\quad \|X_s[\mu]\|_{W^{2,\infty}\big((0,T)\times\mathbb{T}^d\big)},\quad \|\partial_s X_s[\mu]\|_\infty, \leq C_A$$

Proof. Fix $t \in [0,T]$.

(i) Use the second equation in (3.14) to write

$$\Sigma_s^t[\mu] = Id + T\zeta \quad \text{and so} \quad \nabla_q\Sigma_s^t[\mu] = I_d + T\xi,$$

where,

$$T\zeta = \int_s^t \partial_t\Sigma_s^\tau[\mu]d\tau \quad \text{and} \quad T\xi = \int_s^t \nabla_{tq}\Sigma_s^\tau[\mu](q)d\tau.$$

By Theorem 3.8, $\Sigma_s[\mu] \in \mathcal{C}_A$ and so, $\|\xi\|_\infty \leq 3\kappa A/2$. We apply Remark 3.11 to obtain (i) and

$$\|(\nabla_q\Sigma_s^t[\mu])^{-1}\|_\infty \leq c_d. \tag{3.15}$$

(ii) We use the fact that

$$\Sigma_s[\mu] \in \mathcal{C}_A \subset W^{2,\infty}\big((0,T)\times\mathbb{T}^d;\mathbb{T}^d\big) \tag{3.16}$$

and the Sobolev Embedding Theorem to conclude that

$$\Sigma_s[\mu] \in C^1\big((0,T)\times\mathbb{T}^d;\mathbb{T}^d\big).$$

Since $\Sigma_s[\mu] \in \mathcal{C}_A$ implies that $T\zeta(t,\cdot)$ is TA–Lipschitz, and the last inequality in (3.7) yields $TA < 1$, we conclude that $Id + T\zeta$ is one–to–one.

Let $R > 1$ and let $y \in B_{R-1}(0)$, the ball of radius $R-1$ centered at the origin. We use $\Sigma_s[\mu] \in \mathcal{C}_A$ to obtain $T\|\zeta\|_\infty < 1$ and so, for all q on the boundary of the bigger ball $B_R(0)$, $\Sigma_s^s[\mu](q) \neq y$ for any $s \in [0,T]$. Therefore,

$$f(l) := \deg\Big(\Sigma_s^l[\mu],\ B_R(0),\ y\Big)$$

the topological degree of $\Sigma_s^l[\mu]$ is well defined at $y \in B_{R-1}(0)$ (see e.g. [16]). Since f is a continuous function which assumes only integer values, we conclude that $f(l) = f(0) = 1$. This proves that the range of $\Sigma_s^l[\mu]$ contains $B_{R-1}(0)$. Since $R > 1$ is arbitrary, we conclude that the range of $\Sigma_s^l[\mu]$ contains $\mathbb{R}^d$. In particular, taking into account that we have already proved that $\Sigma_s^t[\mu]$ is one–to–one, when $l = t$ we obtain that $\Sigma_s^t[\mu] : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection of class C^1. This, together with $4\det \nabla_q \Sigma_s^t[\mu] \geq 1$ (by (i)), implies that $X_s^t[\mu] : \mathbb{R}^d \to \mathbb{R}^d$, the inverse of $\Sigma_s^t[\mu]$, is of class C^1 and satisfies

$$\nabla_q X_s^t[\mu] = \Big(\nabla_q \Sigma_s^t[\mu]\Big)^{-1} \circ X_s^t[\mu] = \frac{\operatorname{adj}\Big(\nabla_q \Sigma_s^t[\mu]\Big)}{\det\Big(\nabla_q \Sigma_s^t[\mu]\Big)} \circ X_s^t[\mu]. \tag{3.17}$$

Here, if E is a square matrix, adj$E$ is the transposed matrix of the cofactors of E.

(iii) By (3.15) and (3.17) $\|\nabla_q X_s[\mu]\|_\infty \leq c_d$.

Direct computations reveal that

$$\partial_t X_s^t[\mu] = -\nabla_q X_s^t[\mu]\ \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu], \quad \partial_s X_s^t[\mu] = -\nabla_q X_s^t[\mu]\ \partial_s \Sigma_s^t[\mu] \circ X_s^t[\mu]. \tag{3.18}$$

Thus, using the inequality $\|\nabla_q X_s[\mu]\|_\infty \leq c_d$ and the fact that $\Sigma[\mu] \in \mathcal{C}_A^*$, we obtain the third inequality in (iii).

Recall that since $\Sigma_s[\mu]$ is a fixed point for $M_s[\mu]$ we have

$$\partial_t \Sigma_s^t[\mu](q) = \nabla_q u_*\Big(\Sigma_s^0[\mu]q, (\Sigma_s^0[\mu])_\# \mu\Big) - \int_0^t \nabla_q F\Big(\Sigma_s^\tau[\mu]q, \Sigma_s^\tau[\mu]_\# \mu\Big) d\tau. \tag{3.19}$$

We can now differentiate both sides of (3.19) with respect to t, q and use (2.9), (2.15), Remark 3.3 and the fact that $\Sigma_s[\mu] \in \mathcal{C}_A$ to obtain a constant C_A, independent of s and μ, such that the first inequality in (iii) holds.

Finally we use again $\Sigma_s[\mu] \in \mathcal{C}_A$ and differentiate the expressions in (3.17) and (3.18) with respect to t, q, to obtain that the second derivatives of $X_s[\mu]$ are bounded by a constant C_A independent of μ or s. ∎

3.3. ***s*-Orbits passing through μ.** As in Subsections 3.1 and 3.2, we assume that $T > 0, A > 0$ are such that (3.7) holds.

Given $s \in [0,T]$ we define the s-Orbits through μ by

$$\mathcal{O}_s[\mu] = \{\Sigma_s^t[\mu]_\# \mu \mid t \in [0,T]\}.$$

Definition 3.14. For $t \in [0,T]$, $s \in [0,T]$, $q \in \mathbb{T}^d$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$ we define

$$\mathcal{V}_s^t[\mu] := \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu]. \tag{3.20}$$

Lemma 3.15. *Increasing the value of C_A we obtain that for all $\mu \in \mathcal{P}(\mathbb{T}^d)$*

$$\sup_{s\in[0,T]} \|\mathcal{V}_s[\mu]\|_{W^{2,\infty}\big((0,T)\times\mathbb{T}^d\big)}, \quad \|\partial_s \mathcal{V}[\mu]\|_\infty, \quad \le C_A.$$

Proof. Since

$$\mathcal{V}_s^t[\mu] := \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu],$$

the first inequality follows directly from Lemma 3.13. We have

$$\partial_s \mathcal{V}_s^t[\mu] \circ \Sigma_s^t[\mu] = \partial_s \partial_t \Sigma_s^t[\mu] + \nabla_q \partial_t \Sigma_s^t[\mu] \partial_s X_s^t[\mu],$$

and by (3.18)

$$\partial_s X[\mu] \circ \Sigma_s^t[\mu] = -\big(\nabla_q \Sigma_s^t[\mu]\big)^{-1} \partial_s \Sigma_s^t[\mu].$$

Thus, applying once more Lemma 3.13, we obtain the second inequality of the lemma. ∎

Lemma 3.16. *Let $t_0 \in [0,T]$ and set $\sigma_{t_0} = \Sigma_s^{t_0}[\mu]_\# \mu$. We have:*

(i)
$$\Sigma_{t_0}^t[\sigma_{t_0}] \circ \Sigma_s^{t_0}[\mu] = \Sigma_s^t[\mu].$$

(ii) *The maps $\Sigma_{t_0}^t[\sigma_{t_0}]$ and $\Sigma_t^{t_0}[\mu]$ are inverses of each other.*

(iii)
$$\mathcal{V}_s^t[\mu] = \mathcal{V}_{t_0}^t[\sigma_{t_0}].$$

(iv)
$$\partial_s \Sigma_s^t[\mu] = -\nabla_q \Sigma_s^t[\mu] \mathcal{V}_t^s[\mu].$$

Proof. (i) Set

$$\bar{S}_t = S_t \circ S_{t_0}^{-1}, \quad \text{where} \quad S_t = \Sigma_s^t[\mu].$$

Obviously

$$\bar{S}_{t_0} = Id, \tag{3.21}$$

and

$$\partial_t \bar{S}_0 = \partial_t S_0 \circ S_{t_0}^{-1} = \nabla_q u_* \Big(S_0 \circ S_{t_0}^{-1},\ S_{0\,\#}\mu\Big) = \nabla_q u_* \big(\bar{S}_0, \bar{S}_{0\,\#}\sigma_{t_0}\big). \tag{3.22}$$

We exploit the fact that S_t satisfies the second order differential equation in (3.14) to obtain

$$\partial_{tt} \bar{S}_t = -\nabla_q F\big(S_t \circ S_{t_0}^{-1},\ S_{t\,\#}\mu\big) = -\nabla_q F(\bar{S}_t,\ \bar{S}_{t\,\#}\sigma_{t_0}). \tag{3.23}$$

We combine (3.21), (3.22) and (3.23) and apply Remark 3.12 to conclude that $\bar{S}$ is the unique fixed point of $M[\sigma_{t_0}]$. In other words,

$$\bar{S}_t = \Sigma_{t_0}^t[\sigma_{t_0}],$$

which implies the desired conclusion.

(ii) By the fact that $\Sigma[\mu]^s_s = Id$, (i) implies (ii).

(iii) By (i)

$$\partial_t \Sigma^t_s[\mu] = \partial_t \Sigma^t_{t_0}[\sigma_{t_0}] \circ \Sigma^{t_0}_s[\mu].$$

But (i) allows us to compose the left hand–side of the identity with $(\Sigma^t_s[\mu])^{-1}$ and the right hand–side with $(\Sigma^{t_0}_s[\mu])^{-1} \circ (\Sigma^t_{t_0}[\sigma_{t_0}])^{-1}$ to obtain

$$\partial_t \Sigma^t_s[\mu]\circ(\Sigma^t_s[\mu])^{-1} = \partial_t \Sigma^t_{t_0}[\sigma_{t_0}]\circ\Sigma^{t_0}_s[\mu]\circ(\Sigma^{t_0}_s[\mu])^{-1}\circ(\Sigma^t_{t_0}[\sigma_{t_0}])^{-1} = \partial_t \Sigma^t_{t_0}[\sigma_{t_0}]\circ(\Sigma^t_{t_0}[\sigma_{t_0}])^{-1}.$$

This establishes (iii).

(iv) By (i) and (ii) $Id = \Sigma^t_s[\mu] \circ \Sigma^s_t[\sigma_t]$ and so, differentiating both sides of the identity with respect to s we obtain (iv). ∎

Warning 3.17. It is worth pausing for the following remarks.

(i) We would like to warn the reader that in Lemma 3.16 (i) we are not making any claim about the identity $\Sigma^t_{t_0}[\nu] \circ \Sigma^{t_0}_s[\mu] = \Sigma^t_s[\mu]$ for an arbitrary ν. Similarly, in Lemma 3.16 (iii), no claim has been made about an identity as general as $\mathcal{V}^t_s[\mu] = \mathcal{V}^t_{t_0}[\nu]$ for an arbitrary ν.

(ii) We have never attempted to write any identity linking elements of $\mathcal{O}_s[\mu]$ with those of $\mathcal{O}_{\bar{s}}[\mu]$ when $\bar{s} \neq s$.

4. Properties of Σ in the variables (t, s, q); Continuity in μ.

Throughout this section we assume that $T > 0, A > 0$ satisfy (3.7).

Definition 4.1. Let

$$\mathcal{K} := [0, T] \times [0, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

We define the master map $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ by

$$\mathcal{S}(t, s, q, \mu) = \big(t, s, \Sigma^t_s[\mu](q),\ \mu\big).$$

Lemma 4.2. *The following hold :*

(i) *$\mathcal{S}$ is continuous and $\mathcal{S}(\cdot, \cdot, \cdot, \mu)$ is $2A$–Lipschitz.*

(ii) *$\partial_t \mathcal{S}$, $\partial_{tt} \mathcal{S} : \mathcal{K} \to \mathbb{R}^d$ are continuous.*

(iii) *The map $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is a homeomorphism.*

(iv) *$\mathcal{V} : \mathcal{K} \to \mathbb{R}^d$ is continuous.*

Proof. (i) Lemma 3.9 implies that $\mathcal{S}(\cdot, \cdot, \cdot, \mu)$ is $2A$–Lipschitz. To complete the proof of (i) it suffices to show that if $\{\mu^k\}_k \subset \mathcal{P}(\mathbb{T}^d)$ converges to μ, setting

$$S^k = \Sigma(\cdot, \cdot, \cdot, \mu^k),\ S = \Sigma(\cdot, \cdot, \cdot, \mu)$$

then $\{S^k\}_k$ converges uniformly to S. By Lemma 3.9, S^k is $2A$–Lipschitz and so, $\{S^k\}_k$ is equicontinuous. The Ascoli-Arzela lemma ensures the pre–compactness of the sequence in $C\big([0,T]\times[0,T]\times\mathbb{T}^d;\mathbb{T}^d\big)$ and so, the existence of a point of accumulation E. We invoke the continuity of M in all its variables as stated in Remark 3.4 (ii) to conclude that $E(\cdot,s,\cdot)$ is a fixed point of $M_s[\mu]$ for every s. In other words, $E(s,t,q)=\Sigma_s^t[\mu](q)$. Thus there is a unique point of accumulation of $\{S^k\}_k$, and hence we conclude that the whole sequence $\{S^k\}_k$ converges uniformly to S.

(ii) By assumption $\nabla_q F$ and $\nabla_q u_*$ are κ–Lipschitz. By (3.1), $\partial_t\Sigma$ is expressed in terms of Σ. Similarly, by (3.2), $\partial_{tt}\Sigma$ is expressed in terms of Σ. We use the continuity property of $\mathcal{S}$ to conclude that $\partial_t\mathcal{S}$ and $\partial_{tt}\mathcal{S}$ are continuous.

(iii) By Lemma 3.16 (ii), for any $\mu\in\mathcal{P}(\mathbb{T}^d)$, $\mathcal{S}(t,s,\cdot,\mu):\mathbb{T}^d\to\mathbb{T}^d$ is bijective. It thus follows that $\mathcal{S}$ is a bijection which is continuous from the compact set $\mathcal{K}$ into $\mathcal{K}$. Hence, $\mathcal{S}$ is a homeomorphism.

(iv) Recall that $\mathcal{V}_s^t[\mu]=\partial_t\Sigma_s^t[\mu]\circ X_s^t[\mu]$ and so, by (ii) and (iii), $\mathcal{V}$ is continuous. ■

Lemma 4.3. *The following functions are continuous and thus they are bounded:*

(i) $\nabla_q\Sigma:\mathcal{K}\to\mathbb{R}^{d\times d}$.
(ii) $\nabla_q\big(\partial_t\Sigma\big):\mathcal{K}\to\mathbb{R}^{d\times d}$.
(iii) $\partial_s\Sigma:\mathcal{K}\to\mathbb{R}^d$.

Proof. (i) Let $\{s_k\}_k$, $\{t_k\}_k\subset[0,T]$, $\{q_k\}_k\subset\mathbb{T}^d$ and $\{\mu_k\}_k\subset\mathcal{P}(\mathbb{T}^d)$ be sequences converging respectively to s, t, q and μ. We are to show that $\{\nabla_q\Sigma_{s_k}^{t_k}[\mu_k](q_k)\}_k$ converges to $\nabla_q\Sigma[\mu](t,s,q)$. By Theorem 3.8, $\Sigma[\mu_k](\cdot,s_k,\cdot)\in\mathcal{C}_A$ and so, for any $\bar q\in\mathbb{T}^d$

$$\Big|\Sigma_{s_k}^{t_k}[\mu_k](\bar q)-\Sigma_{s_k}^{t_k}[\mu_k](q_k)-\nabla_q\Sigma_{s_k}^{t_k}[\mu_k](q_k)\cdot(\bar q-q_k)\Big|\le\frac{A|\bar q-q_k|^2}{2}\tag{4.1}$$

and

$$|\nabla_q\Sigma_{s_k}^{t_k}[\mu_k](q_k)|\le A.$$

Hence $\{\nabla_q\Sigma_{s_k}^{t_k}[\mu_k](q_k)\}_k$ admits at least one point of accumulation, which we denote by P_0. Since, by Lemma 4.2 (i), Σ is continuous, (4.1) implies

$$\Big|\Sigma_s^t[\mu](\bar q)-\Sigma_s^t[\mu](q)-P_0\cdot(\bar q-q)\Big|\le\frac{A|\bar q-q|^2}{2}.$$

Thus, $P_0=\nabla_q\Sigma_s^t[\mu](q)$ is the unique point of accumulation. This proves (i).

(ii) Since $\Sigma_s[\mu]$ is a fixed point of $M_s[\mu]$, (3.4) yields

$$\begin{aligned}\nabla_{tq}\Sigma_s^t[\mu] &= \nabla_{qq}u_*\big(\Sigma_s^0[\mu](q),\Sigma_s^0[\mu]_\#\mu\big)\nabla_q\Sigma_s^0[\mu](q)\\ &-\int_0^t\nabla_{qq}F\big(\Sigma_s^\tau[\mu](q),\Sigma_s^\tau[\mu]_\#\mu\big)\nabla_q\Sigma_s^\tau[\mu](q)d\tau.\end{aligned}\tag{4.2}$$

Lemma 4.2 (i) ensures the continuity of Σ, while (i) of the current lemma ensures that $\nabla_q \Sigma$ is continuous. Since, $\nabla_{qq} u_*$ is continuous, Remark 3.3 implies that

$$(t, s, q, \mu) \to \nabla_{qq} u_* \big(\Sigma_s^0[\mu] q, \Sigma_s^0[\mu]_\# \mu\big)$$

is continuous. Similarly, we use the fact that $\nabla_{qq} F$ is continuous to obtain that

$$(t, s, q, \mu) \to \int_0^t \nabla_{qq} F\big(\Sigma_s^\tau[\mu](q), \Sigma_s^\tau[\mu]_\# \mu\big) \nabla_q \Sigma_s^\tau[\mu](q) d\tau$$

is continuous. Taking all these facts into consideration, representation formula (4.2) yields the continuity of $\nabla_{tq} \Sigma$.

(iii) Since, by Lemma 4.2 (iv), $\mathcal{V}$ is continuous, and by (ii), $\nabla_q \Sigma$ is continuous, the representation formula for $\partial_s \Sigma_s^t[\mu]$ provided by Lemma 3.16 (iv), ensures that $\partial_s \Sigma$ is continuous. ∎

Lemma 4.4. *The following maps are continuous and thus they are bounded:*

(i) $X : \mathcal{K} \to \mathbb{T}^d$.
(ii) $\nabla_q X : \mathcal{K} \to \mathbb{R}^{d \times d}$.
(iii) $\partial_t X : \mathcal{K} \to \mathbb{R}^d$.

Proof. (i) Lemma 4.2 (iii) gives that X is continuous on the compact set $\mathcal{K}$.

(ii) We use the representation formula (3.17), (i) and Lemma 4.3 (i) to obtain (ii).

(iii) By (3.18)

$$\partial_t X_s^t[\mu] = -\nabla_q X_s^t[\mu] \mathcal{V}_s^t[\mu]$$

and so, (ii) and Lemma 4.2 (iv) yield (iii). ∎

5. Minimality properties of Σ.

Throughout this section we assume that $T > 0, A > 0$ satisfy (3.7). The main result of this section is the following theorem.

Theorem 5.1. *Let $s \in [0, T]$, let $\mu \in \mathcal{P}(\mathbb{T}^d)$ and let $\sigma \in AC^2(0, s; \mathcal{P}(\mathbb{T}^d))$ be a path of velocity $\mathbf{v}$ such that $\sigma_s = \mu$. Then*

$$\mathcal{A}(s; \sigma, \mathbf{v}) \geq \mathcal{A}(s; \bar\sigma, \bar{\mathbf{v}}) + \frac{1 - 3\kappa C_T}{3T^2} \int_0^s W_2^2\big(\sigma_\tau, \bar\sigma_\tau\big) d\tau, \tag{5.1}$$

where

$$\bar\sigma_t = \Sigma_s^t[\mu]_\# \mu, \quad \bar{\mathbf{v}}_t = \mathcal{V}_s^t[\mu].$$

As a consequence $(\bar\sigma, \bar{\mathbf{v}})$ is the unique minimizer of (6.2) which will be later considered in Section 6. Furthermore, for almost every $t \in (0, s)$, $\mathcal{V}_s^t[\mu]$ is the velocity of minimal norm for $\bar\sigma$ and it belongs to $\mathcal{T}_{\bar\sigma_t} \mathcal{P}(\mathbb{T}^d)$.

Corollary 5.2. *Let $s \in [0,T]$, let $\mu \in \mathcal{P}(\mathbb{T}^d)$ and as above set*

$$\bar{\sigma}_t = \Sigma_s^t[\mu]_{\#}\mu, \quad \bar{\mathbf{v}}_t = \mathcal{V}_s^t[\mu].$$

Then:

(i) *If $r \in (0,T]$ and $\sigma \in AC^2\big(0,r;\mathcal{P}(\mathbb{T}^d)\big)$ has velocity $\mathbf{v}$ and $\sigma_r = \bar{\sigma}_r$ then*

$$\mathcal{A}(r;\sigma,\mathbf{v}) > \mathcal{A}(r;\bar{\sigma},\bar{\mathbf{v}})$$

unless $\sigma = \bar{\sigma}$.

(ii) *For every $t \in [0,T]$, $\mathcal{V}_s^t[\mu]$ is the gradient of a function and so, it belongs to $\mathcal{T}_{\bar{\sigma}_t}\mathcal{P}(\mathbb{T}^d)$, and $\nabla_q \mathcal{V}_s^t[\mu]$ is a symmetric matrix.*

We postpone the proof of Theorem 5.1 and first derive Corollary 5.2 from Theorem 5.1. In Subsection 5.1 we will first show a discrete version of (5.1) and then use an approximation argument to prove Theorem 5.1 in its full generality in Subsection 5.2.

Proof of Corollary 5.2. (i) Set

$$\sigma_t^* = \Sigma_r^t[\sigma_r]_{\#}\sigma_r, \quad \mathbf{v}_t^* = \mathcal{V}_r^t[\sigma_r].$$

By Theorem 5.1

$$\mathcal{A}(r;\sigma,\mathbf{v}) > \mathcal{A}(r;\sigma^*,\mathbf{v}^*) \tag{5.2}$$

unless $\sigma = \sigma^*$. By Lemma 3.16 (i)

$$\sigma_t^* = \Sigma_r^t[\sigma_r] \circ \Sigma_s^r[\mu]_{\#}\mu = \Sigma_s^t[\mu]_{\#}\mu = \bar{\sigma}_t. \tag{5.3}$$

By (iii) of the same lemma

$$\mathbf{v}_t^* = \bar{\mathbf{v}}_t. \tag{5.4}$$

We have thus established that $(\bar{\sigma},\bar{\mathbf{v}}) = (\sigma^*,\mathbf{v}^*)$. Using this in (5.2) we conclude the proof of (i). If we set $r = T$ in the above argument, Theorem 5.1 also gives us that there is a set $\mathcal{E} \subset (0,T)$ of full measure such that $\mathcal{V}_s^t[\mu]$ belongs to $\mathcal{T}_{\bar{\sigma}_t}\mathcal{P}(\mathbb{T}^d)$ for every $t \in \mathcal{E}$.

(ii) We divide the proof of (ii) into two steps.

Step 1. Assume $\sigma_s = \varrho_s\mathcal{L}^d$ and $\inf_{\mathbb{T}^d}\varrho_s > 0$. Since $\Sigma_s^t[\mu]_{\#}\mu = \bar{\sigma}_t$, Lemma 3.13 (i) implies that $\bar{\sigma}_t << \mathcal{L}^d$ and so, there exists a nonnegative function $\bar{\varrho}_t \geq 0$ such that $\bar{\sigma}_t = \bar{\varrho}_t\mathcal{L}^d$ and

$$\varrho_s(q) = \bar{\varrho}_t\big(\Sigma_s^t[\mu]\big) \det \nabla_q \Sigma_s^t[\mu]. \tag{5.5}$$

Since $\Sigma_s[\mu] \in \mathcal{C}_A$, we use Remark 2.7 to obtain $\det \nabla_q \Sigma_s^0[\mu] \leq A^d/d^{d/2} \leq A^d$. Thus (5.5) implies

$$0 < \frac{1}{A^d}\inf_{\mathbb{T}^d}\varrho_s \leq \bar{\varrho}_t. \tag{5.6}$$

Therefore, if $t \in \mathcal{E}$ then there exists $\bar{U}_t \in W^{1,2}(\mathbb{T}^d)$ such that $\nabla \bar{U}_t = \bar{\mathbf{v}}_t \in W^{2,\infty}(\mathbb{T}^d)^d$ (by Lemma 3.15) We thus have $\nabla \mathbf{v}_t = \nabla^2 \bar{U}_t$ and so, $\nabla \mathbf{v}_t$ is symmetric. Any $t \in [0,T]$ can be written as the limit of a sequence $\{t_n\}_n \subset \mathcal{E}$. Since $\{\bar{\mathbf{v}}_{t_n}\}_n$ converges uniformly to $\bar{\mathbf{v}}_t$ and there exists $\bar{U}_{t_n} \in W^{1,2}(\mathbb{T}^d)$ such that $\bar{\mathbf{v}}_{t_n} = \nabla \bar{U}_{t_n}$ we obtain a function $\bar{U}_t \in W^{1,2}(\mathbb{T}^d)$ such that $\bar{\mathbf{v}}_t = \nabla \bar{U}_t$. Hence, $\mathbf{v}_t$ belongs to $\mathcal{T}_{\bar{\sigma}_t}\mathcal{P}(\mathbb{T}^d)$ and $\nabla \mathbf{v}_t$ is symmetric for all $t \in [0,T]$.

Step 2. Assume $\mu \in \mathcal{P}(\mathbb{T}^d)$ is arbitrary. Choose a sequence of positive probability densities $\{\varrho_T^n\}_n \subset C(\mathbb{T}^d)$ such that $\inf_{\mathbb{T}^d} \varrho_s^n > 0$ and

$$\lim_{n\to\infty} W_2(\mu^n, \mu) = 0,$$

where we have set $\mu^n = \varrho_s^n \mathcal{L}^d$. Set

$$\sigma_t^n = \Sigma_s^t[\mu^n]_\# \mu^n, \quad \mathbf{v}_t^n = \mathcal{V}_s^t[\mu^n].$$

Since, by Lemma 4.2, $\mathcal{V}$ is continuous, we conclude that $\{\mathcal{V}_s^t[\mu^n]\}_n$ converges pointwise to $\mathcal{V}_s^t[\mu]$ on $\mathbb{T}^d$. By Lemma 3.15 and the Sobolev Imbedding Theorem, $\{\mathcal{V}_s[\mu^n]\}_n$ is pre–compact in $C^1([0,s]\times\mathbb{T}^d)^d$ and hence it converges to $\mathcal{V}_s[\mu]$ in the C^1–topology. Thus $\mathcal{V}_s^t[\mu]$ is the gradient of a function $\bar{U}_t \in C^1(\mathbb{T}^d)$ and $\nabla_q \mathcal{V}_s^t[\mu]$ is symmetric. ■

5.1. **Optimality properties of discrete paths.** Let $s \in (0,T]$, let

$$\bar{x}_1, \cdots, \bar{x}_n \in \mathbb{T}^d$$

and define $x_i : [0,T] \to \mathbb{T}^d$ by

$$x_i(t) = \Sigma_s^t[\mu^{\bar{\mathbf{x}}}](\bar{x}_i)$$

Using (2.6) and (2.12), by the definition of Σ we have

$$\begin{cases} (i) \quad \ddot{x}_i &= -\nabla_\mu \mathcal{F}\big(\mu^{\bar{\mathbf{x}}}\big)\big(\bar{x}_i(t)\big) \\ (ii) \quad x_i(s) &= \bar{x}_i \\ (iii) \quad \dot{x}_i(0) &= \nabla_\mu \mathcal{U}_*\big(\mu^{\mathbf{x}(0)}\big)\big(x_i(0)\big). \end{cases} \tag{5.7}$$

Let

$$y_1, \cdots, y_n \in W^{1,2}\big(0,T;\mathbb{T}^d\big).$$

Reordering and translating the $y_1(s), \cdots, y_n(s)$ if necessary, we may assume that when $t = s$

$$W_2^2\big(\mu^{\mathbf{x}(s)}, \mu^{\mathbf{y}(s)}\big) = \frac{1}{n}\sum_{i=1}^n |x_i(s) - y_i(s)|^2_{\mathbb{T}^d} = \frac{1}{n}\sum_{i=1}^n |x_i(s) - y_i(s)|^2.$$

Set

$$\gamma_t = \frac{1}{n}\sum_{i=1}^n \delta_{(x_i(t), y_i(t))}$$

so that

$$\gamma_s \in \Gamma_0\big(\mu^{\mathbf{x}(s)}, \mu^{\mathbf{y}(s)}\big).$$

We use the identity

$$\frac{|\dot y_i|^2}{2n} = \frac{|\dot x_i|^2}{2n} + \frac{|\dot y_i - \dot x_i|^2}{2n} + \frac{1}{n}(\dot y_i - \dot x_i)\cdot \dot x_i$$

and integrate by parts to obtain

$$\int_0^s \frac{|\dot y_i|^2}{2n} dt = \int_0^s \Big(\frac{|\dot x_i|^2}{2n} + \frac{|\dot y_i - \dot x_i|^2}{2n} - \frac{1}{n}(y_i - x_i)\cdot \ddot x_i \Big) dt + \Big[(y_i - x_i)\cdot \frac{\dot x_i}{n} \Big]_0^s. \tag{5.8}$$

By (2.7)

$$\mathcal{F}\big(\mu^{\mathbf{y}}\big) \le \mathcal{F}\big(\mu^{\mathbf{x}}\big) + \frac{1}{n}\sum_{i=1}^n \nabla_\mu \mathcal{F}\big(\mu^{\mathbf{x}}\big)(x_i)\cdot(y_i - x_i) + \frac{\kappa}{n}\sum_{i=1}^n |y_i - x_i|^2.$$

Hence, using (5.7) (i), we conclude that

$$\int_0^s \mathcal{F}\big(\mu^{\mathbf{y}}\big) dt \le \int_0^s \bigg(\mathcal{F}\big(\mu^{\mathbf{x}}\big) - \frac{1}{n}\sum_{i=1}^n \ddot x_i \cdot (y_i - x_i) + \frac{\kappa}{n}\sum_{i=1}^n |y_i - x_i|^2 \bigg) dt. \tag{5.9}$$

Similarly, (2.13) and (5.7) (iii) imply

$$\mathcal{U}_*\big(\mu^{\mathbf{y}(0)}\big)\big) \ge \mathcal{U}_*\big(\mu^{\mathbf{x}(0)}\big) + \frac{1}{n}\sum_{i=1}^n [\dot x_i(0)\cdot (y_i(0) - x_i(0)) - \kappa |y_i(0) - x_i(0)|^2]. \tag{5.10}$$

Let $\mathbf{v}$ be a velocity for $\mu^{\mathbf{x}}$ and let $\mathbf{w}$ be a velocity for $\mu^{\mathbf{y}}$. In fact $\mathbf{w}_t$ is uniquely determined for almost all $t \in (0,T)$. We combine (5.8), (5.9) and (5.10) to conclude that

$$\begin{aligned} \mathcal{A}(s;\mu^{\mathbf{y}},\mathbf{w}) - \mathcal{A}(s;\mu^{\mathbf{x}},\mathbf{v}) \quad &\ge \quad \frac{1}{n}\sum_{i=1}^n \dot x_i(s)\cdot (y_i(s) - x_i(s)) \\ &- \quad \frac{\kappa}{n}\sum_{i=1}^n |y_i(0) - x_i(0)|^2 \\ &+ \quad \frac{1}{2n}\sum_{i=1}^n \int_0^s \Big(|\dot y_i - \dot x_i|^2 - 2\kappa |y_i - x_i|^2 \Big) dt. \end{aligned} \tag{5.11}$$

Set

$$\Delta_i(t) := y_i(t) - x_i(t) = y_i(s) - x_i(s) + \int_s^t (\dot y_i(\tau) - \dot x_i(\tau)) d\tau.$$

We have for $0 \le t \le s$

$$|\Delta_i(t)| \le |\Delta_i(s)| + \int_0^s |\dot{\Delta}_i(\tau)| d\tau$$

and so,

$$|\Delta_i(t)|^2 \le 3|\Delta_i(s)|^2 + \frac{3}{2} T \int_0^s |\dot{\Delta}_i(\tau)|^2 d\tau$$

This proves that

$$|\Delta_i(0)|^2 \le 3|\Delta_i(s)|^2 + \frac{3}{2} T \int_0^s |\dot{\Delta}_i(\tau)|^2 d\tau \tag{5.12}$$

and

$$\int_0^s |\Delta_i(t)|^2 dt \le 3T|\Delta_i(s)|^2 + \frac{3}{2} T^2 \int_0^s |\dot{\Delta}_i(\tau)|^2 d\tau. \tag{5.13}$$

We combine (5.11), (5.12) and (5.13) to obtain

$$\begin{aligned} \mathcal{A}(s;\mu^{\mathbf{y}},\mathbf{w}) - \mathcal{A}(s;\mu^{\mathbf{x}},\mathbf{v}) \quad \ge \quad & \frac{1}{n}\sum_{i=1}^n \dot{x}_i(s)\cdot \Delta_i(s) \\ + \quad & \frac{1-3\kappa C_T}{2n} \sum_{i=1}^n \int_0^s |\dot{\Delta}_i|^2 d\tau \\ - \quad & \frac{\kappa}{n}(1+3T)\sum_{i=1}^n |\Delta_i(s)|^2. \end{aligned} \tag{5.14}$$

We use again (5.13) in (5.14) to obtain

$$\begin{aligned} \mathcal{A}(s;\mu^{\mathbf{y}},\mathbf{w}) - \mathcal{A}(s;\mu^{\mathbf{x}},\mathbf{v}) \quad \ge \quad & \frac{1}{n}\sum_{i=1}^n \dot{x}_i(s)\cdot \Delta_i(s) \\ + \quad & \frac{1-3\kappa C_T}{3T^2 n} \sum_{i=1}^n \int_0^s |\Delta_i|^2 d\tau \\ - \quad & \frac{1-3\kappa C_T}{Tn} \sum_{i=1}^n |\Delta_i(s)|^2 \\ - \quad & \frac{\kappa}{n}(1+3T)\sum_{i=1}^n |\Delta_i(s)|^2. \end{aligned} \tag{5.15}$$

Hence,

$$
\begin{aligned}
\mathcal{A}(s;\mu^{\mathbf{y}},\mathbf{w})-\mathcal{A}(s;\mu^{\mathbf{x}},\mathbf{v}) \quad \geq \quad & -\frac{1}{n}\sqrt{\sum_{i=1}^{n}\|\dot{x}_i\|_\infty^2}\sqrt{\sum_{i=1}^{n}|\Delta_i(s)|^2} \\
+ \quad & \frac{1-3\kappa C_T}{3T^2 n}\sum_{i=1}^{n}\int_0^s |\Delta_i|^2 d\tau \\
- \quad & \frac{1-3\kappa C_T}{Tn}\sum_{i=1}^{n}|\Delta_i(s)|^2 \\
- \quad & \frac{\kappa}{n}(1+3T)\sum_{i=1}^{n}|\Delta_i(s)|^2. \qquad (5.16)
\end{aligned}
$$

We now use in (5.16) that $\Sigma_s \in \mathcal{C}_A$ and

$$
\gamma_t \in \Gamma\big(\mu^{\mathbf{x}(t)},\mu^{\mathbf{y}(t)}\big), \quad \gamma_s \in \Gamma_0\big(\mu^{\mathbf{x}(s)},\mu^{\mathbf{y}(s)}\big), \quad |\Delta_i(s)| = |\Delta_i(s)|_{\mathbb{T}^d}
$$

to obtain

$$
\begin{aligned}
\mathcal{A}(s;\mu^{\mathbf{y}},\mathbf{w})-\mathcal{A}(s;\mu^{\mathbf{x}},\mathbf{v}) \quad \geq \quad & -AW_2\big(\mu^{\mathbf{x}(s)},\mu^{\mathbf{y}(s)}\big)-B_T W_2^2\big(\mu^{\mathbf{x}(s)},\mu^{\mathbf{y}(s)}\big) \\
+ \quad & \frac{1-3\kappa C_T}{3T^2}\int_0^s W_2^2\big(\mu^{\mathbf{x}(\tau)},\mu^{\mathbf{y}(\tau)}\big)d\tau. \qquad (5.17)
\end{aligned}
$$

Here, B_T is a constant depending only on T and κ.

5.2. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. For each integer $m \geq 1$, let $\mathcal{P}^m(\mathbb{T}^d)$ denote the set of averages of m Dirac masses on $\mathbb{T}^d$. Let $AC^2(0,s;\mathcal{P}^m(\mathbb{T}^d))$ denote the set of pairs $(\sigma,\mathbf{w})$ such that $\sigma \in AC^2(0,s;\mathcal{P}(\mathbb{T}^d))$ and $(\sigma,\mathbf{w})$ satisfy the following properties: There exist

$$
y_i \in W^{1,2}(0,s;\mathbb{T}^d), \quad \text{for} \quad i=1,\cdots,m
$$

such that

$$
\sigma_t = \frac{1}{m}\sum_{i=1}^{m}\delta_{y_i(t)} \quad \text{and} \quad \mathbf{w}_t \circ y_i = \dot{y}_i, \quad i=1,\cdots,m \quad \text{a.e..}
$$

Let $s \in [0,T]$, let $\mu \in \mathcal{P}(\mathbb{T}^d)$ and let $\sigma \in AC^2(0,s;\mathcal{P}(\mathbb{T}^d))$ be a path of velocity $\mathbf{w}$ such that $\sigma_s = \mu$. Proposition 5.1 of [21] provides us with a sequence of pairs $(\sigma^m,\mathbf{w}^m)$ and a sequence of real numbers $\{r^m\}_m \subset (0,1)$ decreasing to 0 such that

$$
\sup_{t\in[0,s]} W_2(\sigma_t^m,\sigma_t) \leq r^m, \quad \frac{1}{2}\int_0^s ||\mathbf{w}^m||^2_{\sigma_t^m} dt \leq \frac{1}{2}\int_0^s ||\mathbf{w}||^2_{\sigma_t} dt + r^m. \qquad (5.18)
$$

We combine (5.17) and (5.18) and use the fact that $\mathcal{F}$ and $\mathcal{U}_*$ are κ–Lipschitz to obtain

$$\mathcal{A}(s;\sigma^m,\mathbf{w}^m) \leq (1+s\kappa)r^m + \kappa r^m + \mathcal{A}(s;\sigma,\mathbf{w}). \tag{5.19}$$

Let $\{x_i^m(s)\}_{i=1}^m \subset \mathbb{T}^d$ be such that

$$\lim_{m\to\infty} \bar r^m = 0, \quad \text{where} \quad \bar r^m = W_2(\bar\sigma_s^m,\mu) \quad \text{and} \quad \bar\sigma_s^m = \frac{1}{m}\sum_{i=1}^m \delta_{x_i^m(s)}.$$

Set

$$x_i^m(t) = \Sigma_s^t[\bar\sigma_s^m](x_i^m(s)), \quad \bar{\mathbf{v}}_t^m = \mathcal{V}_s^t[\bar\sigma_s^m], \quad \bar\sigma = \Sigma_s^t[\mu]_\#\mu, \quad \bar{\mathbf{v}}_t = \mathcal{V}_s^t[\mu].$$

Note that

$$\bar\sigma_t^m = \Sigma_s^t[\bar\sigma_s^m]_\#\bar\sigma_s^m \tag{5.20}$$

and thus

$$\|\bar{\mathbf{v}}_t^m\|_{\sigma_t^m} = \|\partial_t\Sigma_s^t[\bar\sigma_s^m]\|_{\bar\sigma_s^m} \quad \text{and} \quad ||\bar{\mathbf{v}}_t||_{\bar\sigma_t} = ||\partial_t\Sigma_s^t[\mu]||_\mu. \tag{5.21}$$

Because $\{\bar\sigma_s^m\}_m$ converges to μ, and $\Sigma_s[\bar\sigma_s^m] \in \mathcal{C}_A$ we obtain that $\{\Sigma_s[\bar\sigma_s^m]\}$ is equicontinuous and so, $\{\Sigma_s[\bar\sigma_s^m]\}$ converges uniformly to $\Sigma_s[\mu]$ on $[0,s]\times\mathbb{T}^d$. Hence,

$$\lim_{m\to\infty} \tilde r_1^m = 0, \quad \text{where} \quad r_1^m = \sup_{t\in[0,s]} W_2^2(\bar\sigma_t^m,\bar\sigma_t). \tag{5.22}$$

Similarly, $\{\partial_t\Sigma_s[\bar\sigma_s^m]\}$ converges uniformly to $\partial_t\Sigma_s[\mu]$ on $[0,s]\times\mathbb{T}^d$. Consequently, using the identities in (5.21), we have

$$\lim_{m\to\infty} r_2^m = 0, \quad \text{where} \quad r_2^m = \sup_{t\in[0,s]}\Big| ||\bar{\mathbf{v}}_t^m||^2_{\sigma_t^m} - ||\bar{\mathbf{v}}_t||^2_{\bar\sigma_t}\Big|. \tag{5.23}$$

Hence, since $\mathcal{U}_*$ and $\mathcal{F}$ are Lipschitz, (5.22) and (5.23) imply

$$\lim_{m\to\infty} \mathcal{A}(s;\bar\sigma^m,\bar{\mathbf{v}}^m) = \mathcal{A}(s;\bar\sigma,\bar{\mathbf{v}}) \tag{5.24}$$

We now apply inequality (5.17) to get

$$\begin{aligned} \mathcal{A}(s;\sigma^m,\mathbf{w}^m) - \mathcal{A}(s;\bar\sigma^m,\bar{\mathbf{v}}^m) \quad &\geq \quad -AW_2\big(\sigma_s^m,\bar\sigma_s^m\big) - B_T W_2^2(\sigma_s^m,\bar\sigma_s^m) \\ &+ \quad \frac{1-3\kappa C_T}{3T^2}\int_0^s W_2^2\big(\sigma_\tau^m,\bar\sigma_\tau^m\big)d\tau. \end{aligned} \tag{5.25}$$

Letting m tend to ∞ in (5.25), we use (5.18), (5.19), (5.22) and (5.24) to obtain

$$\mathcal{A}(s;\sigma,\mathbf{w}) \geq \mathcal{A}(s;\bar\sigma,\bar{\mathbf{v}}) + \frac{1-3\kappa C_T}{3T^2}\int_0^s W_2^2\big(\sigma_\tau,\bar\sigma_\tau\big)d\tau.$$

This concludes the proof of (5.1).

A straightforward consequence of (5.1) is that $(\bar\sigma,\bar{\mathbf{v}})$ is the unique minimizer in (6.2).

Denote by $|\bar\sigma'|$ the metric derivative of σ (see e.g. [4]). By Proposition 8.3.1 of [4] there exists a velocity $\mathbf{v}^*$ for $\bar\sigma$ such that for almost every $t \in (0,s)$

$$||\mathbf{v}^*_t||_{\bar\sigma_t} \le |\sigma'|(t) \le ||\bar{\mathbf{v}}_t||_{\bar\sigma_t}. \tag{5.26}$$

This implies that all inequalities in (5.26) are equalities as otherwise we would get

$$\mathcal{A}(s;\bar\sigma,\mathbf{v}^*) < \mathcal{A}(s;\bar\sigma,\bar{\mathbf{v}})$$

which would contradict the minimality property of $(\bar\sigma,\bar{\mathbf{v}})$. By Proposition 8.4.5 of [4], since we have equalities in (5.26) for almost every $t \in (0,s)$, we get $\bar{\mathbf{v}}_t \in \mathcal{T}_{\bar\sigma_t}\mathcal{P}(\mathbb{T}^d)$ for almost every $t \in (0,s)$. ∎

6. Hamilton–Jacobi equation on $\mathcal{P}(\mathbb{T}^d)$.

Throughout this section we assume that $T > 0, A > 0$ satisfy (3.7). We assume (see Example 2.9) to be given $U^0 \in C^3(\mathbb{T}^d)$, U^1, $\phi \in C^3(\mathbb{T}^d)$ such that the latter two functions are even and the three functions satisfy

$$||\phi||_{C^3(\mathbb{T}^d)}, \quad 2||U^0||_{C^3(\mathbb{T}^d)}, \quad 2||U^1||_{C^3(\mathbb{T}^d)} \le \kappa. \tag{6.1}$$

We assume that for any $q \in \mathbb{T}^d$ and any $\mu \in \mathcal{P}(\mathbb{T}^d)$

$$F(q,\mu) = \phi * \mu(q), \qquad \mathcal{F}(\mu) = \int_{\mathbb{T}^d} \frac{1}{2}\phi * \mu(y)\mu(dy),$$

so that

$$\nabla_q F(q,\mu) = \nabla\phi * \mu(q).$$

We set

$$u_*(q,\mu) = U^0(q) + U^1 * \mu(q), \qquad \mathcal{U}_*(\mu) = \int_{\mathbb{T}^d} \big(U^0 + \frac{1}{2}U^1 * \mu\big)(y)\mu(dy).$$

For $s \in [0,T], \mu \in \mathcal{P}(\mathbb{T}^d)$ we define the value function

$$\mathcal{U}(s,\mu) = \inf\Big\{\int_0^s \mathcal{L}(\sigma,\mathbf{v})dt + \mathcal{U}_*(\sigma_0) \mid \sigma_s = \mu\Big\}, \tag{6.2}$$

where the infimum is taken over the set of all pairs $(\sigma,\mathbf{v})$ such that $\sigma \in AC^2(0,s;\mathcal{P}(\mathbb{T}^d))$ and $\mathbf{v}$ is a velocity for σ. Recall that $\mathcal{L}$ is defined by (2.1).

Using the terminology of [2] and [20], $\mathcal{U}$ is the unique metric viscosity solution to

$$\begin{cases} \partial_t\mathcal{U} + \mathcal{H}(\mu,\nabla_\mu\mathcal{U}) = 0 \quad \text{in} \quad (0,T)\times\mathcal{P}(\mathbb{T}^d) \\ \mathcal{U}(0,\cdot) = \mathcal{U}_* \quad \text{on} \quad \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{6.3}$$

Furthermore, $\mathcal{U}$ satisfies the semigroup property (the so–called dynamic programming principle): For any $r \in [0,s]$

$$\mathcal{U}(s,\mu) = \inf\Big\{\int_r^s \mathcal{L}(\sigma,\mathbf{v})ds + \mathcal{U}(r,\sigma_r) \mid \sigma_s = \mu\Big\}. \tag{6.4}$$

Proposition 6.1. *Fix $s \in [0,T]$, $\mu \in \mathcal{P}(\mathbb{T}^d)$ and set*

$$\bar\sigma_t = \Sigma_s^t[\mu]_\#\mu, \quad \bar{\mathbf{v}}_t = \mathcal{V}_s^t[\mu], \qquad \forall\, t \in [0,T].$$

Then, for any $r \in [0,T]$, we have

$$\mathcal{U}(r, \bar\sigma_r) = \mathcal{A}(r; \bar\sigma, \bar{\mathbf{v}}),$$

in particular $\mathcal{U}(s,\mu) = \mathcal{A}(s; \bar\sigma, \bar{\mathbf{v}})$.

Proof. The result is a direct consequence of Corollary 5.2. ■

Remark 6.2. We recall that $\mathcal{U}$ is Lipschitz continuous on $[0,T] \times \mathcal{P}(\mathbb{T}^d)$ (see [20]).

6.1. **Semiconvexity/semiconcavity properties of the value function.** Fix a positive integer n. For $\mathbf{q} = (q_1, \cdots, q_n) \in (\mathbb{T}^d)^n$ and $\mathbf{p} = (p_1, \cdots, p_n) \in (\mathbb{R}^d)^n$ we define

$$\mathcal{U}_0^n(\mathbf{q}) = \frac{1}{n}\sum_{i=1}^n U^0(q_i) + \frac{1}{2n^2}\sum_{i,j=1}^n U^1(q_i - q_j)$$

and

$$L_n(q,p) = \frac{|\mathbf{p}|^2}{2n} - \frac{1}{2n^2}\sum_{i,j=1}^n \phi(q_i - q_j).$$

We notice that for any $\mathbf{q} \in (\mathbb{T}^d)^n, \mathbf{p} \in (\mathbb{R}^d)^n$

$$-\nabla_q F(q_i, \mu^{\mathbf{q}}) = n\nabla_{q_i} L_n(\mathbf{q}, \mathbf{p}), \tag{6.5}$$

$$\nabla_q u_*(q_i, \mu^{\mathbf{q}}) = n\nabla_{q_i}\mathcal{U}_0^n(\mathbf{q}). \tag{6.6}$$

For $\mathbf{x} = (x_1, \cdots, x_n) \in (\mathbb{T}^d)^n$ we define

$$\mathcal{U}^n(s, \mathbf{x}) = \inf_{\mathbf{y}}\Big\{\int_0^s L_n(\mathbf{y}, \dot{\mathbf{y}})dt + \mathcal{U}_0^n(\mathbf{y}(0)) \mid \mathbf{y}(0) = \mathbf{x}\Big\}.$$

Proposition 6.1 implies that $\mathcal{U}^n(s,\mathbf{x}) = \mathcal{U}(s, \mu^{\mathbf{x}})$.

Lemma 6.3. *Let $s \in (0,T)$, let $\mathbf{x} = (x_1, \cdots, x_n) \in (\mathbb{T}^d)^n, \mathbf{x}^* = (x_1^*, \cdots, x_n^*) \in (\mathbb{T}^d)^n$. Then there exists $\gamma \in \Gamma_0\big(\mu^{\mathbf{x}}, \mu^{\mathbf{x}^*}\big)$ such that*

(i)

$$\begin{aligned}\mathcal{U}\big(s, \mu^{\mathbf{x}^*}\big) \;&\le\; \mathcal{U}\big(s, \mu^{\mathbf{x}}\big) + \int_{\mathbb{T}^d\times\mathbb{T}^d} \mathcal{V}_s^s[\mu^{\mathbf{x}}](q)\cdot(b-q)\gamma(dq,db) \\ &+\; \kappa(1+s)W_2^2\big(\mu^{\mathbf{x}}, \mu^{\mathbf{x}^*}\big).\end{aligned} \tag{6.7}$$

(ii)

$$\begin{aligned} \mathcal{U}\big(s,\mu^{\mathbf{x}^*}\big) &\geq \mathcal{U}\big(s,\mu^{\mathbf{x}}\big) + \int_{\mathbb{T}^d\times\mathbb{T}^d} \mathcal{V}_s^s[\mu^{\mathbf{x}}](q)\cdot(b-q)\gamma(dq,db) \\ &- \frac{1}{2}W_2^2\big(\mu^{\mathbf{x}},\mu^{\mathbf{x}^*}\big)\Big(\frac{1}{(T-s)}+\kappa(T-s)\Big). \end{aligned} \tag{6.8}$$

Proof. (i) For any $t\in[0,T]$ we set

$$z_i(t)=\Sigma_s^t[\mu^{\mathbf{x}}](x_i), \quad i=1,\cdots,n.$$

Then Proposition 6.1 yields

$$\mathcal{U}^n(s,\mathbf{x})=\mathcal{U}(s,\mu^{\mathbf{x}})=\int_0^s L_n(\mathbf{z},\dot{\mathbf{z}})dt+\mathcal{U}_0^n(\mathbf{z}(0)). \tag{6.9}$$

Since

$$\dot{z}_i(0)=\partial_t\Sigma_s^0[\mu^{\mathbf{x}}](x_i)=\nabla_q u_*\big(z_i(0),\mu^{\mathbf{z}(0)}\big), \quad \ddot{z}_i(t)=-\nabla_q F(z_i(t),\mu^{\mathbf{z}(t)}),$$

(6.5) and (6.6) imply

$$\nabla\mathcal{U}_0^n(\mathbf{z}(0))=\frac{\dot{z}(0)}{n}, \quad \nabla_{q_i}L_n(\mathbf{z}(t),\dot{\mathbf{z}}(t))=\frac{\ddot{z}(t)}{n}. \tag{6.10}$$

It follows from (2.7) and (2.13) that the functions $\mathcal{U}_0^n$ and $L_n(\cdot,\mathbf{p})$ for every $\mathbf{p}\in(\mathbb{R}^d)^n$ are $2\kappa/n$–concave. We now set

$$y_i(t)=z_i(t)+x_i^*-x_i.$$

We have

$$\mathcal{U}^n(s,\mathbf{x}^*)\leq\int_0^s L_n(\mathbf{y},\dot{\mathbf{y}})dt+\mathcal{U}_0^n(\mathbf{y}(0)). \tag{6.11}$$

Hence, using (6.9), (6.10), (6.11) and the semiconcavity of $\mathcal{U}_0^n, L_n(\cdot,\dot{\mathbf{z}})$, we obtain

(6.12)

$$\begin{aligned} \mathcal{U}^n(s,\mathbf{x}^*) &\leq \mathcal{U}^n(s,\mathbf{x})+\int_0^s[L_n(\mathbf{y},\dot{\mathbf{y}})-L_n(\mathbf{z},\dot{\mathbf{z}})]dt+\mathcal{U}_0^n(\mathbf{y}(0))-\mathcal{U}_0^n(\mathbf{z}(0)) \\ &\leq \mathcal{U}^n(s,\mathbf{x})+\int_0^s\frac{\ddot{\mathbf{z}}}{n}\cdot(\mathbf{x}^*-\mathbf{x})dt+\frac{\kappa s}{n}|\mathbf{x}^*-\mathbf{x}|^2+\frac{\dot{\mathbf{z}}(0)}{n}\cdot(\mathbf{x}^*-\mathbf{x})+\frac{\kappa}{n}|\mathbf{x}^*-\mathbf{x}|^2 \\ &=\mathcal{U}^n(s,\mathbf{x})+\frac{\dot{\mathbf{z}}(s)}{n}\cdot(\mathbf{x}^*-\mathbf{x})+\frac{\kappa(1+s)}{n}|\mathbf{x}^*-\mathbf{x}|^2. \end{aligned}$$

Reordering the points $x_1^*,\cdots,x_n^*$ and translating them if necessary, we may assume without loss of generality that

$$\frac{1}{n}\sum_{i=1}^n|x_i-x_i^*|^2=W_2^2\big(\mu^{\mathbf{x}},\mu^{\mathbf{x}^*}\big) \tag{6.13}$$

and thus

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{(x_i,x_i^*)}\in\Gamma_0\big(\mu^{\mathbf{x}},\mu^{\mathbf{x}^*}\big).$$

Therefore (6.12) and (6.13) imply (6.7).

(ii) By Proposition 6.1

$$\mathcal{U}^n(T,\mathbf{z}(T))=\int_0^T L_n\mathbf{z},\dot{\mathbf{z}})dt+\mathcal{U}_0^n(\mathbf{z}(0))=\int_s^T L_n(\mathbf{z},\dot{\mathbf{z}})dt+\mathcal{U}^n(s,\mathbf{x}). \tag{6.14}$$

We set

$$y_i(t)=z_i(t)+\frac{T-t}{T-s}(x_i^*-x_i),\quad t\in[s,T].$$

Since $\mathbf{y}(s)=\mathbf{x}^*$ and $\mathbf{y}(T)=\mathbf{z}(T)$ use the fact that $\mathcal{U}$ satisfies the semigroup property (6.4) to obtain

$$\mathcal{U}^n(s,\mathbf{x}^*)\ge\mathcal{U}^n(T,\mathbf{z}(T))-\int_s^T L_n(\mathbf{y},\dot{\mathbf{y}})dt$$

This, together with (6.14) implies

$$\mathcal{U}^n(s,\mathbf{x}^*)\ge\mathcal{U}^n(s,\mathbf{x})+\int_s^T\big(L_n(\mathbf{z},\dot{\mathbf{z}})-L_n(\mathbf{y},\dot{\mathbf{y}})\big)dt.$$

Therefore, by the semiconcavity of $L_n(\cdot,\dot{\mathbf{z}})$ and (6.10), we obtain

$$\begin{aligned}\mathcal{U}^n(s,\mathbf{x}^*)&\ge\mathcal{U}^n(s,\mathbf{x})-\frac{1}{2n}\frac{|\mathbf{x}^*-\mathbf{x}|^2}{T-s}+\frac{1}{n}\int_s^T\frac{\dot{\mathbf{z}}\cdot(\mathbf{x}^*-\mathbf{x})}{T-s}dt\\&\quad-\frac{1}{n}\int_s^T\ddot{\mathbf{z}}\cdot(\mathbf{x}^*-\mathbf{x})\frac{T-t}{T-s}dt-\frac{\kappa}{n}|\mathbf{x}^*-\mathbf{x}|^2\frac{T-s}{3}\\&\ge\mathcal{U}^n(s,\mathbf{x})+\frac{\dot{\mathbf{z}}(s)\cdot(x^*-x)}{n}-\frac{|x^*-x|^2}{2n}\Big(\frac{1}{T-s}+\kappa(T-s)\Big).\end{aligned}\tag{6.15}$$

We conclude the proof arguing as in part (i). ∎

Theorem 6.4. *Let $\mu,\mu^*\in\mathbb{T}^d$ and let $\gamma\in\Gamma_0\big(\mu,\mu^*\big)$.*

(i) *If $s\in[0,T]$ then*

$$\begin{aligned}\mathcal{U}\big(s,\mu^*\big)&\le\mathcal{U}\big(s,\mu\big)+\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathcal{V}_s^s[\mu](q)\cdot(b-q)\gamma(dq,db)\\&+\kappa(1+s)W_2^2\big(\mu,\mu^*\big).\end{aligned}\tag{6.16}$$

(ii) *If* $s \in [0,T)$

$$\begin{aligned} \mathcal{U}\big(s,\mu^*\big) \quad &\geq \quad \mathcal{U}\big(s,\mu\big) + \int_{\mathbb{T}^d\times\mathbb{T}^d} \mathcal{V}_s^s[\mu](q)\cdot(b-q)\gamma(dq,db) \\ &- \quad \frac{1}{2}W_2^2\big(\mu,\mu^*\big)\Big(\frac{1}{(T-s)}+\kappa(T-s)\Big). \end{aligned} \tag{6.17}$$

(iii) *For any* $s \in (0,T)$ *and* $t \in [0,T)$ *we have*

$$\nabla_\mu \mathcal{U}(t,\bar\sigma_t) = \mathcal{V}_s^t[\mu],$$

where $\bar\sigma_t = \Sigma_s^t[\mu]_\# \mu$.

Proof. (i) The function $\mathcal{U}$ is continuous by Remark 6.2. Since, by Lemma 4.2, $\mathcal{V}$ is continuous, it suffices to prove (6.16) for $s,t \in (0,T)$. Using again the fact that $\mathcal{U}$ and $\mathcal{V}$ are continuous, since every μ and μ^* can be approximated by averages of Dirac masses, it follows from Lemma 6.3 that there exists $\gamma \in \Gamma_0\big(\mu,\mu^*\big)$ such that (i) holds. It remains to show that (i) holds for all $\gamma \in \Gamma_0\big(\mu,\mu^*\big)$. We fix such γ, and let μ^*_λ be the geodesic defined by

$$\int_{\mathbb{T}^d} \varphi(q)\mu^*_\lambda(dq) = \int_{\mathbb{T}^d\times\mathbb{T}^d} \varphi\big((1-\lambda)q+\lambda b\big)\gamma(dq,db).$$

By Lemma 7.2.1 of [4], for $\lambda \in (0,1)$, $\Gamma_0\big(\mu,\mu^*_\lambda\big)$ contains a unique element γ_l. Thus,

$$\mathcal{U}(s,\mu^*_\lambda) \leq \mathcal{U}(s,\mu) + \int_{\mathbb{T}^d\times\mathbb{T}^d} \mathcal{V}_s^s[\mu](q)\cdot(b-q)\gamma_\lambda(dq,db) + \kappa W_2^2\big(\mu,\mu^*_\lambda\big)\big(1+s\big).$$

Letting λ tend to 1 we obtain (6.16). Similar arguments yield (6.17).

(iii) By (i), $\mathcal{V}_s^s[\mu]$ belongs to $\partial^\cdot\mathcal{U}(s,\mu)\cap\partial_\cdot\mathcal{U}(s,\mu)$, whereas Corollary 5.2 ensures that $\mathcal{V}_s^s[\mu] \in \mathcal{T}_\mu\mathcal{P}(\mathbb{T}^d)$. Therefore, by the remark in Definition 2.5 (iii), $\nabla_\mu\mathcal{U}(s,\mu) = \mathcal{V}_s^s[\mu]$. Using Lemma 3.16 (iii) we now have

$$\nabla_\mu \mathcal{U}(t,\bar\sigma_t) = \mathcal{V}_t^t[\bar\sigma_t] = \mathcal{V}_s^t[\mu].$$

■

Remark 6.5. Lemma 6.3 and Theorem 6.4 correct and sharpen the statements of Theorem 5.1 (iii) and Theorem 5.2 (iv) of [19].

Lemma 6.6. *Let* $s \in (0,T]$, *let* $\mu \in \mathcal{P}(\mathbb{T}^d)$ *and set*

$$\mathbf{v}_t = \mathcal{V}_s^t[\mu], \quad \sigma_t = \Sigma_s^t[\mu]_\#\mu.$$

Then:

(i) $(\sigma, \mathbf{v})$ *satisfies the following system of equations, where the first identity in (6.18) holds pointwise,*

$$\begin{cases} \partial_t \mathbf{v} + \nabla \mathbf{v} \mathbf{v} = -\nabla_q F(\cdot, \sigma_t) \\ \mathbf{v}_0 = \nabla_\mu \mathcal{U}_*[\sigma_0] = \nabla U^0 + \nabla U^1 * \sigma_0. \end{cases} \tag{6.18}$$

(ii) $\nabla \mathbf{v}_t$ *is the gradient of a function and thus it is a symmetric matrix for any* $t \in [0, T]$.

Proof. Observe first that by Lemma 3.15 and the Sobolev Imbedding Theorem, $\mathbf{v}$ is continuously differentiable on $(0, T) \times \mathbb{T}^d$.

(i) By the definition of $\mathbf{v}$ we have $\partial_t \Sigma_s^t[\mu] = \mathbf{v}_t(\Sigma_s^t[\mu])$ and so, differentiating with respect to t and using the first equation in (3.14) we obtain

$$\begin{aligned} -\nabla_q F\Big(\Sigma_s^t[\mu](q),\, \Sigma_s^t[\mu]_{\#}\mu\Big) &= \partial_{tt} \Sigma_s^t[\mu](q) \\ &= \partial_t \mathbf{v}_t\big(\Sigma_s^t[\mu]q\big) + \nabla \mathbf{v}_t\big(\Sigma_s^t[\mu]q\big) \partial_t \Sigma_s^t[\mu](q) \end{aligned}$$

Thus

$$-\nabla_q F\Big(q,\, \Sigma_s^t[\mu]_{\#}\mu\Big) = \partial_t \mathbf{v}_t(q) + \nabla \mathbf{v}_t(q) \mathbf{v}_t(q),$$

which gives the first identity in (6.18). The second identity in (6.18) follows from Theorem 6.4 (iii).

Part (ii) is already stated in Corollary 5.2. ∎

7. Weak solution to the first order Mean Field equations.

Throughout this section we assume that $T > 0, A > 0$ satisfy (3.7). We also assume that F, u_* and $\mathcal{U}_*$ are given through functions ϕ, U^0, and U^1 satisfying the assumptions imposed in Section 6.

Given $s \in [0, T]$, $q \in \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathbb{T}^d)$, we define

$$u(s, q, \mu) = u_*\big(q, \Sigma_s^0[\mu]_{\#}\mu\big) - \int_0^s \bigg(\frac{|\mathcal{V}_s^\tau[\mu](q)|^2}{2} + F\big(q, \Sigma_s^\tau[\mu]_{\#}\mu\big) \bigg) d\tau, \tag{7.1}$$

and set

$$\bar{\sigma}_t = \Sigma_s^t[\mu]_{\#}\mu \quad \text{and} \quad \bar{\mathbf{v}}_t = \mathcal{V}_s^t[\mu] \qquad \forall\, t \in [0, T]. \tag{7.2}$$

Since, by Lemma 3.16,

$$\mathcal{V}_t^\tau[\bar{\sigma}_t] = \mathcal{V}_s^\tau[\mu] \quad \text{and} \quad \Sigma_t^\tau[\bar{\sigma}_t]_{\#}\bar{\sigma}_t = \Sigma_t^\tau[\bar{\sigma}] \circ \Sigma_s^t[\mu]_{\#}\mu = \Sigma_s^\tau[\mu]_{\#}\mu = \bar{\sigma}_\tau,$$

we conclude that

$$u(t, q, \bar{\sigma}_t) = u_*(q, \bar{\sigma}_0) - \int_0^t \bigg(\frac{|\mathcal{V}_s^\tau[\mu](q)|^2}{2} + F\big(q, \Sigma_s^\tau[\mu]_{\#}\mu\big) \bigg) d\tau, \qquad \forall t \in [0, T] \tag{7.3}$$

Lemma 7.1. *We have, for every* $s \in [0, T], \mu \in \mathcal{P}(\mathbb{T}^d)$*:*

(i) *for any* $(t,q) \in (0,T)$

$$u(0,\cdot,\mu) = u_*(\cdot,\mu) \quad \textit{and} \quad \nabla_q u(t,q,\bar\sigma_t) = \bar{\mathbf{v}}_t(q) = \nabla_\mu \mathcal{U}_t(t,\bar\sigma_t)(q).$$

(ii) $t \to u\big(t,q,\bar\sigma_t\big)$ *is continuously differentiable and*

$$\partial_t\big(u(t,q,\bar\sigma_t)\big) + \frac{|\nabla_q u(t,q,\bar\sigma_t)|^2}{2} + F(q,\bar\sigma_t) = 0, \quad \forall\, (t,q) \in (0,T)\times\mathbb{T}^d.$$

Proof. (i) The identity $u(0,\cdot,\mu) = u_*(\cdot,\mu)$ is straightforward to check.

We substitute $\mathcal{V}_s^\tau[\mu]$ by $\bar{\mathbf{v}}_\tau$ in (7.3) and differentiate the subsequent identity with respect to q to obtain

$$\nabla_q u(t,q,\bar\sigma_t) = \nabla_q u_*(q,\bar\sigma_0) - \int_0^t \Big(\nabla^T \bar{\mathbf{v}}_\tau(q)\bar{\mathbf{v}}_\tau(q) + \nabla_q F(q,\bar\sigma_\tau)\Big)d\tau.$$

We use that, by Lemma 6.6 (ii), $\nabla\bar{\mathbf{v}}_\tau$ is symmetric and then use Lemma 6.6 (i) to conclude that

$$\begin{aligned} \nabla_q u(t,q,\bar\sigma_t) &= \nabla_q u_*(q,\bar\sigma_0) - \int_0^t \Big(\nabla \bar{\mathbf{v}}_\tau(q)\bar{\mathbf{v}}_s(q) + \nabla_q F(\cdot,\bar\sigma_\tau)\Big)d\tau \\ &= \nabla_q u_*(q,\bar\sigma_0) + \int_0^t \partial_t \bar{\mathbf{v}}_\tau(q)d\tau. \end{aligned}$$

We combine this with the fact that, by Lemma 6.6 (i), $\nabla_q u_*(q,\bar\sigma_0) = \bar{\mathbf{v}}_0$, and use Theorem 6.4 (iii) to obtain

$$\nabla_q u(t,q,\bar\sigma_t) = \nabla_q u_*(q,\bar\sigma_0) + \bar{\mathbf{v}}_t(q) - \bar{\mathbf{v}}_0(q) = \bar{\mathbf{v}}_t(q) = \nabla_\mu \mathcal{U}(t,\bar\sigma_t)(q). \tag{7.4}$$

(ii) By Lemma 4.2 (iv), $\mathcal{V}$ is continuous in all its variables. Since ϕ and Σ are continuous we conclude that for every y,

$$\tau \to F(q,\bar\sigma_\tau) = \int_{\mathbb{T}^d} \phi\big(q - \Sigma_s^\tau[\mu](y)\big)\mu(dy)$$

is continuous. Using the representation formula provided by (7.3) we thus conclude that the function $t \to u\big(t,q,\bar\sigma_t\big)$ is continuously differentiable and

$$\partial_t\big(u(t,q,\bar\sigma_t)\big) + \frac{|\mathcal{V}_s^t[\mu](q)|^2}{2} + F(q,\bar\sigma_t) = 0.$$

We now substitute $\mathcal{V}_s^t[\mu]$ by $\nabla_q u(t,q,\bar\sigma_t)$ to conclude the proof of (ii). ∎

We now fix s and μ and define the function U (which depends on s and μ) by

$$U(t,q) \equiv U_{s,\mu}(t,q) = u\big(t,q,\Sigma_s^t[\mu]_\#\mu\big)$$

so that, by (7.3),

$$U(t,q) = u_*(q,\bar\sigma_0) - \int_0^t \Big(\frac{|\bar{\mathbf{v}}_\tau(q)|^2}{2} + F(q,\bar\sigma_\tau)\Big)d\tau.$$

Corollary 7.2. *The following hold:*

(i)
$$U \in W^{2,\infty}\big((0,T)\times\mathbb{T}^d\big)$$

(ii) *U is a classical solution (hence also a viscosity solution) to (7.5) (a), where*

$$\begin{cases} (a) \quad \partial_t U(t,q) + \frac{|\nabla U(t,q)|^2}{2} + F(q,\bar\sigma_t) = 0 \\ (b) \quad \partial \bar\sigma_t + \nabla\cdot(\bar\sigma_t \nabla U) = 0 \quad in \quad \mathcal{D}'\big((0,T)\big)\times\mathbb{T}^d\big) \\ (c) \quad U_0 = u_*(q,\bar\sigma_0), \quad \bar\sigma_s = \mu. \end{cases} \tag{7.5}$$

Moreover, if μ has a density with respect to the Lebesgue measure, then so does $\bar\sigma_t$ for every $t\in[0,T]$.

Proof. By Lemma 7.1, (7.5) (a) holds in the classical sense and

$$\partial_t U(t,q) = -\frac{|\mathcal{V}_s^t[\mu](q)|^2}{2} - \int_{\mathbb{T}^d}\phi\big(q-\Sigma_s^\tau[\mu](y)\big)\mu(dy), \quad \nabla_q U(t,q) = \mathcal{V}_s^t[\mu](q). \tag{7.6}$$

Lemma 3.15 guarantees that $\mathcal{V}_s[\mu]$ belongs to $W^{2,\infty}\big((0,T)\times\mathbb{T}^d\big)^d$, while Theorem 3.8 (ii) ensures that $\Sigma_s[\mu]$ belongs to $W^{2,\infty}\big((0,T)\times\mathbb{T}^d;\mathbb{T}^d\big)$. Hence, by (7.6),

$$U \in W^{2,\infty}\big((0,T)\big)\times\mathbb{T}^d\big)$$

and, using the fact that $\mathbf{v}$ is a velocity for σ and that $\nabla_q U_t = \bar{\mathbf{v}}_t$, we obtain (7.5) (b). The two identities in (7.5) (c) follow from Lemma 7.1.

It is clear from the definition of $\bar\sigma_t$ and the regularity of $X_s^t[\mu]$ (the inverse of $\Sigma_s^t[\mu]$) given by Lemma 3.13 (iii), that if μ has a density with respect to the Lebesgue measure, then so does $\bar\sigma_t$ for every $t\in[0,T]$. ∎

We notice that if $u(\cdot,q,\cdot)$ is regular enough then, by Lemma 9.8,

$$\partial_t\big(u(t,q,\bar\sigma_t)\big) = \partial_t u(t,q,\bar\sigma_t) + \int_{\mathbb{T}^d}\nabla_\mu u(t,q,\bar\sigma_t)(z)\cdot\bar{\mathbf{v}}_t\,\bar\sigma_t(dz). \tag{7.7}$$

When $t=s$ then $\bar\sigma_t=\mu$ and so, we may then use Lemma 7.1 (i) to substitute $\nabla_q u(s,z,\mu)$ for $\bar{\mathbf{v}}_s(z)$ in (7.7) and then use Lemma 7.1 (ii) to obtain

$$\partial_s u(s,q,\mu) + \int_{\mathbb{T}^d}\nabla_\mu u(s,q,\mu)(z)\cdot\nabla_q u(s,z,\mu)\;\mu(dz) + \frac{|\nabla_q u(s,q,\mu)|^2}{2} + F(q,\mu) = 0.$$

Thus to prove that u is a pointwise (strong) solution to the master equation (1.1), it suffices to show that u is regular enough in all its variables.

The above comments suggest the following definition of a weak solutions to (1.1). Let

$$u:[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}$$

be a continuous function such that $u(0,\cdot,\cdot)=u_*$.

Definition 7.3. We say that u is a weak solution to (1.1) if for every $s \in (0,T)$ and every $\mu \in \mathcal{P}(\mathbb{T}^d)$ there exists a path $\sigma \in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$ with a velocity $\mathbf{v}$ such that the following hold:

(i) for almost every $t \in (0,T)$, $\nabla_q u(t,\cdot,\sigma_t)$ exists σ_t–almost everywhere.

(ii) $\sigma_s = \mu$ and for almost every $t \in (0,T)$

$$\mathbf{v}_t = \nabla_q u(t,\cdot,\sigma_t), \quad \sigma_t\text{–almost everywhere.}$$

(iii) $U_\mu(t,q) := u(t,q,\sigma_t)$ is a viscosity solution to (7.5) (a).

8. Regularity properties of $\Sigma(t,s,q,\cdot)$; A discretization approach.

Throughout this section we assume that $T>0, A>0$ satisfy (3.7) and that F, u_* and $\mathcal{U}_*$ are given through functions ϕ, U^0, and U^1 satisfying the assumptions imposed in Section 6.

We recall that Σ and $\mathcal{S}$ are given by Definitions 3.10 and 4.1.

Remark 8.1. Let $\{a^k\}_k \subset [0,\infty)$ be a sequence and let α and $\beta<1$ be two nonnegative numbers such that $a^k \le \alpha + \beta a^{k-1}$ for all natural numbers k. Then

$$a^k \le \alpha \sum_{i=1}^{k-1} \beta^i + \beta^k a^0 \le \frac{\alpha}{1-\beta} + a^0.$$

8.1. **Spatial derivatives of the discrete master map.** Throughout this subsection n is a fixed natural number. To $\mathbf{x} = (x_1,\cdots,x_n) \in (\mathbb{T}^d)^n$ we associate the measure

$$\mu^{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}.$$

For $s \in [0,T]$ recall that $M_s[\mu]$ is the map defined in (2.17). For $\mathbf{x} \in (\mathbb{T}^d)^n$ and any continuous map

$$S : [0,T]\times[0,T]\times\mathbb{T}^d\times(\mathbb{T}^d)^n \to \mathbb{T}^d,$$

we define

$$M_s[\mathbf{x}](S) := M_s[\mu^{\mathbf{x}}](S(\cdot,s,\cdot,\mathbf{x})),$$

i.e.

$$\begin{aligned} M_s[\mathbf{x}](S)(t,q) &= q + (t-s)\nabla U^0\big(S(0,s,q,\mathbf{x})\big) \\ &+ (t-s)\frac{1}{n}\sum_{j=1}^n \nabla U^1\big(S(0,s,q,\mathbf{x}) - S(0,s,x_j,\mathbf{x})\big) \\ &+ \frac{1}{n}\sum_{j=1}^n \int_t^s ds \int_0^s \nabla\phi\big(S(\tau,s,q,\mathbf{x}) - S(\tau,s,x_j,\mathbf{x})\big)d\tau. \end{aligned} \tag{8.1}$$

Corollary 8.2. *Let $S^0(t,s,q,\mathbf{x}) \equiv q$. Defining inductively*

$$S^k(t,s,q,\mathbf{x}) = M_s[\mathbf{x}]\big(S^{k-1}\big)(t,q),$$

the following hold:

(i) $S^k(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}_A^*$ *for every* $\mathbf{x} \in (\mathbb{T}^d)^n$.

(ii) *There is a constant C_A independent of k such that for any $i \in \{1,\cdots,n\}$*

$$\|\nabla_{x_i} S^k\|_\infty, \quad \|\nabla_{x_i}\partial_t S^k\|_\infty \le \frac{C_A}{n}. \tag{8.2}$$

Proof. Since $S^0(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}_A^*$, Lemma 3.9 implies (i).

(ii) We have

$$\begin{aligned}
&\nabla_{x_i}\big(S^k(t,s,q,\mathbf{x})\big) = (t-s)\nabla^2 U^0\big(S^{k-1}(0,s,q,\mathbf{x})\big)\nabla_{x_i} S^{k-1}(0,s,q,\mathbf{x}) \\
-\;& \frac{1}{n}(t-s)\nabla^2 U^1\big(S^{k-1}(0,s,q,\mathbf{x}) - S^{k-1}(0,s,x_i,\mathbf{x})\big)\nabla_q S^{k-1}(0,s,x_i,\mathbf{x}) \\
+\;& \frac{1}{n}(t-s)\sum_{j=1}^n \nabla^2 U^1\big(S^{k-1}(0,s,q,\mathbf{x}) - S^{k-1}(0,s,x_i,\mathbf{x})\big)\Delta_i(0,q,x_j,\mathbf{x}) \\
-\;& \frac{1}{n}\int_t^s dl \int_0^l \nabla^2\phi\Big(S^{k-1}(\tau,s,q,\mathbf{x}) \\
&\qquad\qquad -S^{k-1}(\tau,s,x_i,\mathbf{x})\Big)\nabla_q S^{k-1}(\tau,s,x_i,\mathbf{x})d\tau \\
+\;& \frac{1}{n}\sum_{j=1}^n\int_t^s dl \int_0^l \nabla^2\phi\Big(S^{k-1}(\tau,s,q,\mathbf{x}) \\
&\qquad\qquad -S^{k-1}(\tau,s,x_j,\mathbf{x})\Big)\Delta_i(\tau,q,x_j,\mathbf{x})d\tau,
\end{aligned} \tag{8.3}$$

where we do not display the s dependence in

$$\Delta_i(\tau,q,x_j,\mathbf{x}) := \nabla_{x_i} S^{k-1}(\tau,s,q,\mathbf{x}) - \nabla_{x_i} S^{k-1}(\tau,s,x_j,\mathbf{x}).$$

We exploit (8.3) to infer

$$\|\nabla_{x_i} S^k\|_\infty \le \frac{\kappa C_T}{2n}\|\nabla_q S^{k-1}\|_\infty + \kappa T(2+T)\|\nabla_{x_i} S^{k-1}\|_\infty.$$

Since $S^{k-1}(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}_A^*$, we conclude that

$$\|\nabla_{x_i} S^k\|_\infty \le \frac{\kappa A C_T}{2n} + 2\kappa C_T\|\nabla_{x_i} S^{k-1}\|_\infty. \tag{8.4}$$

We apply Remark 8.1 to (8.4) and use the fact that $\nabla_{x_i} S^0 \equiv 0$ to obtain

$$\|\nabla_{x_i} S^k\|_\infty \le \frac{\kappa A C_T}{2n(1-2\kappa C_T)}. \tag{8.5}$$

Direct differentiation yields

$$\begin{aligned}
&\nabla_{x_i}\big(\partial_t S^k(t,s,q,\mathbf{x})\big) = \nabla^2 U^0\big(S^{k-1}(0,s,q,\mathbf{x})\big)\nabla_{x_i} S^{k-1}(0,s,q,\mathbf{x})\\
- \quad & \frac{1}{n}\nabla^2 U^1\big(S^{k-1}(0,s,q,\mathbf{x}) - S^{k-1}(0,s,x_i,\mathbf{x})\big)\nabla_q S^{k-1}(0,s,x_i,\mathbf{x})\\
+ \quad & \frac{1}{n}\sum_{j=1}^n \nabla^2 U^1\big(S^{k-1}(0,s,q,\mathbf{x}) - S^{k-1}(0,s,x_i,\mathbf{x})\big)\Delta_i(0,q,x_j,\mathbf{x})\\
+ \quad & \frac{1}{n}\int_0^t \nabla^2\phi\big(S^{k-1}(\tau,s,q,\mathbf{x}) - S^{k-1}(\tau,s,x_i,\mathbf{x})\big)\nabla_q S^{k-1}(\tau,s,x_i,\mathbf{x})d\tau\\
- \quad & \frac{1}{n}\sum_{j=1}^n\int_0^t \nabla^2\phi\big(S^{k-1}(\tau,s,q,\mathbf{x}) - S^{k-1}(\tau,s,x_j,\mathbf{x})\big)\Delta_i(\tau,q,x_j,\mathbf{x})d\tau.
\end{aligned}$$

Estimating we thus obtain

$$\|\nabla_{x_i}\partial_t S^k\|_\infty \le 2\kappa(1+T)\|\nabla_{x_i} S^{k-1}\|_\infty + \frac{\kappa A(1+T)}{n}.$$

This, together with (8.5), yields

$$\|\nabla_{x_i}\partial_t S^k\|_\infty \le 2\kappa(1+T)\frac{\kappa A C_T}{2n(1-2\kappa C_T)} + \frac{\kappa A(1+T)}{n}.$$

We can choose C_A in terms of T,κ and A to conclude the proof of the lemma. ∎

Corollary 8.3. *The sequence $\{S^k\}_k$ defined in Corollary 8.2 converges uniformly on $[0,T]\times[0,T]\times\mathbb{T}^d\times(\mathbb{T}^d)^n$ to the function S defined by $S(t,s,q,\mathbf{x}) = \Sigma_s^t[\mu^{\mathbf{x}}](q)$. Furthermore, the following hold:*

(i) *S satisfies (8.2), $S(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}_A^*$ for every $\mathbf{x}\in(\mathbb{T}^d)^n$, and $S(t,s,q,\mathbf{x}) = S(t,s,q,\bar{\mathbf{x}})$ if $\bar{\mathbf{x}}$ is a permutation of $\mathbf{x}$.*

(ii) *Increasing the value of C_A if necessary we have:*

$$\|\nabla_{qx_i} S\|_\infty \le \frac{C_A}{n}, \quad i=1,...,n. \tag{8.6}$$

$$\|\nabla_{x_jx_i} S\|_\infty \le \frac{C_A}{n^2}, \quad i,j=1,...,n, i\neq j. \tag{8.7}$$

$$\|\nabla_{x_ix_i} S\|_\infty \le \frac{C_A}{n}, \quad i=1,...,n. \tag{8.8}$$

Proof. By Lemma 3.9, for each $\mathbf{x}$ fixed, $\{S^k(\cdot,\cdot,\cdot,\mathbf{x})\}_k$ converges uniformly to $S(\cdot,\cdot,\cdot)$ defined by $S(t,s,q,\mathbf{x}) = \Sigma_s^t[\mu^{\mathbf{x}}](q)$. Thanks to the Arzela–Ascoli Lemma, the bounds on $\{S^k\}_k$ and its derivatives provided by Corollary 8.2 imply that $\{S^k\}_k$ converges uniformly to S on $[0,T]\times[0,T]\times\mathbb{T}^d\times(\mathbb{T}^d)^n$.

(i) The fact that S satisfies (8.2) and $S(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}_A^*$ follows from Corollary 8.2. Since for t, s and q fixed, $S(t,s,q,\cdot)$ depends only on $\mu^{\mathbf{x}}$ then $S(t,s,q,\mathbf{x}) = S(t,s,q,\bar{\mathbf{x}})$ if $\bar{\mathbf{x}}$ is a permutation of $\mathbf{x}$.

(ii) We differentiate both sides of (8.3) with respect to q to obtain an identity from which we derive the upper bound

$$\|\nabla_{qx_i} S^k\|_\infty \le \kappa C_T \Big(2\|\nabla_q S^{k-1}\|_\infty \|\nabla_{x_i} S^{k-1}\|_\infty + \|\nabla_{q,x_i} S^{k-1}\|_\infty \Big) + \frac{\kappa C_T}{2n} \|\nabla_q S^{k-1}\|_\infty^2.$$

This, together with Corollary 8.2 implies

$$\|\nabla_{qx_i} S^k\|_\infty \le \kappa C_T \|\nabla_{q,x_i} S^{k-1}\|_\infty + \frac{2\kappa A C_A C_T}{n} + \frac{\kappa A^2 C_T}{n}.$$

We apply Remark 8.1 and use the fact that $\|\nabla_{qx_i} S^0\|_\infty = 0$ and then replace C_A by an appropriate larger constant, still denoted by C_A, such that

$$\|\nabla_{qx_i} S^k\|_\infty \le \frac{C_A}{n}.$$

Letting k tend to ∞ we obtain (8.6).

For (8.7) we differentiate both sides of (8.3) with respect to $x_j, j \neq i$, and estimate the subsequent expression to obtain

$$\begin{aligned} \|\nabla_{x_j x_i} S^k\|_\infty \;\; &\le \;\; 2\kappa C_T \|\nabla_{x_j x_i} S^{k-1}\|_\infty + A_1 \|\nabla_{x_i} S^{k-1}\|_\infty \cdot \|\nabla_{x_j} S^{k-1}\|_\infty \\ &+ \;\; \frac{A_2}{n} \Big(\|\nabla_{x_j} S^{k-1}\|_\infty \cdot \|\nabla_q S^{k-1}\|_\infty + \|\nabla_{x_j q} S^{k-1}\|_\infty \Big). \end{aligned}$$

for some A_1, A_2 depending only on κ, T. We then use (i), (8.2) and (8.6) to get

$$\|\nabla_{x_j x_i} S^k\|_\infty \le 2\kappa C_T \|\nabla_{x_j x_i} S^{k-1}\|_\infty + \frac{A_3}{n^2}$$

for some A_3 depending only on κ, T, A, C_A. We now apply Remark 8.1 and use the fact that $\|\nabla_{x_l x_i} S^0\|_\infty = 0$ and to obtain a constant that we still denote by C_A such that

$$\|\nabla_{x_j x_i} S^k\|_\infty \le \frac{C_A}{n^2}.$$

Letting k tend to ∞ yields (8.7).

To obtain (8.8) we differentiate both sides of (8.3) with respect to x_i and repeat similar arguments. However now the differentiation of the second and fourth lines in (8.3) will produce terms that can only be bounded by

$$\frac{C}{n} \Big(\|\nabla_q S^{k-1}\|_\infty^2 + \|\nabla_{qq} S^{k-1}\|_\infty \Big)$$

for some constant C. This is the reason why we obtain a weaker estimate than (8.7). ∎

Corollary 8.4. *Let*

$$s \in (0,T), \quad \bar t, \hat t \in [0,T], \quad \bar q, \hat q \in \mathbb{T}^d, \quad \bar{\mathbf{x}}, \hat{\mathbf{x}} \in (\mathbb{T}^d)^n$$

be such that

$$W_2^2\big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}}\big) = \frac{1}{n}\sum_{i=1}^n |\hat x_i - \bar x_i|^2_{\mathbb{T}^d}$$

Set

$$\bar S = S(\bar t, s, \bar q, \bar{\mathbf{x}}), \quad \hat S = S(\hat t, s, \hat q, \hat{\mathbf{x}}), \tag{8.9}$$

and

$$\partial_t \bar S = \partial_t S(\bar t, s, \bar q, \bar{\mathbf{x}}), \quad \nabla_q \bar S = \nabla_q S(\bar t, s, \bar q, \bar{\mathbf{x}}), \quad \nabla_{x_i} \bar S = \nabla_{x_i} S((\bar t, s, \bar q, \bar{\mathbf{x}}). \tag{8.10}$$

Then there is a constant, still denoted by C_A, such that

$$\begin{aligned} &|\hat S - \bar S - (\hat t - \bar t)\partial_t \bar S - \nabla_q \bar S(\hat q - \bar q) - \sum_{i=1}^n \nabla_{x_i} \bar S(\hat x_i - \bar x_i)| \\ \le\ & C_A\Big((\hat t - \bar t)^2 + |\hat q - \bar q|^2_{\mathbb{T}^d} + W_2^2\big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}}\big)\Big). \end{aligned} \tag{8.11}$$

Proof. We write the Taylor expansion of S around $(\bar t, \bar q, \bar{\mathbf{x}})$ to obtain that the expression in the left hand side of (8.11) is bounded by $L + K = L + K_1 + K_2$, where

$$L = \frac{1}{2}\Bigg((\hat t - \bar t)^2 ||\partial_{tt} S||_\infty + \sum_{i,j=1}^n |\hat x_i - \bar x_i|_{\mathbb{T}^d} \cdot |\hat x_j - \bar x_j|_{\mathbb{T}^d} \|\nabla_{x_i x_j} S\|_\infty + |\hat q - \bar q|^2_{\mathbb{T}^d} \|\nabla_{qq} S\|_\infty\Bigg)$$

and

$$K_1 = \|\nabla_q \partial_t S\|_\infty |\hat t - \bar t| \cdot |\hat q - \bar q| + \sum_{i=1}^n \|\nabla_{x_i} \partial_t S\|_\infty |\hat t - \bar t| \cdot |\hat x_i - \bar x_i|_{\mathbb{T}^d}$$

and

$$K_2 = \sum_{i=1}^n \|\nabla_{q x_i} S\|_\infty |\hat q - \bar q|_{\mathbb{T}^d} \cdot |\hat x_i - \bar x_i|_{\mathbb{T}^d}.$$

Corollaries 8.2 and 8.3 provide us with upper bounds on the partial derivatives of S up to the second order. These bounds yield

$$\sum_{i\neq j} |\hat{x}_i - \bar{x}_i|_{\mathbb{T}^d} \cdot |\hat{x}_j - \bar{x}_j|_{\mathbb{T}^d} \|\nabla_{x_i x_j} S\|_\infty \leq \frac{C_A}{2n^2} \sum_{i\neq j} \left(|\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d} + |\hat{x}_j - \bar{x}_j|^2_{\mathbb{T}^d} \right)$$

$$\leq \frac{C_A}{n} \sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d} \tag{8.12}$$

and

$$\sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d} \|\nabla_{x_i x_i} S\|_\infty \leq \frac{C_A}{n} \sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d}. \tag{8.13}$$

We also have

$$(\hat{t} - \bar{t})^2 \|\partial_{tt} S\|_\infty + |\hat{q} - \bar{q}|^2_{\mathbb{T}^d} \|\nabla_{qq} S\|_\infty \leq \kappa (\tilde{t} - \bar{t})^2 + A |\tilde{q} - \bar{q}|^2_{\mathbb{T}^d}. \tag{8.14}$$

We combine (8.12), (8.13) and (8.14) to conclude that

$$L \leq \tilde{C}_A \Big((\hat{t} - \bar{t})^2 + |\hat{q} - \bar{q}|^2_{\mathbb{T}^d} + W_2^2 \big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}} \big) \Big) \tag{8.15}$$

for some constant $\tilde{C}_A$.

Since

$$\|\nabla_{x_i} \partial_t S\|_\infty |\hat{t} - \bar{t}| \cdot |\hat{x}_i - \bar{x}_i|_{\mathbb{T}^d} \leq \frac{C_A}{n} \frac{|\hat{t} - \bar{t}|^2 + |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d}}{2},$$

summing up we have

$$\sum_{i=1}^n \|\nabla_{x_i} \partial_t S\|_\infty |\hat{t} - \bar{t}| \cdot |\hat{x}_i - \bar{x}_i|_{\mathbb{T}^d} \leq \frac{C_A}{2} |\hat{t} - \bar{t}|^2 + \frac{C_A}{2n} \sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d}. \tag{8.16}$$

Similarly,

$$\sum_{i=1}^n \|\nabla_{q x_i} S\|_\infty |\hat{q} - \bar{q}|_{\mathbb{T}^d} \cdot |\hat{x}_i - \bar{x}_i|_{\mathbb{T}^d} \leq \frac{C_A}{2} |\hat{q} - \bar{q}|^2_{\mathbb{T}^d} + \frac{C_A}{2n} \sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2_{\mathbb{T}^d}. \tag{8.17}$$

Notice also that

$$\|\nabla_q \partial_t S\|_\infty |\hat{t} - \bar{t}| \cdot |\hat{q} - \bar{q}| \leq \frac{3\kappa A}{4} \left(|\hat{t} - \bar{t}|^2 + |\hat{q} - \bar{q}|^2_{\mathbb{T}^d} \right). \tag{8.18}$$

We combine (8.16), (8.17) and (8.18) to conclude that

$$K \leq \tilde{D}_A \Big((\hat{t} - \bar{t})^2 + |\hat{q} - \bar{q}|^2_{\mathbb{T}^d} + W_2^2 \big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}} \big) \Big)$$

for some constant $\tilde{D}_A$. This, together with (8.15), completes the proof of the lemma. ∎

Corollary 8.5. *Increasing the value of C_A if necessary we have:*

(i)

$$\|\partial_{ttt}S\|_\infty,\ \ \|\nabla_q\partial_{tt}S\|_\infty,\ \ \|\nabla^2_{qq}\partial_t S\|_\infty \le C_A.$$

(ii) *For* $i = 1, ..., n$

$$\|\nabla_{x_i}\partial_{tt}S\|_\infty,\ \ \|\nabla_{q,x_i}\partial_t S\|_\infty \le \frac{C_A}{n}.$$

(iii) *For* $i, j = 1, ..., n, i \neq j$

$$\frac{1}{n}\|\nabla_{x_ix_i}\partial_t S\|_\infty, \quad \|\nabla_{x_ix_j}\partial_t S\|_\infty \le \frac{C_A}{n^2}.$$

Proof. (i) Since

$$\begin{aligned} \partial_{tt}S(t,s,q,\mathbf{x}) &= -\nabla_q F(S(t,s,q,\mathbf{x}), S(t,s,\cdot,\mathbf{x})_\# \mu^{\mathbf{x}}) \\ &= -\frac{1}{n}\sum_{j=1}^n \nabla\phi\big(S(t,s,q,\mathbf{x}) - S(t,s,x_j,\mathbf{x})\big), \end{aligned} \tag{8.19}$$

we have

$$\begin{aligned} &\partial_{ttt}S(t,s,q,\mathbf{x}) \\ = \ &-\frac{1}{n}\sum_{j=1}^n \nabla^2\phi\big(S(t,s,q,\mathbf{x}) - S(t,s,x_j,\mathbf{x})\big)\big(\partial_t S(t,s,q,\mathbf{x}) - \partial_t S(t,s,x_j,\mathbf{x})\big). \end{aligned}$$

We use the fact that $S(\cdot,\cdot,\cdot,\mathbf{x}) \in \mathcal{C}^*_A$ (see Corollary 8.3 (i)) to conclude that

$$\|\partial_{ttt}S\|_\infty \le 2\kappa A.$$

Similarly, we obtain the remaining inequalities in (i).

(ii) Differentiating both sides of (8.19) with respect to x_i we obtain

$$\begin{aligned} \nabla_{x_i}\partial_{tt}S(t,s,q,\mathbf{x}) &= \frac{1}{n}\nabla^2\phi\big(S(t,s,q,\mathbf{x}) - S(t,s,x_i,\mathbf{x})\big)\nabla_q S(t,s,x_i,\mathbf{x}) \\ -\frac{1}{n}\sum_{j=1}^n &\nabla^2\phi\big(S(t,s,q,\mathbf{x}) - S(t,s,x_j,\mathbf{x})\big)\big(\nabla_{x_i}S(t,s,q,\mathbf{x}) - \nabla_{x_i}S(t,s,x_j,\mathbf{x})\big). \end{aligned}$$

This, together with Corollary 8.3 (i), gives us

$$\|\nabla_{x_i}\partial_{tt}S\|_\infty \le \frac{\kappa A}{n} + \frac{2\kappa C_A}{n}.$$

The other inequality in (ii) is proved similarly.

(iii) We differentiate $\nabla_{x_j}\partial_t S$ with respect to x_i and use Corollary 8.3. ∎

Corollary 8.6. *Let*

$$\bar t, \hat t \in [0,T], \quad \bar q, \hat q \in \mathbb{T}^d, \quad \bar{\mathbf{x}}, \hat{\mathbf{x}} \in (\mathbb{T}^d)^n$$

be such that

$$W_2^2\big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}}\big) = \frac{1}{n}\sum_{i=1}^n |\hat x_i - \bar x_i|^2_{\mathbb{T}^d}.$$

Using the notation (8.9) and (8.10), increasing the value of C_A if necessary, we have

$$|\partial_t \hat S - \partial_t \bar S - (\hat t - \bar t)\partial_{tt}\bar S - \nabla_q \partial_t \bar S(\hat q - \bar q) - \sum_{i=1}^n \nabla_{x_i}\partial_t \bar S(\hat x_i - \bar x_i)|$$

$$\leq \quad C_A\Big((\hat t - \bar t)^2 + |\hat q - \bar q|^2_{\mathbb{T}^d} + W_2^2\big(\mu^{\bar{\mathbf{x}}}, \mu^{\hat{\mathbf{x}}}\big)\Big). \tag{8.20}$$

Proof. We exploit Corollary 8.5 to complete the proof in exactly the same way Corollaries 8.2 and 8.3 are used to prove Corollary 8.4. ∎

8.2. **Spatial derivatives of the inverse of the master map.** Throughout this section n is a fixed natural number. We define

$$\mathcal{S}(t,s,q,\mathbf{x}) := (t,s,\Sigma^t_s[\mu^{\mathbf{x}}](q),\mathbf{x}) = (t,s,S(t,s,q,\mathbf{x}),\mathbf{x}),$$

$$R(t,s,b,\mathbf{x}) := X^t_s[\mu^{\mathbf{x}}](b).$$

By Corollary 8.3, for every $s \in [0,T]$,

$$\mathcal{S}(\cdot,s,\cdot,\cdot) \in W^{2,\infty}\Big((0,T)\times\mathbb{T}^d\times(\mathbb{T}^d)^n;(0,T)\times\mathbb{T}^d\times(\mathbb{T}^d)^n\Big).$$

Denote by $I \in \mathbb{R}^{d\times d}$ the identity matrix.

We exploit Lemma 4.2 (iii) to infer that $\mathcal{S}$ is a homeomorphism and we denote its inverse by $\mathcal{X}$. Observe that

$$\mathcal{X}(t,s,b,\mathbf{x}) = \big(t,s,R(t,s,b,\mathbf{x}),\mathbf{x}\big).$$

Denoting the null $d\times d$ matrix by $\vec 0$ and the $d\times 1$ null matrix by $\bar 0$ we have

$$\nabla_{(t,q,\mathbf{x})}\mathcal{S} = \begin{bmatrix} 1 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & \cdots & 0\cdots 0 \\ \bar 0 & \nabla_q S & \nabla_{x_1} S & \nabla_{x_2} S & \cdots & \nabla_{x_n} S \\ \bar 0 & \vec 0 & I & \vec 0 & \cdots & \vec 0 \\ \bar 0 & \vec 0 & \vec 0 & I & \cdots & \vec 0 \\ \vdots & \vdots & & \ddots & I & \\ \bar 0 & \vec 0 & \vec 0 & \vec 0 & \cdots & I \end{bmatrix}$$

and so, exploiting Lemma 3.13 (i), we obtain

$$\det \nabla_{(t,q,\mathbf{x})}\mathcal{S} = \det \nabla_q S \geq \frac{1}{4}. \tag{8.21}$$

We use the Inverse Function Theorem to conclude that for every $s \in [0,T]$

$$\mathcal{X}(\cdot, s, \cdot, \cdot) \in W^{2,\infty}\Big((0,T) \times \mathbb{T}^d \times (\mathbb{T}^d)^n; (0,T) \times \mathbb{T}^d \times (\mathbb{T}^d)^n\Big). \tag{8.22}$$

Denote by adj(A) the adjugate of a $d \times d$ matrix A so that

$$\mathrm{adj}(A)A = (\det A)I.$$

We have

$$\nabla_q R = \frac{\mathrm{adj}(\nabla_q S)}{\det \nabla_q S} \circ R, \tag{8.23}$$

$$\partial_t R = -\Big(\frac{\mathrm{adj}\, \nabla_q S}{\det \nabla_q S} \partial_t S\Big) \circ R, \tag{8.24}$$

and

$$\nabla_{x_i} R = -\frac{\mathrm{adj}(\nabla_q S)\nabla_{x_i} S}{\det \nabla_q S} \circ R. \tag{8.25}$$

Theorem 8.7. *There exists a constant $\bar{D}_A$ independent of n such that the following hold:*

(i)

$$||\partial_s R||_\infty, \quad ||\partial_t R||_\infty, \quad ||\nabla_q R||_\infty \quad ||\partial_{tt} R||_\infty, \quad ||\nabla_{qq} R||_\infty, \quad ||\partial_t \nabla_q R||_\infty \leq \bar{D}_A.$$

(ii) *For $i = 1, ..., n$*

$$||\nabla_{x_i} R||_\infty, \quad ||\nabla_{x_i q} R||_\infty, \quad ||\nabla_{x_i x_i} R||_\infty, \quad ||\partial_t \nabla_{x_i} R||_\infty \leq \frac{\bar{D}_A}{n}.$$

(iii) *For $i, j = 1, ..., n, i \neq j$*

$$||\nabla_{x_i x_j} R||_\infty \leq \frac{\bar{D}_A}{n^2}.$$

Proof. We obtain (i) as a consequence of Lemmas 3.13 and 4.4. We differentiate the expression in (8.25) successively with respect to q, x_j, t, use the fact that $R(t, s, \cdot, \mathbf{x})$ and $S(t, s, \cdot, \mathbf{x})$ are inverses of each other, and then we use the bounds obtained in Corollary 8.3 to conclude that (ii) and (iii) hold. ∎

Remark 8.8. Denoting $V(t, s, b, \mathbf{x}) := \mathcal{V}_s^t[\mu^{\mathbf{x}}](b)$, we have

$$\begin{aligned}\nabla_{x_i} V(t, s, b, \mathbf{x}) &= \nabla_q \partial_t S(t, s, R(t, s, b, \mathbf{x}), \mathbf{x}) \nabla_{x_i} R(t, s, b, \mathbf{x}) \\ &+ \nabla_{x_i} \partial_t S(t, s, R(t, s, b, \mathbf{x}), \mathbf{x}).\end{aligned}$$

Therefore, using Corollary 8.3 (i) and the first inequality in Theorem 8.7 (i), we conclude that, increasing the value of the constant $\bar{D}_A$ if necessary,

$$\|\nabla_{x_i} V\|_\infty \leq \frac{\bar{D}_A}{n}.$$

Corollary 8.9. *For every $t, r, s, \tau \in [0,T]$, $b, \bar{b} \in \mathbb{T}^d$ and $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, we have*

$$|X^r_\tau[\nu](\bar{b}) - X^t_s[\mu](b)| \leq \bar{D}_A\Big(|\tau - s| + |r - t| + |\bar{b} - b|_{\mathbb{T}^d} + W_2(\nu, \mu)\Big).$$

Proof. Let $t, r, s, \tau \in [0,T]$, $q, \bar{q} \in \mathbb{T}^d$ and $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$. Let $\gamma \in \Gamma_0(\mu, \nu)$. Choose sequences $\{(x^n_i, y^n_i)\}_n$ in the support of γ such that (8.35) holds and

$$W_2\big(\mu^{\mathbf{x}^n}, \mu^{\mathbf{y}^n}\big) = \frac{1}{n}\sum_{i=1}^n |x^n_i - y^n_i|^2_{\mathbb{T}^d}. \tag{8.26}$$

It follows from Theorem 8.7 and (8.26) that

$$|X^r_\tau[\mu^{\bar{\mathbf{x}}^n}](\bar{b}) - X^t_s[\mu^{\mathbf{x}^n}](b)|_{\mathbb{T}^d} \leq \bar{D}_A\Big(|\tau - s| + |r - t| + |\bar{b} - b|_{\mathbb{T}^d} + W_2(\mu^{\bar{\mathbf{x}}^n}, \mu^{\mathbf{x}^n})\Big).$$

Since, by Lemma 4.4, X is continuous, it remains to let n tend to ∞ and use (8.35). ∎

8.3. **Regularity properties of the master map.** Let

$$\mathcal{B} = [0,T] \times [0,T] \times \mathbb{T}^d \times \Big\{(y_i, \mu^{\mathbf{y}}) \ : \ i \in \{1, \cdots, n\}\Big\}.$$

Let $f : \mathcal{B} \to \mathbb{R}$ be a continuous function such that if

$$s, t \in [0,T], \quad q \in \mathbb{T}^d, \quad i, j \in \{1, \cdots, n\}, \quad \mathbf{x}, \mathbf{y} \in (\mathbb{T}^d)^n \tag{8.27}$$

then

$$|f(t, s, q, (y_j, \mu^{\mathbf{y}})) - f(t, s, q, (x_i, \mu^{\mathbf{x}}))| \leq C_A\Big(|x_i - y_j|_{\mathbb{T}^d} + W_2\big(\mu^{\mathbf{x}}, \mu^{\mathbf{y}}\big) + \frac{1}{n}\Big). \tag{8.28}$$

For $z \in \mathbb{T}^d$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$ we define

$$g(t, s, q, z, \mu) = \inf\Big\{f(t, s, q, (y_j, \mu^{\mathbf{y}})) + C_A\Big(|z - y_j|_{\mathbb{T}^d} + W_2\big(\mu, \mu^{\mathbf{y}}\big)\Big)\Big\},$$

where the infimum is performed over the set of $(i, \mathbf{y})$ such that

$$i \in \{1, \cdots, n\}, \ \mathbf{y} \in (\mathbb{T}^d)^n. \tag{8.29}$$

Lemma 8.10. *Suppose (8.28) holds whenever (8.27) holds. Suppose that for any $\mathbf{x} \in (\mathbb{T}^d)^n$ $f(\cdot, \cdot, \cdot, (x_i, \mu^{\mathbf{x}}))$ is C_A–Lipschitz. Then*

(i) *g is $\sqrt{3}C_A$–Lipschitz.*
(ii) *On $\mathcal{B}$ we have $|g - f| \leq C_A/n$.*

Proof. (i) For every $t, s \in [0,T], q \in \mathbb{T}^d$, $g(t, s, q, \cdot, \cdot)$ is $\sqrt{2}C_A$–Lipschitz since it is the infimum of $\sqrt{2}C_A$–Lipschitz functions. This, together with the fact that, for every $\mathbf{x} \in (\mathbb{T}^d)^n$, $f(\cdot, \cdot, \cdot, (x_i, \mu^{\mathbf{x}}))$ is C_A–Lipschitz, gives that g is $\sqrt{3}C_A$–Lipschitz.

(ii) It follows from the definition of g that, for every $t, s \in [0,T], q \in \mathbb{T}^d, \mathbf{x} \in (\mathbb{T}^d)^n, i = 1, ..., n,$

$$g(t,s,q,x_i,\mu^{\mathbf{x}}) \leq f(t,s,q,(x_i,\mu^{\mathbf{x}})).$$

Moreover, by (8.28),

$$\begin{aligned} g(t,s,q,x_i,\mu^{\mathbf{x}}) &= \inf\Big\{ f(t,s,q,(y_j,\mu^{\mathbf{y}})) + C_A\Big(|x_i - y_j|_{\mathbb{T}^d} + W_2\big(\mu^{\mathbf{x}},\mu^{\mathbf{y}}\big)\Big)\Big\} \\ &\geq \inf\Big\{ f(t,s,q,(x_i,\mu^{\mathbf{x}})) + C_A\Big(|x_i - y_j|_{\mathbb{T}^d} + W_2\big(\mu^{\mathbf{x}},\mu^{\mathbf{y}}\big)\Big)\Big\} \\ &\quad -C_A\Big(|x_i - y_j| + W_2\big(\mu^{\mathbf{x}},\mu^{\mathbf{y}}\big) + \frac{1}{n}\Big) = f(t,s,q,(x_i,\mu^{\mathbf{x}})) - \frac{C_A}{n}. \end{aligned}$$

■

We now set for $t, s \in [0,T]$, $q \in \mathbb{T}^d, \mathbf{x} \in (\mathbb{T}^d)^n, j = 1, ..., n,$

$$\zeta^n(t,s,q,(x_j,\mu^{\mathbf{x}})) = n\nabla_{x_j} S(t,s,q,\mathbf{x}). \tag{8.30}$$

The map ζ^n is well defined on $\mathcal{B}$. In particular it is periodic in q and if $\mathbf{x} \in \mathbb{R}^d, \bar{\mathbf{x}} \in \mathbb{R}^d$ are such that $|\mathbf{x} = \bar{\mathbf{x}}|_{\mathbb{T}^d} = 0$ then $\zeta^n(t,s,q,(x_j,\mu^{\mathbf{x}})) = \zeta^n(t,s,q,(\bar{x}_j,\mu^{\bar{\mathbf{x}}}))$.

Corollary 8.11. *For each natural number n, ζ^n admits and extension*

$$\chi^n : [0,T] \times [0,T] \times \mathbb{T}^d \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^{d\times d}$$

such that, increasing the value of C_A, we have

(i) *χ^n is C_A–Lipschitz.*
(ii) *On $\mathcal{B}$ we have $|\chi^n - \zeta^n| \leq C_A/n$.*

Proof. We first check the Lipschitz property of $\zeta^n(\cdot,\cdot,\cdot,(x_j,\mu^{\mathbf{x}}))$ for $j \in \{1,\cdots,n\}$ and $\mathbf{x} \in (\mathbb{T}^d)^n$. We differentiate with respect to x_i the expressions in Lemma 3.16 (iv) to obtain

$$\nabla_{x_i}\partial_s\Sigma_s^t[\mu^{\mathbf{x}}] = -\nabla_{q\,x_i}\Sigma_s^t[\mu^{\mathbf{x}}]\mathcal{V}_t^s[\mu^{\mathbf{x}}] - \nabla_q\Sigma_s^t[\mu^{\mathbf{x}}]\nabla_{x_i}\mathcal{V}_t^s[\mu^{\mathbf{x}}].$$

We use the bound on $|\nabla_{qx_i}\Sigma|$ provided by Corollary 8.3, the bound on $|\mathcal{V}|$ provided by Lemma 3.15, the bound on $|\nabla_{x_i}\mathcal{V}_s^t[\mu^{\mathbf{x}}](q)|$ provided by Remark 8.8, and the fact that $\|\nabla_q\Sigma\|_\infty \leq A$ to conclude that $\zeta^n(t,\cdot,q,x_j,\mu^{\mathbf{x}})$ is $\tilde{C}_A$–Lipschitz for some constant $\tilde{C}_A$. Corollary 8.3 (i) gives that $\zeta^n(\cdot,s,q,x_j,\mu^{\mathbf{x}})$ is C_A–Lipschitz, while (8.6) ensures that $\zeta^n(t,s,\cdot,x_j,\mu^{\mathbf{x}})$ is C_A–Lipschitz. Corollaries 8.2 and 8.3 imply that ζ^n is continuous. It remains to show that ζ^n satisfies (8.28) whenever (8.27) holds, so that we may apply Lemma 8.10 to each component of ζ^n to conclude the proof.

Let $t \in [0,T]$, $s \in (0,T)$, and $q \in \mathbb{T}^d$. Let $\mathbf{x}, \mathbf{y} \in (\mathbb{T}^d)^n$ and $1 \leq i < j \leq n$. Since $S(t,s,q,\mathbf{x})$ is invariant under the permutation of the $x_1,\cdots,x_n$, and $\nabla_{x_j}S(t,s,q,\mathbf{x})$

is periodic in the $\mathbf{x}$ variables, rearranging and translating the points, we can assume that

$$\sum_{k\neq i,j} |x_k - y_k|^2 \leq W_2^2\left(\mu^{\mathbf{x}}, \mu^{\mathbf{y}}\right)$$

and

$$|x_j - y_i| = |x_j - y_i|_{\mathbb{T}^d}, \quad |x_i - y_j| = |x_i - y_j|_{\mathbb{T}^d}.$$

Moreover, using again the invariance under permutations, we have

$$\begin{aligned} \nabla_{x_j} S(t,s,q,\mathbf{y}) \\ = \nabla_{x_1} S(t,s,q,y_j,y_i,y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_{j-1},y_{j+1},\cdots,y_n), \end{aligned} \tag{8.31}$$

and

$$\begin{aligned} \nabla_{x_i} S(t,s,q,\mathbf{x}) \\ = \nabla_{x_1} S(t,s,q,x_i,x_j,x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_{j-1},x_{j+1},\cdots,x_n). \end{aligned} \tag{8.32}$$

We combine (8.31) and (8.32) to obtain

$$\begin{aligned} |\nabla_{x_j} S(t,s,q,\mathbf{y}) - \nabla_{x_i} S(t,s,q,\mathbf{x})| &\leq \|\nabla_{x_1,x_1} S\|_\infty |y_j - x_i| \\ &+ \|\nabla_{x_1,x_2} S\|_\infty |y_i - x_j| \\ &+ \sum_{k=1}^{i-1} \|\nabla_{x_1,x_{k+2}} S\|_\infty |y_k - x_k| \\ &+ \sum_{k=i+1}^{j-1} \|\nabla_{x_1,x_{k+1}} S\|_\infty |y_k - x_k| \\ &+ \sum_{k=j+1}^{n} \|\nabla_{x_1,x_k} S\|_\infty |y_k - x_k|. \end{aligned}$$

Therefore, by Corollary 8.3,

$$\begin{aligned} |\nabla_{x_j} S(t,s,q,\mathbf{y}) &- \nabla_{x_i} S(t,s,q,\mathbf{x})| \leq \frac{C_A}{n}|y_j - x_i| + \frac{C_A}{n^2}|y_i - x_j| \\ &+ \frac{C_A}{n^2}\sum_{k\neq i,j}|y_k - x_k| \\ &\leq \frac{C_A}{n}|y_j - x_i|_{\mathbb{T}^d} + \frac{C_A}{n^2}|y_i - x_j|_{\mathbb{T}^d} \\ &+ \frac{C_A\sqrt{n}}{n^2}\sqrt{\sum_{k\neq i,j}|y_k - x_k|^2}. \end{aligned}$$

$$\leq \frac{C_A}{n}\Big(|y_j - x_i|_{\mathbb{T}^d} + \frac{\sqrt{d}}{2n} + W_2\left(\mu^{\mathbf{x}}, \mu^{\mathbf{y}}\right)\Big), \tag{8.33}$$

where we used the fact that the diameter of $\mathbb{T}^d$ is $\sqrt{d}/2$. Consequently,

$$n|\nabla_{x_j} S(t,s,q,\mathbf{y}) - \nabla_{x_i} S(t,s,q,\mathbf{x})| \leq \sqrt{d} C_A \Big(|y_j - x_i|_{\mathbb{T}^d} + W_2\big(\mu^{\mathbf{x}}, \mu^{\mathbf{y}}\big) + \frac{1}{n} \Big),$$

which proves the each component of ζ^n satisfies (8.28). ∎

Theorem 8.12. *The following hold:*

(i) *For $s \in [0,T]$, Σ_s is differentiable on $[0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.*

(ii) *Increasing the value of C_A, there is a C_A–Lipschitz map*

$$\bar{\nabla}_\mu \Sigma : [0,T] \times [0,T] \times \mathbb{T}^d \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^{d \times d}$$

such that for any $s,r,t \in [0,T]$, $\bar{q}, q \in \mathbb{T}^d$, $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, and any $\gamma \in \Gamma_0(\mu,\nu)$

$$\begin{aligned} \Big| \Sigma_s^r[\nu](\bar{q}) - \Sigma_s^t[\mu](q) - \partial_t \Sigma_s^t[\mu](q)(r-t) - \nabla_q \Sigma_s^t[\mu](q)(\bar{q}-q) & \\ - \int_{\mathbb{T}^d \times \mathbb{T}^d} \bar{\nabla}_\mu \Sigma_s^t[\mu](q,z)(y-z)\gamma(dz,dy) \Big| & \\ \leq C_A \Big((r-t)^2 + |\bar{q}-q|^2_{\mathbb{T}^d} + W_2^2\big(\mu,\nu\big) \Big). & \end{aligned} \tag{8.34}$$

Proof. Let $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, $r,s,t \in [0,T]$ and $\bar{q}, q \in \mathbb{T}^d$. Choose a sequence

$$\gamma^n = \frac{1}{n} \sum_{i=1}^n \delta_{(x_i^n, y_i^n)}$$

such that for each $i \in \{1, \cdots, n\}$, (x_i^n, y_i^n) belongs to the support of γ and $\{\gamma^n\}_n$ converges narrowly to γ in $\mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d)$. Note that since each (x_i^n, y_i^n) belongs to the support of γ, $\{(x_i^n, y_i^n)\}_{i=1}^n$ is $|\cdot|_{\mathbb{T}^d}$–monotone (cf. [4]) and so

$$\gamma^n \in \Gamma_0\Big(\mu^{\mathbf{x}^n}, \mu^{\mathbf{y}^n}\Big).$$

Furthermore,

$$\lim_{n\to\infty} W_2\big(\mu, \mu^{\mathbf{x}^n}\big) = \lim_{n\to\infty} W_2\big(\nu, \mu^{\mathbf{y}^n}\big) = 0. \tag{8.35}$$

Let the map ζ^n be defined as in (8.30) and let χ^n be as in Corollary 8.11. By Corollary 8.4

$$\begin{aligned} \Big| \Sigma_s^r[\mu^{\mathbf{y}^n}](\bar{q}) - \Sigma_s^t[\mu^{\mathbf{x}^n}](q) - \partial_t \Sigma_s^t[\mu^{\mathbf{x}^n}](q)(r-t) & \\ - \nabla_q \Sigma_s^t[\mu^{\mathbf{x}^n}](q)(\bar{q}-q) - \int_{\mathbb{T}^d \times \mathbb{T}^d} \zeta^n(t,s,q,z,\mu^{\mathbf{x}^n})(y-z)\gamma^n(dz,dy) \Big| & \\ \leq C_A \Big((r-t)^2 + |\bar{q}-q|^2 + W_2^2\big(\mu^{\mathbf{x}^n}, \mu^{\mathbf{y}^n}\big) \Big). & \end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\Sigma_s^r[\mu^{\mathbf{y}^n}](\bar q) - \Sigma_s^t[\mu^{\mathbf{x}^n}](q) - \partial_t\Sigma_s^t[\mu^{\mathbf{x}^n}](q)(r-t) \\
&\qquad -\nabla_q\Sigma_s^t[\mu^{\mathbf{x}^n}](q)\cdot(\bar q - q) - \int_{\mathbb{T}^d\times\mathbb{T}^d}\chi^n(t,q,z,\mu^{\mathbf{x}^n})(y-z)\gamma^n(dz,dy)\Big| \\
&\qquad \le C_A\Big((s-t)^2 + |\bar q - q|^2 + W_2^2\big(\mu^{\mathbf{x}^n},\mu^{\mathbf{y}^n}\big)\Big) \\
&\qquad +\Big|\int_{\mathbb{T}^d\times\mathbb{T}^d}\big(\xi^n(t,q,z,\mu^{\mathbf{x}^n}) - \zeta^n(t,q,z,\mu^{\mathbf{x}^n})\big)(y-z)\gamma^n(dz,dy)\Big|. \\
&\qquad \le C_A\Big((r-t)^2 + |\bar q - q|^2 + W_2^2\big(\mu^{\mathbf{x}^n},\mu^{\mathbf{y}^n}\big)\Big) + \frac{C_A}{n}W_2\big(\mu^{\mathbf{x}^n},\mu^{\mathbf{y}^n}\big). \qquad (8.36)
\end{aligned}$$

By Theorem 8.7 (ii), $\{\chi^n\}_n$ is uniformly bounded, and by Corollary 8.11 (i) the sequence is C_A–Lipschitz. Thus the Arzela–Ascoli lemma provides us with a subsequence $\{\chi^{n_m}\}_m$ which converges uniformly to a C_A–Lipschitz map

$$\bar\nabla_\mu\Sigma : [0,T]\times[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}^{d\times d}.$$

By Lemma 4.2, Σ and $\partial_t\Sigma$ are continuous while, by Lemma 4.3, $\nabla_q\Sigma$ is continuous. Thus, replacing n by n_m in (8.36) and then letting m tend to ∞ we obtain that (ii) is satisfied. ■

Using Corollaries 8.5 and 8.6 and arguing similarly as we did to obtain Corollary 8.11 and then Theorem 8.12, we can derive the following theorem. We leave the details to the readers.

Theorem 8.13. *The following hold:*

(i) *For $s\in[0,T]$, $\partial_t\Sigma_s$ is differentiable on $(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.*
(ii) *Increasing the value of C_A, there is a C_A–Lipschitz map*

$$\xi : [0,T]\times[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}^{d\times d}$$

such that for any $s,r,t\in(0,T)$, $\bar q, q\in\mathbb{T}^d$, $\mu,\nu\in\mathcal{P}(\mathbb{T}^d)$, and any $\gamma\in\Gamma_0(\mu,\nu)$ we have

$$\begin{aligned}
&\Big|\partial_t\Sigma_s^r[\nu](\bar q) - \partial_t\Sigma_s^t[\mu](q) - \partial_{tt}\Sigma_s^t[\mu](q)(r-t) - \nabla_q\partial_t\Sigma_s^t[\mu](q)(\bar q - q) \\
&\qquad\qquad - \int_{\mathbb{T}^d\times\mathbb{T}^d}\xi_s^t[\mu](q,z)(y-z)\gamma(dz,dy)\Big| \\
&\qquad\qquad \le C_A\Big((r-t)^2 + |\bar q - q|^2 + W_2^2\big(\mu,\nu\big)\Big). \qquad (8.37)
\end{aligned}$$

8.4. **Regularity properties of the inverse master map.** In this subsection, we use the notation of Subsection 8.2.

Theorem 8.14. *The following hold:*

(i) *X is continuously differentiable on $[0,T]\times[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.*

(ii) *The maps $\partial_t X$, $\nabla_b X$ are continuous on $[0,T]\times[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$ and the map*

$$\bar\nabla_\mu X_s^t[\mu](b,z) := -\nabla_b X_s^t[\mu](b)\ \bar\nabla_\mu \Sigma_s^t[\mu]\big(X_s^t[\mu](b),z\big)$$

is continuous on $[0,T]\times[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.

(iii) *Increasing suitably the value of $\bar D_A$, we obtain that for any $r,t\in[0,T]$, $\bar b, b\in\mathbb{T}^d$, $\mu,\nu\in\mathcal{P}(\mathbb{T}^d)$, and any $\gamma\in\Gamma_0(\mu,\nu)$, we have*

$$\begin{aligned}\Big| X_s^r[\nu](\bar b) - X_s^t[\mu](b) - \partial_t X_s^t[\mu](b)(r-t) - \nabla_b X_s^t[\mu](b)(\bar b - b)\\ - \int_{\mathbb{T}^d\times\mathbb{T}^d} \bar\nabla_\mu X_s^t[\mu](b,z)(y-z)\gamma(dz,dy)\Big|\\ \le \bar D_A\Big((r-t)^2 + |\bar b - b|^2 + W_2^2\big(\mu,\nu\big)\Big).\end{aligned}\tag{8.38}$$

Proof. (i) Part (i) will follow once we show (ii) and (iii).

(ii) By Lemma 4.4, X, $\partial_t X$ and $\nabla_b X$ are continuous. Since, by Theorem 8.12 (ii), $\bar\nabla_\mu\Sigma$ is continuous, we conclude the proof of (ii).

(iii) Let $\mu,\nu\in\mathcal{P}(\mathbb{T}^d)$, $r,t\in[0,T]$ and $\bar b, b\in\mathbb{T}^d$. Set

$$q = X_s^t[\mu](b),\quad \bar q = X_s^r[\nu](\bar b),\quad \text{i.e.}\ \ b = \Sigma_s^t[\mu](q),\quad \bar b = \Sigma_s^r[\nu](\bar q).$$

Recall that, by (3.17) and (3.18), we have

$$\nabla_b X_s^t[\mu](b) = \big(\nabla_q \Sigma_s^t[\mu]\big)^{-1}\circ X_s^t[\mu](b) =: E,$$

$$\partial_t X_s^t[\mu](b) = -\nabla_b X_s^t[\mu](b)\partial_t\Sigma_s^t[\mu]\circ X_s^t[\mu](b) = -E\partial_t\Sigma_s^t[\mu]\circ X_s^t[\mu](b),$$

and, by (3.15), $|E|\le c_d$. Therefore

$$\begin{aligned}&\Big| X_s^r[\nu](\bar b) - X_s^t[\mu](b) - \partial_t X_s^t[\mu](b)(r-t) + \nabla_b X_s^t[\mu](b)(\bar b - b)\\ &\qquad - \int_{\mathbb{T}^d\times\mathbb{T}^d} \bar\nabla_\mu X_s^t[\mu](b,z)(y-z)\gamma(dz,dy)\Big|\\ =\ &\Big| E\Big[\nabla_q\Sigma_s^t[\mu](q)(\bar q - q) + \partial_t\Sigma_s^t[\mu](q)(r-t) - \big(\Sigma_s^r[\nu](\bar q) - \Sigma_s^t[\mu](q)\big)\\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{T}^d}\bar\nabla_\mu\sigma_s^t[\mu](q,z)(y-z)\gamma(dz,dy)\Big]\Big|\\ \le\ & c_d C_A\Big((r-t)^2 + |X_s^t[\nu](\bar b) - X_s^r[\mu](b)|^2 + W_2^2\big(\mu,\nu\big)\Big).\end{aligned}$$

It remains to use Corollary 8.9 to find an appropriately large constant $\bar{D}_A$ so that (8.38) is satisfied. ■

8.5. **First order expansion of $\mathcal{V}$.** Recall that, by Definition 3.14, for $t, s \in [0,T]$, $q \in \mathbb{T}^d$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\mathcal{V}_s^t[\mu] := \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu].$$

We can differentiate the above expression by Lemma 3.13 (iii) to obtain:

$$\partial_t \mathcal{V}_s^t[\mu] = \partial_{tt} \Sigma_s^t[\mu] \circ X_s^t[\mu] + \nabla_q \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu] \; \partial_t X_s^t[\mu], \tag{8.39}$$

$$\nabla_b \mathcal{V}_s^t[\mu] = \nabla_q \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu] \; \nabla_b X_s^t[\mu]. \tag{8.40}$$

Set

$$\bar{\nabla}_\mu \mathcal{V}_s^t[\mu](b,z) = \xi_s^t[\mu]\big(X_s^t[\mu](b), z\big) + \nabla_q \partial_t \Sigma_s^t[\mu] \circ X_s^t[\mu](b) \; \bar{\nabla}_\mu X_s^t[\mu](b,z).$$

Corollary 8.15. *The following hold:*

(i) *$\partial_t \mathcal{V}$ and $\nabla_b \mathcal{V}$ are continuous on $[0,T] \times [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.*

(ii) *$\bar{\nabla}_\mu \mathcal{V}$ is continuous on $[0,T] \times [0,T] \times \mathbb{T}^d \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.*

(iii) *Increasing the value of $\bar{D}_A$, we obtain that for any $s, r, t \in (0,T)$, $\bar{b}, b \in \mathbb{T}^d$, $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, and any $\gamma \in \Gamma_0(\mu,\nu)$, we have*

$$\begin{aligned}\Big| \mathcal{V}_s^r[\nu](\bar{b}) - \mathcal{V}_s^t[\mu](b) - \partial_t \mathcal{V}_s^t[\mu](b)(r-t) - \nabla_b \mathcal{V}_s^t[\mu](b)(\bar{b}-b) \\ - \int_{\mathbb{T}^d \times \mathbb{T}^d} \bar{\nabla}_\mu \mathcal{V}_s^t[\mu](b,z)(y-z)\gamma(dz,dy) \Big| \\ \leq \bar{D}_A \Big((r-t)^2 + |\bar{b}-b|^2 + W_2^2(\mu,\nu) \Big).\end{aligned} \tag{8.41}$$

Proof. (i) By Lemma 4.4, X, $\partial_t X$ and $\nabla_b X$ are continuous. By Lemma 4.3 (ii), $\nabla_q \partial_t \Sigma$ is continuous, while Lemma 4.2 (ii) ensures that $\partial_{tt}\Sigma$ is continuous. Using the representation formulas (8.39) and (8.40) we thus conclude the proof of (i).

(ii) Since X, $\nabla_q \partial_t \Sigma$, ξ and $\bar{\nabla}_\mu X$ are all continuous, we obtain that $\bar{\nabla}_\mu \mathcal{V}$ is continuous.

(iii) We combine Theorems 8.13 and 8.14, to obtain

$$\begin{aligned}&\Big| \mathcal{V}_s^r[\nu](\bar{b}) - \mathcal{V}_s^t[\mu](b) - \partial_t \mathcal{V}_s^t[\mu](b)(r-t) - \nabla_b \mathcal{V}_s^t[\mu](b)(\bar{b}-b) \\ &\qquad - \int_{\mathbb{T}^d \times \mathbb{T}^d} \bar{\nabla}_\mu \mathcal{V}_s^t[\mu](b,z)(y-z)\gamma(dz,dy) \Big| \\ \leq\; & C_A \Big((r-t)^2 + |X_s^r[\mu](\bar{b}) - X_s^t[\mu](b)|^2 + W_2^2(\mu,\nu) \Big) \\ &\qquad + \bar{D}_A \|\nabla_q \partial_t \Sigma\|_\infty \Big((r-t)^2 + |\bar{b}-b|^2 + W_2^2(\mu,\nu) \Big) \\ \leq\; & \bar{D}_A \Big((r-t)^2 + |\bar{b}-b|^2 + W_2^2(\mu,\nu) \Big),\end{aligned}$$

for some appropriately large constant, still denoted by $\bar{D}_A$. Above we used Corollary 8.9 and the fact that $\Sigma_s[\mu] \in \mathcal{C}_A$ to obtain the last inequality. ∎

8.6. **Smoothness properties of the velocity $|\mathcal{V}_s|^2$.** We set

$$\bar{\mathcal{V}} = \bar{\nabla}_\mu \mathcal{V}\mathcal{V}$$

Corollary 8.16. *The function $|\mathcal{V}_s|^2$ is is continuously differentiable on $(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$ for every $s \in (0,T)$. More precisely, increasing the value of $\bar{D}_A$ as necessary, the following hold:*

(i) *$\partial_t\mathcal{V}\cdot\mathcal{V}$, $\nabla_q\mathcal{V}\mathcal{V}$ and $\bar{\mathcal{V}}$ are continuous.*

(ii) *For any $s,r,t \in (0,T)$, $\bar{b}, b \in \mathbb{T}^d$, $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$, and any $\gamma \in \Gamma_0(\mu,\nu)$ we have*

$$\begin{aligned} &\Big|\frac{|\mathcal{V}_s^r[\nu](\bar{b})|^2}{2} - \frac{|\mathcal{V}_s^t[\mu](b)|^2}{2} - (\partial_t\mathcal{V}\cdot\mathcal{V})_s^t[\mu](b)(r-t) \\ &-(\nabla_q\mathcal{V}\mathcal{V})_s^t[\mu](b)\cdot(\bar{b}-b) - \int_{\mathbb{T}^d\times\mathbb{T}^d}\bar{\mathcal{V}}_s^t[\mu](b,z)\cdot(y-z)\gamma(dz,dy)\Big| \\ &\leq \bar{D}_A\Big((r-t)^2 + |\bar{b}-b|^2 + W_2^2\big(\mu,\nu\big)\Big). \end{aligned} \tag{8.42}$$

Proof. The continuity of $\partial_t\cdot\mathcal{V}\mathcal{V}, \nabla_q\mathcal{V}\mathcal{V}$ and $\bar{\mathcal{V}}$ follows from the continuity of $\mathcal{V}$, $\partial_t\mathcal{V}$, $\nabla_q\mathcal{V}$ and $\bar{\nabla}_\mu\mathcal{V}$. Part (ii) is a direct consequence of Corollary 8.15. ∎

9. Strong solutions to the master equation.

Throughout this section we assume that $T>0, A>0$ satisfy (3.7). As in Section 6 we assume that $F, \mathcal{F}, u_*$ and $\mathcal{U}_*$ are given through functions ϕ, U^0, and U^1 satisfying the assumptions imposed in that section. Using the notation of Section 7, we define

$$u : [0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

by

$$u(s,q,\mu) = u_*\big(q, \Sigma_s^0[\mu]_{\#}\mu\big) - \int_0^s\Big(\frac{|\mathcal{V}_s^l[\mu](q)|^2}{2} + F\big(q, (\Sigma_s^l[\mu])_{\#}\mu\big)\Big)dl \tag{9.1}$$

for $s \in [0,T]$, $q \in \mathbb{T}^d$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$.

9.1. **Regularity of u with respect to the μ variable.** We set

$$\mathcal{N}_s^t[\mu](q) = -\int_{\mathbb{T}^d}\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot\partial_t\Sigma_s^t[\mu](y)\mu(dy)$$

and

$$\begin{aligned}\bar{\mathcal{N}}_s^t[\mu](q,y) &= -(\nabla_q \Sigma_s^t[\mu](y))^T \nabla\phi\big(q-\Sigma_s^t[\mu](y)\big) \\ &- \int_{\mathbb{T}^d} (\bar{\nabla}_\mu \Sigma_s^t[\mu](u,y))^T \nabla\phi\big(q-\Sigma_s^t[\mu](u)\big)\mu(du),\end{aligned}$$

where $(\nabla_q \Sigma_s^t[\mu](y))^T$ and $(\bar{\nabla}_\mu \Sigma_s^t[\mu](u,y))^T$ are the transpositions of the matrices $\nabla_q \Sigma_s^t[\mu](y)$ and $\bar{\nabla}_\mu \Sigma_s^t[\mu](u,y)$.

Lemma 9.1. *The function $(t,s,q,\mu) \to F\big(q, \Sigma_s^t[\mu]_\# \mu\big)$ is continuously differentiable in the following sense:*

(i) *$\mathcal{N}$ and $(t,s,q,\mu) \to \nabla_q F\big(q,\Sigma_s^t[\mu]_\#\mu\big)$ are continuous on $[0,T]\times[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$, and $\bar{\mathcal{N}}$ is continuous on $[0,T]\times[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.*

(ii) *Suitably increasing the value of $\bar{D}_A$ we have that for any*

$$(s,t,r,q,\bar{q},\mu,\nu) \in [0,T]\times[0,T]\times[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\times\mathcal{P}(\mathbb{T}^d)$$

and any $\gamma \in \Gamma_0(\mu,\nu)$,

$$\begin{aligned}&\Big|F\big(\bar{q},\bar{\sigma}_r\big) - F(q,\sigma_t) - (r-t)\mathcal{N}_s^t[\mu](q) - \nabla_q F(q,\sigma_t)\cdot(\bar{q}-q) \\ &\qquad - \int_{\mathbb{T}^d\times\mathbb{T}^d} \bar{\mathcal{N}}_s^t[\mu](q,y)\cdot(z-y)\gamma(dy,dz)\Big| \\ &\le \bar{D}_A\Big((r-t)^2 + |\bar{q}-q|^2 + W_2^2(\mu,\nu)\Big). \qquad (9.2)\end{aligned}$$

Here,

$$\sigma_t = \Sigma_s^t[\mu]_\#\mu, \quad \bar{\sigma}_r = \Sigma_s^r[\nu]_\#\nu.$$

Proof. (i) The continuity of $\mathcal{N}$ and $(t,s,q,\mu) \to \nabla_q F\big(q,\Sigma_s^t[\mu]_\#\mu\big)$ follows from the continuity of $\nabla\phi$, $\mathcal{V}$ and Σ. Since in addition $\nabla_q\Sigma$ and $\bar{\nabla}_\mu\Sigma$ are continuous, we conclude that $\bar{\mathcal{N}}$ is continuous.

(ii) We have

$$F\big(\bar{q},\bar{\sigma}_r\big) - F(q,\sigma_t) = \int_{\mathbb{T}^d\times\mathbb{T}^d} \Big(\phi\big(\bar{q}-\Sigma_s^r[\nu](z)\big) - \phi\big(q-\Sigma_s^t[\mu](y)\big)\Big)\,\gamma(dy,dz).$$

Since $\|\nabla^2\phi\|_\infty \le \kappa$, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^d\times\mathbb{T}^d}\Big[\phi\big(\bar q-\Sigma_s^r[\nu](z)\big)-\phi\big(q-\Sigma_s^t[\mu](y)\big)\\
&-\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot\Big((\bar q-q)-\big(\Sigma_s^r[\nu](z)-\Sigma_s^t[\mu](y)\big)\Big)\Big]\gamma(dy,dz)\Big|\\
&\le \frac{\kappa}{2}\int_{\mathbb{T}^d\times\mathbb{T}^d}\Big|\bar q-q-\Sigma_s^r[\nu](z)+\Sigma_s^t[\mu](y)\Big|^2\gamma(dy,dz)\\
&\le \tilde C_A\int_{\mathbb{T}^d\times\mathbb{T}^d}\Big((r-t)^2+|\bar q-q|^2+|z-y|^2+W_2^2(\mu,\nu)\Big)\gamma(dy,dz)\\
&= \tilde C_A\Big((r-t)^2+|\bar q-q|^2+2W_2^2(\mu,\nu)\Big), \qquad (9.3)
\end{aligned}$$

for some independent constant $\tilde C_A$. We notice that

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot(\bar q-q)\gamma(dy,dz)=\nabla_q F(q,\sigma_t)\cdot(\bar q-q).$$

Using Theorem 8.12 (ii) we now have

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^d\times\mathbb{T}^d}\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot\Big[\Sigma_s^r[\nu](z)-\Sigma_s^t[\mu](y)-\partial_t\Sigma_s^t[\mu](y)(r-t)\\
&-\nabla_q\Sigma_s^t[\mu](y)(z-y)-\int_{\mathbb{T}^d\times\mathbb{T}^d}\bar\nabla_\mu\Sigma_s^t[\mu](y,w)(u-w)\gamma(dw,du)\Big]\gamma(dy,dz)\Big|\\
&\le \kappa C_A\int_{\mathbb{T}^d\times\mathbb{T}^d}\Big((r-t)^2+|z-y|^2+W_2^2(\mu,\nu)\Big)\gamma(dy,dz)\\
&= \kappa C_A\Big((r-t)^2+2W_2^2(\mu,\nu)\Big). \qquad (9.4)
\end{aligned}$$

It remains to combine (9.3) and (9.4) and notice that

$$\begin{aligned}
-\mathcal{N}_s^t[\mu](q) &= \int_{\mathbb{T}^d}\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot\partial_t\Sigma_s^t[\mu](y)\mu(dy)\\
&= \int_{\mathbb{T}^d\times\mathbb{T}^d}\nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot\partial_t\Sigma_s^t[\mu](y)\gamma(dy,dz).
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{T}^d\times\mathbb{T}^d} \nabla\phi\big(q-\Sigma_s^t[\mu](y)\big)\cdot \Big[\nabla_q\Sigma_s^t[\mu](y)(z-y) \\
+\int_{\mathbb{T}^d\times\mathbb{T}^d}\bar\nabla_\mu\Sigma_s^t[\mu](y,w)(u-w)\gamma(dw,du)\Big]\gamma(dy,dz) \\
=-\int_{\mathbb{T}^d\times\mathbb{T}^d}\bar{\mathcal{N}}_s^t[\mu](q,y)\cdot(z-y)\gamma(dy,dz).
\end{aligned}$$

To obtain the last equality we first changed the order of integration and then renamed the variables. ■

Denote

$$\begin{aligned}
\bar{\mathcal{M}}_s[\mu](q,y) &= -(\nabla_q\Sigma_s^0[\mu](y))^T\nabla U^1\big(q-\Sigma_s^0[\mu](y)\big) \\
&\quad -\int_{\mathbb{T}^d}(\bar\nabla_\mu\Sigma_s^0[\mu](u,y))^T\nabla U^1\big(q-\Sigma_s^0[\mu](u)\big)\mu(du).
\end{aligned}$$

Repeating the same proof as this of Lemma 9.1 we also obtain the following result.

Lemma 9.2. *The function $(s,q,\mu)\to u_*\big(q,\Sigma_s^0[\mu]_\#\mu\big)$ is continuously differentiable in the following sense:*

(i) *The function $(s,q,\mu)\to\nabla_q u_*\big(q,\Sigma_s^0[\mu]_\#\mu\big)$ is continuous on $[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$, and $\bar{\mathcal{M}}$ is continuous on $[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.*

(ii) *Suitably increasing the value of $\bar D_A$ we have that for any*

$$(s,q,\bar q,\mu,\nu)\in[0,T]\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\times\mathcal{P}(\mathbb{T}^d)\quad\text{and}\quad\gamma\in\Gamma_0(\mu,\nu)$$

$$\begin{aligned}
\Big|u_*\big(\bar q,\bar\sigma_0\big)-u_*(q,\sigma_0)-\nabla_q u_*(q,\sigma_0)\cdot(\bar q-q) \\
-\int_{\mathbb{T}^d\times\mathbb{T}^d}\bar{\mathcal{M}}_s^0[\mu](q,y)\cdot(z-y)\gamma(dy,dz)\Big| \\
\le \bar D_A\Big(|\bar q-q|^2+W_2^2(\mu,\nu)\Big),
\end{aligned}\tag{9.5}$$

where $\sigma_t,\bar\sigma_r$ are as in Lemma 9.1.

We now set

$$\Upsilon_s[\mu](q,y)=\int_0^s\Big(\bar{\mathcal{V}}_s^t[\mu](q,y)+\bar{\mathcal{N}}_s^t[\mu](q,y)\Big)dt+\bar{\mathcal{M}}_s^0[\mu](q,y).$$

The following corollary is a direct consequence of Corollary 8.16 and Lemmas 9.1 and 9.2.

Corollary 9.3. *The following hold:*

(i) Υ *is continuous on* $[0,T] \times \mathbb{T}^d \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

(ii) *Suitably increasing the value of* $\bar{D}_A$ *we have that for any*

$$(s,q,\mu,\nu) \in [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d) \quad \textit{and} \quad \gamma \in \Gamma_0(\mu,\nu)$$

$$\left| u(s,q,\nu) - u(s,q,\mu) - \int_{\mathbb{T}^d \times \mathbb{T}^d} \Upsilon_s[\mu](q,y) \cdot (z-y)\gamma(dy,dz) \right| \leq \bar{D}_A W_2^2(\mu,\nu).$$

Proof. (i) Since $\bar{\mathcal{V}}, \bar{\mathcal{N}}$ and $\bar{\mathcal{M}}$ are continuous, we obtain that Υ is continuous.

(ii) We combine Corollary 8.16 and Lemmas 9.1 and 9.2 to obtain (ii). ∎

Remark 9.4.

(i) We combine Remark 2.6 and Corollary 9.3 to obtain that the gradient of $u(s,q,\cdot)$ at μ is the orthogonal projection of $\Upsilon(s,q,y,\mu)$ onto the tangent space $\mathcal{T}_\mu \mathcal{P}(\mathbb{T}^d)$. Since the combination of Lemma 6.6 (ii) and Lemma 7.1 (i) gives that $\mathbf{v}_s$ belongs to the tangent space and $\mathbf{v}_s = \nabla_q u(s,\cdot,\mu)$, we obtain

$$\int_{\mathbb{T}^d} \Upsilon_s[\mu](q,y) \cdot \mathbf{v}_s(y)\mu(dy) = \int_{\mathbb{T}^d} \nabla_\mu u(s,q,\mu)(y) \cdot \nabla_q u(s,\cdot,\mu)\mu(dy).$$

(ii) Since Υ is continuous on a compact set, it is bounded and so, $\nabla_\mu u$ is bounded. Using Corollary 9.3 we thus obtain that there exists a constant κ_1 which is independent of $s \in [0,T]$ and $q \in \mathbb{T}^d$, such that $u(s,q,\cdot)$ is κ_1–Lipschitz.

9.2. **Existence of a strong solution to the master equation.**

Theorem 9.5. *The function* u *defined in (9.1) is Lipschitz on* $[0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$*, differentiable with respect to each of its variables, and* $u(0,\cdot,\cdot) = u_*$*. Furthermore,* u *satisfies the following:*

(i) *For any* $s \in [0,T]$ *and any* $\mu \in \mathcal{P}(\mathbb{T}^d)$*, there exists* $\bar{\sigma} \in AC^2(0,T;\mathcal{P}(\mathbb{T}^d))$ *such that* $\bar{\sigma}_s = \mu$ *and*

$$\partial_t \bar{\sigma}_t + \nabla \cdot (\bar{\sigma}_t \nabla_q u(t,q,\bar{\sigma}_t)) = 0 \qquad \textit{in} \quad \mathcal{D}'\big((0,T) \times \mathbb{T}^d\big).$$

(ii) *The gradient* $\nabla_q u$ *and the derivative* $\partial_t u$ *are continuous on* $(0,T) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ *and, for every* $(s,q,\mu) \in (0,T) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$*,*

$$\partial_t u(s,q,\mu) + \int_{\mathbb{T}^d} \nabla_\mu u(s,q,\mu)(y) \cdot \nabla_q u(s,y,\mu)\mu(dy) + \frac{|\nabla_q u(s,q,\mu)|^2}{2} - F(q,\mu) = 0.$$

Proof. We know from Lemma 7.1 and Corollary 9.3 that $\nabla_q u$ and $\nabla_\mu u$ exist on $(0,T) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$, and Lemma 4.2 (ii) guarantees that $\nabla_q u$ is continuous. Moreover, Lemma 3.15 and Remark 9.4 allow us to conclude that the functions $u(t,\cdot,\cdot)$ are Lipschitz continuous with a Lipschitz constant which is independent of $t \in [0,T]$. It thus suffices to show that the derivative $\partial_t u$ exists for every $(t,q,\mu) \in (0,T) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ and that (ii) is satisfied to conclude that u is Lipschitz.

We will use the following notation:

$$\bar{\mathbf{v}}_t(q) = \mathcal{V}_s^t[\mu], \quad \bar\sigma_t = \Sigma_s^t[\mu]_{\#}\mu,$$

and

$$\hat\sigma_t = \Big(Id + (t-s)\mathbf{v}_s\Big)_{\#}\bar\sigma_s = \Big(\Sigma_s^s[\mu] + (t-s)\partial_t\Sigma_s^{t=s}[\mu]\Big)_{\#}\mu.$$

(i) By Lemma 7.1, $\nabla_q u(t,q,\bar\sigma_t) = \bar{\mathbf{v}}_t(q)$ and so, since the function U used in Corollary 7.2 satisfies $\nabla_q u(t,q,\bar\sigma_t) = \nabla_q U(t,q)$, we use (ii) of the same corollary to conclude that (i) holds for $\bar\sigma_t$.

(ii) If $s+h \in [0,T]$ then

$$g_h := \Big(\Sigma_s^{s+h}[\mu] \times \big(Id + h\bar{\mathbf{v}}_s\big)\Big)_{\#}\mu \in \Gamma(\bar\sigma_{s+h}, \hat\sigma_{s+h})$$

, and thus,

$$\begin{aligned} W_2^2(\bar\sigma_{s+h},\hat\sigma_{s+h}) &\le \int_{\mathbb{T}^d}\Big|\Sigma_s^{s+h}[\mu](q) - \Sigma_s^s[\mu](q) - h\partial_t\Sigma_s^s[\mu](q)\Big|^2_{\mathbb{T}^d}\mu(dq) \\ &\le \frac{h^4}{4}||\partial_{tt}\Sigma||^2_\infty. \end{aligned} \tag{9.6}$$

By Lemma 3.15 $\bar{\mathbf{v}} \in C^1(\mathbb{T}^d)^d$ and so, by Lemma 6.6 (ii), $\bar{\mathbf{v}}$ is the gradient of a function which belongs to $C^2(\mathbb{T}^d)$. Thus, for $|h|$ small enough, $Id + h\bar{\mathbf{v}}_s$ is the gradient of a convex function. Consequently (see e.g. [4])

$$\gamma_h := \big(Id \times (Id + h\bar{\mathbf{v}}_s)\big)_{\#}\mu \ \in \ \Gamma_0(\mu, \hat\sigma_{s+h}), \tag{9.7}$$

and so

$$W_2^2(\hat\sigma_{s+h},\mu) = \int_{\mathbb{T}^d\times\mathbb{T}^d}|a-b|^2\gamma_h(da,db) = h^2\|\bar{\mathbf{v}}_s\|^2_\mu. \tag{9.8}$$

Furthermore, using (9.7) in Corollary 9.3, we have

$$\begin{aligned} &\left|u\big(s+h,q,\hat\sigma_{s+h}\big) - u(s+h,q,\mu) - h\int_{\mathbb{T}^d}\Upsilon_{s+h}[\mu](q,y)\cdot\bar{\mathbf{v}}_s(y)\mu(dy)\right| \\ &\qquad \le \bar D_A W_2^2(\hat\sigma_{s+h},\mu). \end{aligned}$$

This, together with the fact that Υ is continuous on a compact set and thus admits a modulus of continuity ω, yields

$$\begin{aligned} &\left|u\big(s+h,q,\hat\sigma_{s+h}\big) - u(s+h,q,\mu) - h\int_{\mathbb{T}^d}\Upsilon_s[\mu](q,y)\cdot\bar{\mathbf{v}}_s(y)\mu(dy)\right| \\ &\qquad \le |h|\omega(|h|)\|\bar{\mathbf{v}}_s\|_\mu + \bar D_A W_2^2(\hat\sigma_{s+h},\mu). \end{aligned} \tag{9.9}$$

By Remark 9.4, for any $t \in [0,T]$, $u(t,q,\cdot)$ is κ_1–Lipschitz and so, thanks to (9.6),

$$\Big| u\big(s+h, q, \hat\sigma_{s+h}\big) - u\big(s+h, q, \bar\sigma_{s+h}\big)\Big| \leq \frac{\kappa_1 h^2}{2} \|\partial_{tt}\Sigma\|_\infty. \tag{9.10}$$

By Lemma 7.1

$$\left| u\big(s+h, q, \bar\sigma_{s+h}\big) - u(s,q,\mu) + h\left(\frac{1}{2}|\nabla_q u(s,q,\mu)|^2 + F(q,\mu)\right)\right| = o(h). \tag{9.11}$$

Writing

$$\begin{aligned} u(s+h,q,\mu) - u(s,q,\mu) &= u(s+h,q,\mu) - u\big(s+h,q,\hat\sigma_{s+h}\big) \\ &+ u\big(s+h,q,\hat\sigma_{s+h}\big) - u\big(s+h,q,\bar\sigma_{s+h}\big) \\ &+ u\big(s+h,q,\bar\sigma_{s+h}\big) - u(s,q,\mu), \end{aligned} \tag{9.12}$$

we obtain

$$\begin{aligned} &\Big| u(s+h,q,\mu) - u(s,q,\mu) + h\int_{\mathbb{T}^d} \Upsilon_s[\mu](q,y)\cdot \mathbf{v}_s(y)\mu(dy) \\ &\qquad\qquad + h\Big(\frac{1}{2}|\nabla_q u(s,q,\mu)|^2 + F(q,\mu)\Big)\Big| \\ &\leq \Big| u(s+h,q,\mu) - u\big(s+h,q,\hat\sigma_{s+h}\big) + h\int_{\mathbb{T}^d} \Upsilon_s[\mu](q,y)\cdot \bar{\mathbf{v}}_s(y)\mu(dy)\Big| \\ &+ \big| u\big(s+h,q,\hat\sigma_{s+h}\big) - u\big(s+h,q,\bar\sigma_{s+h}\big)\big| \\ &+ \Big| u\big(s+h,q,\bar\sigma_{s+h}\big) - u(s,q,\mu) + h\Big(\frac{1}{2}|\nabla_q u(s,q,\mu)|^2 + F(q,\mu)\Big)\Big|. \end{aligned} \tag{9.13}$$

We combine (9.8), (9.9), (9.10) and (9.13) to conclude that if we set

$$\bar u := -\int_{\mathbb{T}^d} \Upsilon_s[\mu](q,y)\cdot \bar{\mathbf{v}}_s(y)\mu(dy) - \frac{1}{2}|\nabla_q u(s,q,\mu)|^2 - F(q,\mu)$$

then

$$|u(s+h,q,\mu) - u(s,q,\mu) - h\bar u| = o(h).$$

This proves that $u(\cdot,q,\mu)$ is differentiable at s, $\partial_t u(s,q,\mu) = \bar u$, and $\partial_t u$ is continuous on $(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$. In other words,

$$\partial_t u(s,q,\mu) + \int_{\mathbb{T}^d} \Upsilon_s[\mu](q,y)\cdot \bar{\mathbf{v}}_s(y)\mu(dy) + \frac{|\nabla_q u(s,q,\mu)|^2}{2} + F(q,\mu) = 0.$$

We now use Remark 9.4 (i) to complete the proof. ∎

Definition 9.6. We say that a continuous function $v : [0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a strong solution to (1.1) if $v(0,\cdot,\cdot) = u_*$, v is differentiable with respect to each of its variables on $(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$, $\partial_t v, \nabla_q v$ are bounded and continuous on

$(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$, there exists a bounded and continuous map $\Upsilon:(0,T)\times\mathbb{T}^d\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}^d$ satisfying for every $(t,q,\mu,\nu)\in(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\times\mathcal{P}(\mathbb{T}^d)$

(9.14)
$$\sup_{\gamma\in\Gamma_0(\mu,\nu)}\left|v(t,q,\nu)-v(t,q,\mu)-\int_{\mathbb{T}^d\times\mathbb{T}^d}\Upsilon_s[\mu](q,y)\cdot(z-y)\gamma(dy,dz)\right|=o\left(W_2(\mu,\nu)\right),$$

and we have

$$\partial_t v(t,q,\mu)+\int_{\mathbb{T}^d}\nabla_\mu v(t,q,\mu)(y)\cdot\nabla_q v(t,y,\mu)\mu(dy)+\frac{|\nabla_q v(t,q,\mu)|^2}{2}-F(q,\mu)=0$$

for every $(t,q,\mu)\in(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$.

A direct consequence of Theorem 9.5 is the following corollary.

Corollary 9.7. *The function u defined in (9.1) is a strong solution to (1.1).*

We finish this section with a chain rule for functions regular enough to be strong solutions to (1.1). In the rest of this section T is any positive number.

Lemma 9.8. *Let $T>0$ and let $v:(0,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}$ have the regularity properties required for a strong solution to (1.1). Let $Q\in W^{1,2}\big(0,T;\mathbb{T}^d\big),\sigma\in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$. Let $s\in(0,T)$ be such that $\dot Q_s$ exists and there exists a velocity of minimal norm $\mathbf{v}_s$ for σ. (Recall that by Theorem 8.3.1 and Proposition 8.4.5 of* [4], *$\mathbf{v}_t$ exists for a.e. t and $\mathbf{v}_t\in\mathcal{T}_{\sigma_t}\mathcal{P}(\mathbb{T}^d)$.) Then the function $t\to v(t,S_t,\sigma_t)$ is differentiable at $t=s$ and*

$$\begin{aligned}\frac{d}{dt}v(t,S_t,\sigma_t)_{|t=s}&=\partial_t v(s,Q_s,\sigma_s)+\nabla_q v(s,Q_s,\sigma_s)\cdot\dot Q_s\\&\quad+\int_{\mathbb{T}^d}\nabla_\mu v(s,Q_s,\sigma_s)(y)\cdot\mathbf{v}_s(y)\sigma_s(dy).\end{aligned}\tag{9.15}$$

Proof. The proof is similar to the proof of Theorem 9.5. We define

$$\hat\sigma_t:=\Big(Id+(t-s)\mathbf{v}_s\Big)_{\#}\sigma_s.$$

Then

$$\gamma_h:=\big(Id\times(Id+h\bar{\mathbf{v}}_s)\big)_{\#}\sigma_s\ \in\ \Gamma_0(\sigma_s,\hat\sigma_{s+h}),$$

and, by (9.8),

$$W_2(\hat\sigma_{s+h},\sigma_s)=|h|\|\bar{\mathbf{v}}_s\|_{\sigma_s}.\tag{9.16}$$

Moreover, by Proposition 8.4.6 of [4],

$$W_2(\sigma_{s+h},\hat\sigma_{s+h})=o(h).\tag{9.17}$$

We have

$$\begin{aligned} v(s+h, Q_{s+h}, \sigma_{s+h}) - v(s, Q_s, \sigma_s) &= v(s+h, Q_{s+h}, \sigma_{s+h}) - v(s, Q_{s+h}, \sigma_{s+h}) \\ &+ v(s, Q_{s+h}, \sigma_{s+h}) - v(s, Q_s, \sigma_{s+h}) \\ &+ v(s, Q_s, \sigma_{s+h}) - v(s, Q_s, \sigma_s). \end{aligned} \tag{9.18}$$

By the mean value theorem there is $\tau \in [s, s+h]$ such that

$$v(s+h, Q_{s+h}, \sigma_{s+h}) - v(s, Q_{s+h}, \sigma_{s+h}) = h\partial_t v(\tau, Q_{s+h}, \sigma_{s+h}).$$

Therefore, by the continuity of $\partial_t v$,

$$\left| \frac{v(s+h, Q_{s+h}, \sigma_{s+h}) - v(s, Q_{s+h}, \sigma_{s+h})}{h} - \partial_t v(s, Q_s, \sigma_s) \right| = \frac{o(h)}{|h|}. \tag{9.19}$$

Using the mean value theorem again, there is z in the line segment connecting Q_s and Q_{s+h} such that

$$v(s, Q_{s+h}, \sigma_{s+h}) - v(s, Q_s, \sigma_{s+h}) = \nabla_q v(s, z, \sigma_{s+h}) \cdot (Q_{s+h} - Q_s).$$

Therefore, by the continuity of $\nabla_q v$,

$$\left| \frac{v(s, Q_{s+h}, \sigma_{s+h}) - v(s, Q_s, \sigma_{s+h})}{h} - \nabla_q v(s, Q_s, \sigma_s) \cdot \dot{Q}_s \right| = \frac{o(h)}{|h|}. \tag{9.20}$$

Now

$$\begin{aligned} v(s, Q_s, \sigma_{s+h}) - v(s, Q_s, \sigma_s) &= v(s, Q_s, \sigma_{s+h}) - v(s, Q_s, \hat{\sigma}_{s+h}) \\ &+ v(s, Q_s, \hat{\sigma}_{s+h}) - v(s, Q_s, \sigma_s). \end{aligned} \tag{9.21}$$

Since it is easy to see that v is Lipschitz, there is a constant $L > 0$, such that

$$\left| \frac{v(s, Q_s, \sigma_{s+h}) - v(s, Q_s, \hat{\sigma}_{s+h})}{h} \right| \leq \frac{L W_2(\sigma_{s+h}, \hat{\sigma}_{s+h})}{|h|} = \frac{o(h)}{|h|}, \tag{9.22}$$

where we used (9.17). Finally, if Υ is a function from (9.14) for v, we obtain

$$\begin{aligned} &\Big| \frac{v(s, Q_s, \hat{\sigma}_{s+h}) - v(t, q, \sigma_s)}{h} - \int_{\mathbb{T}^d} \Upsilon_s[\sigma_s](Q_s, y) \cdot \mathbf{v}_s(y) \sigma_s(dy) \Big| \\ &= \Big| \frac{v(s, Q_s, \hat{\sigma}_{s+h}) - v(t, q, \sigma_s)}{h} - \frac{1}{h} \int_{\mathbb{T}^d \times \mathbb{T}^d} \Upsilon_s[\sigma_s](Q_s, y) \cdot (z - y) \gamma_h(dy, dz) \Big| \\ &\qquad = \frac{o\left(W_2^2(\sigma_s, \hat{\sigma}_{s+h})\right)}{|h|} = \frac{o(|h| \|\bar{\mathbf{v}}_s\|_{\sigma_s})}{|h|} = \frac{o(h)}{|h|}, \end{aligned} \tag{9.23}$$

where we used (9.16). Therefore, combining (9.18), (9.19), (9.20), (9.21), (9.22) and (9.23), we obtain

$$\frac{d}{dt}v(t,S_t,\sigma_t)_{|t=s} = \partial_t v(s,Q_s,\sigma_s) + \nabla_q v(s,Q_s,\sigma_s)\cdot \dot{Q}_s + \int_{\mathbb{T}^d} \Upsilon_s[\sigma_s](Q_s,y)\cdot \mathbf{v}_s(y)\sigma_s(dy).$$

It remains to notice that $\mathbf{v}_s \in \mathcal{T}_{\sigma_s}\mathcal{P}(\mathbb{T}^d)$, and thus

$$\int_{\mathbb{T}^d} \Upsilon_s[\sigma_s](Q_s,y)\cdot \mathbf{v}_s(y)\sigma_s(dy) = \int_{\mathbb{T}^d} \nabla_\mu v(s,Q_s,\sigma_s)(y)\cdot \mathbf{v}_s(y)\sigma_s(dy).$$

■

9.3. **Connection with MFG equations and existence of a Nash equilibrium.** Our study does not establish whether strong solutions to (1.1) are unique and thus we cannot exclude the possibility that there may be another strong solution to (1.1) not given by the representation formula (9.1). For this reason, in this subsection, without appealing to that representation formula, we explain how any strong solution to (1.1) can be used to obtain a solution to the First Order Mean Field Games equations (1.2) and obtain the existence of an analogue of a Nash equilibrium for a game with a continuum of players.

We begin with a lemma that provides a rigorous connection between strong solutions to the master equation equation (1.1) and the First Order Mean Field Games equations (1.2).

Lemma 9.9. *Let $T > 0$, let u be a strong solution to (1.1) (see Definition 9.6), and let $\mu \in \mathcal{P}(\mathbb{T}^d)$. Then:*

(i) *There exist $\bar\sigma \in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$ such that*

$$\begin{cases} \partial_t\bar\sigma_t + \nabla\cdot\big(\nabla_q u(t,q,\bar\sigma_t)\bar\sigma_t\big) = 0 & \text{in } \mathcal{D}'\big((0,T)\times\mathbb{T}^d\big), \\ \bar\sigma_T = \mu. \end{cases} \tag{9.24}$$

The solution $\bar\sigma$ is given by $\bar\sigma_t = S(t,\cdot)_{\#}\mu$ for $t \in [0,T]$ and if μ is non–atomic, so is $\bar\sigma_t$ for $t \in [0,T]$. In particular if μ has a density with respect to the Lebesgue measure, so does $\bar\sigma_t$ for $t \in [0,T]$. Here, S is the flow uniquely determined by the system of differential equations

$$\begin{cases} \partial_t S(t,q) &= \nabla_q u(t,S(t,q),\bar\sigma_t), \qquad q \in \mathbb{T}^d, \quad t \in (0,T) \\ S(s,q) &= q \qquad\qquad q \in \mathbb{T}^d. \end{cases} \tag{9.25}$$

(ii) *If $U(t,q) := u(t,q,\bar\sigma_t)$, then $U \in C([0,T]\times\mathbb{T}^d)\cap C^1\big((0,T)\times\mathbb{T}^d\big)$ and the pair $(\bar\sigma,U)$ satisfies the system of equations (1.2), in fact U is a classical solution to the HJ equation in this system.*

Proof. We sketch the proof. Since $\nabla_q u$ is continuous and bounded, for any $\sigma \in AC_2(0,T,\mathcal{P}(\mathbb{T}^d))$ there exists $\sigma^* \in AC^2\big(0,T;\mathcal{P}(\mathbb{T}^d)\big)$ such that

$$
(9.26) \qquad \begin{cases} \partial_t \sigma^*_t + \nabla \cdot \big(\nabla_q u(t,q,\sigma_t)\sigma^*_t\big) = 0 & \text{in } \mathcal{D}'\big((0,T)\times\mathbb{T}^d\big), \\ \sigma^*_s = \mu. \end{cases}
$$

In other words, we have defined a map which to each σ, associates σ^*. Using iterations one checks that this map has a fixed point $\bar\sigma$; in other words, $\bar\sigma$ satisfies (9.24).

It is obvious that $U(t,q) := u(t,q,\bar\sigma_t)$ satisfies $U \in C([0,T]\times\mathbb{T}^d)$. By Lemma 9.8 and the fact that u is a strong solution to (1.1) we have, for a.e. $t \in (0,T)$ and every $q \in \mathbb{T}^d$,

$$
\begin{aligned}
\partial_t U(t,q) &= \frac{d}{dt} u(t,q,\bar\sigma_t) \\
&= \partial_t u(t,q,\bar\sigma_t) + \int_{\mathbb{T}^d} \nabla_\mu u(t,q,\bar\sigma_t)(y)\cdot \mathbf{v}_t(y)\bar\sigma_t(dy) \\
&= -\frac{|\nabla_q u(t,q,\bar\sigma_t)|^2}{2} - F(q,\bar\sigma_t), \qquad (9.27)
\end{aligned}
$$

since $\mathbf{v}_t = \nabla_q u(t,q,\bar\sigma_t)$ for a.e. t. Noticing that the right hand side of (9.27) is continuous, we conclude that $\partial_t U(t,q)$ must exist for every $t \in (0,T)$ and every $q \in \mathbb{T}^d$ and be equal to the right hand side of (9.27). Thus $U \in C^1((0,T)\times\mathbb{T}^d)$ and the pair $(\bar\sigma, U)$ solves (1.2). In particular U is a classical solution to the HJ equation

$$
(9.28) \qquad \partial_t U(t,q) + \frac{|\nabla_q U(t,q)|^2}{2} + F(q,\bar\sigma_t) = 0, \quad U(0,q) = u_*(0,q,\bar\sigma_0).
$$

Moreover it is a standard that, under our assumptions on u_* and F, we have $\nabla_{qq} U \le C I_d$ for some $C > 0$. Using these facts it then follows from the theory of HJ equations and ODE theory that (9.25) has a unique solution on $(0,T)$ and the flow satisfies $|S(t,q_1) - S(t,q_2)|_{\mathbb{T}^d} \ge C_1 |q_1 - q_2|_{\mathbb{T}^d}$ for $t \in [0,T], q_1, q_2 \in \mathbb{T}^d$, for some $C_1 > 0$, and thus if μ is non–atomic, so is $\hat\sigma_t := S(t,\cdot)_\# \mu$ for $t \in [0,T]$. In particular if μ has a density, so does $\hat\sigma_t := S(t,\cdot)_\#\mu$ for $t \in [0,T]$. Moreover, by results on the continuity equation, $\hat\sigma$ is the unique solution to

$$
(9.29) \qquad \begin{cases} \partial_t \hat\sigma_t + \nabla \cdot \big(\nabla_q u(t,q,\bar\sigma_t)\hat\sigma_t\big) = 0 & \text{in } \mathcal{D}'\big((0,T)\times\mathbb{T}^d\big), \\ \hat\sigma_T = \mu. \end{cases}
$$

For the proofs of these statements we refer the reader to [12], Section 4.1, Lemmas 4.11 and 4.13, and Section 4.2, Theorem 4.18 (see also [1]). The uniqueness of solutions of (9.29) thus implies $\bar\sigma_t = \hat\sigma_t = S(t,\cdot)_\#\mu$, which completes the proof. ∎

We assume in the rest of this subsection that $T > 0$ and u is a strong solution to (1.1).

We are now ready to explain what we mean by a Nash equilibrium.

Let $s \in [0,T]$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$ be a non–atomic measure; for instance, we may assume that μ has a density. Given paths

$$Q \in W^{1,2}\big(0,s;\mathbb{T}^d\big), \quad \sigma \in AC^2\big(0,s;\mathcal{P}(\mathbb{T}^d)\big),$$

we define the action

$$\mathcal{J}_s(Q,\sigma) = \int_0^s \Big(\frac{1}{2}|\dot{Q}_\tau|^2 - F(Q_\tau,\sigma_\tau)\Big)d\tau$$

and the augmented action

$$\mathcal{J}_s^o(Q,\sigma) = \int_0^s \Big(\frac{|\dot{Q}_t|^2}{2} - F\big(Q_t,\sigma_t\big)\Big)dt + u_*(Q_0,\sigma_0).$$

Suppose we have infinitely many players represented by points on $\mathbb{T}^d$, the position of an average player at time s is at $q \in \mathbb{T}^d$, and μ is the given probability distribution of the players at time s. The players try to choose paths to minimize their utility functions given by the augmented actions. It is assumed that each player knows the overall distribution of all the players at each time t, which is represented by the measure $\sigma[S]_t$, and which is determined by the overall flow of all players. However neither player can change the distribution of the players by his/her own actions alone as it can be changed only by a collective action of the players. This is why the phrase continuum of players is used and games with this kind of structure are also called non-atomic. We are thus looking for a map $S(t,q)$ such that, $S(\cdot,q) \in W^{1,2}(0,s;\mathbb{T}^d)$ for every $q \in \mathbb{T}^d$, and

$$\mathcal{J}_s^o\big(S(\cdot,q),\sigma[S]\big) \le \mathcal{J}_s^o\big(Q,\sigma[S]\big)$$

for every $q \in \mathbb{T}^d, Q \in W^{1,2}\big(0,s;\mathbb{T}^d\big)$ such that $Q(s) = q$, where

$$\sigma[S]_t = S(t,\cdot)_{\#}\mu.$$

The measure $\sigma[S]_t$ gives the distribution of the players at time t determined by the flow S, and the path $S(\cdot,q)$ is then optimal for the player which is located at q at time s for every $q \in \mathbb{T}^d$. This is what we mean by a Nash equilibrium.

Let S, $\bar{\sigma}$ be as in Lemma 9.9 (applied with $T = s$) so that (9.24) and (9.25) are satisfied and $\bar{\sigma}_t$ has a density with respect to the Lebesgue measure for all $t \in [0,s]$.

We recall that if u is the strong solution constructed in Section 9.2 then the pair

$$S(t,q) = \Sigma_s^t[\mu](q), \quad \bar{\sigma}_t = S(t,\cdot)_{\#}\mu$$

solves (9.24)-(9.25).

We refer to $\nabla_q u(t,q,\bar{\sigma}_t)$ as a closed loop feedback control strategy. We claim that map S constructed this way gives a Nash equilibrium for the game in the sense described above, and $u(t, q, \mu)$ is the payoff function for the player which is at y at time s, i.e., for every y,

$$u(s, q, \mu) = \mathcal{J}^o_s(S(\cdot, q), \bar{\sigma}) \leq \mathcal{J}^o_s(Q, \bar{\sigma})$$

for every path $Q : [0, s] \to \mathbb{T}^d$ is such that $Q_s = q$. We refer the reader to [12] and [22] for more on the concept of a Nash equilibrium for games with large numbers of players.

Lemma 9.10. *Let $Q \in W^{1,2}\big(0, s; \mathbb{T}^d\big)$ and be differentiable at $t \in (0, s)$. Then*

$$\frac{d}{dt}\Big(u(t, Q_t, \bar{\sigma}_t) - \mathcal{J}_t(Q, \bar{\sigma})\Big) < 0$$

unless $\dot{Q}_t = \nabla_q u(t, Q_t, \bar{\sigma}_t)$, in which case equality holds.

Proof. We remind that $\bar{\mathbf{v}}_t = \nabla_q u(t, \cdot, \bar{\sigma}_t)$ is the velocity of minimal norm for $\bar{\sigma}$ for every t. We thus have, by (9.28),

$$\begin{aligned}
&\frac{d}{dt}\Big(u(t, Q_t, \bar{\sigma}_t) - \mathcal{J}_t(Q, \bar{\sigma})\Big) = \frac{d}{dt}\Big(U(t, Q_t) - \mathcal{J}_t(Q, \bar{\sigma})\Big) \\
&= \partial_t U(t, Q_t) + \dot{Q}_t \cdot \nabla_q U(t, Q_t) - \frac{|\dot{Q}_t|^2}{2} + F(Q_t, \bar{\sigma}_t) \\
&= \partial_t U(t, Q_t) + \frac{1}{2}|\nabla_q U(t, Q_t)|^2 + F(Q_t, \bar{\sigma}_t) - \frac{1}{2}|\dot{Q}_t - \nabla_q U(t, Q_t)|^2 \\
&= -\frac{1}{2}|\dot{Q}_t - \nabla_q u(t, Q_t, \bar{\sigma}_t)|^2.
\end{aligned}$$

This completes the proof. ∎

Corollary 9.11 (Existence of a Nash equilibrium)**.** *Assume $Q \in W^{1,2}\big(0, s; \mathbb{T}^d\big)$ is such that $Q_s = q$. Then*

$$u(s, q, \mu) = \mathcal{J}^o_s(S(\cdot, q), \bar{\sigma}) < \mathcal{J}^o_s(Q, \bar{\sigma}), \tag{9.30}$$

unless $Q \equiv S(\cdot, q)$.

Proof. By Lemma 9.10, unless $\dot{Q}_t \equiv \nabla_q u(t, Q_t, \bar{\sigma}_t)$ for a.e. $t \in [0, s]$, we have

$$u(s, Q_s, \mu) - \mathcal{J}_s(Q, \bar{\sigma}) - u(0, Q_0, \bar{\sigma}_0) < 0$$

and

$$u(s, S(s, q), \mu) - \mathcal{J}_s(S(\cdot, q), \bar{\sigma}) - u(0, S(0, q), \bar{\sigma}_0) = 0.$$

We now use that

$$u(s, Q_s, \mu) = u(s, q, \mu) = u(s, S(s, q), \mu)$$

and

$$u(0, Q_0, \bar{\sigma}_0) = u_*(Q_0, \bar{\sigma}_0), \quad u(0, S(0, q), \bar{\sigma}_0) = u_*(S(0, q), \bar{\sigma}_0),$$

to obtain (9.30). Since $S(\cdot, q)$ is the unique solution to (9.25), $\dot{Q}_t \equiv \nabla_q u(t, Q_t, \bar{\sigma}_t)$ for a.e. $t \in [0, s]$, implies $S(\cdot, q) \equiv Q$ on $[0, s]$. ∎

Acknowledgments

The research of W. Gangbo was supported by NSF grant DMS–1160939. W. Gangbo would like to acknowledge the generous support provided through the Eisenbud Professorship at MSRI during the Fall 2013 program on Optimal Transportation: Geometry and Dynamics, where part of this work was done. W. Gangbo would also like to thank R. Jensen, R. Hynd and K. Trivisa for many extremely fruitful discussions at MSRI. Both authors would like to thank P. Cardialaguet for his input on the first draft of the manuscript.

References

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2004), no. 2, 227–260.
[2] L. Ambrosio and J. Feng, On a class of first order Hamilton-Jacobi equations in metric spaces, J. Differential Equations 256 (2014), no. 7, 219–2245.
[3] L. Ambrosio and W. Gangbo, Hamiltonian ODEs in the Wasserstein space of probability measures, Comm. Pure Applied Math. 61 (2008), no. 1, 18–53.
[4] L. Ambrosio, N. Gigli and G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.
[5] R. J. Aumann, Markets with a continuum of traders, Econometrica 32 (1964), 39–50.
[6] R. J. Aumann and L. S. Shapley, Values of non-atomic games, A Rand Corporation Research Study, Princeton University Press, Princeton, New Jersey, 1974.
[7] M. Balandat and C. Tomlin, On efficiency in mean field differential games, 2013 American Control Conference (ACC), 2013.
[8] T. Basar, H. Tembine and Q. Zhu, Risk-sensitive Mean Field games, preprint, arXiv:1210.2806, 2012.
[9] A. Bensoussan, J. Frehse and S. C. P. Yam, Mean field games and mean field type control, Springer Briefs in Mathematics, 2013.
[10] A. Bensoussan, J. Frehse and S. C. P. Yam, The Master equation in mean field theory, J. Math. Pures Appl. (9) 103 (2015), 1441–1474.
[11] R. Buckdahn, J. Li, S. Peng and C. Rainer, Mean-field stochastic differential equations and associated PDEs, preprint, arXiv:1407.1215, 2014.
[12] P. Cardialaguet, Notes on mean-field games, (from P.-L. Lions' lectures at Collège de France), 2013, https://www.ceremade.dauphine.fr/∼cardalia/MFG20130420.pdf.
[13] R. Carmona and F. Delarue, The Master equation for large population equilibriums, Stochastic Analysis and Applications, 2014, Springer Proceedings in Mathematics & Statistics, vol. 100, 77–128.
[14] J.-F. Chassagneux, D. Crisan and F. Delarue, A Probabilistic approach to classical solutions of the master equation for large population equilibria, preprint, arXiv:1411.3009v2, 2015.
[15] E. Feleqi, The derivation of ergodic Mean Field Game equations for several populations of players, Dynamic Games and Applications, to appear.
[16] I. Fonseca and W. Gangbo, Degree Theory in Analysis and Its Applications, GMT, Oxford University Press, 1995.

[17] W. Gangbo, T. Nguyen and A. Tudorascu, Euler-Poisson systems as action-minimizing paths in the Wasserstein space, Arch. Ration. Mech. Anal. 192 (2009), no. 3, 419–452.
[18] W. Gangbo, T. Nguyen and A. Tudorascu, Hamilton-Jacobi equations in the Wasserstein space, Methods Appl. Anal. 15 (2008), no. 2, 155–183.
[19] W. Gangbo and A. Święch, Optimal Transport and Large Number of Particles, Discrete Contin. Dyn. Syst. 34 (2014), no. 4, 1397–1441.
[20] W. Gangbo and A. Święch, Metric viscosity solutions of Hamilton-Jacobi equations depending on local slopes, preprint, http://people.math.gatech.edu/∼swiech/Gangbo-Swiech.2014.pdf.
[21] W. Gangbo and A. Tudorascu, Weak KAM theory on the Wasserstein torus with multi–dimensional underlying space, Comm. Pure Appl. Math. 67 (2014), no. 3, 408–463.
[22] D. A. Gomes, Mean field games, lecture notes based upon P. L. Lions course at College de France, preprint.
[23] D. A. Gomes and S. Patrizi, Obstacle and weakly coupled systems problem in mean–field games, preprint.
[24] D. A. Gomes, S. Patrizi and V. Voskanyan, On the existence of classical solutions for stationary extended Mean Field Games, Nonlinear Anal. 99 (2014), 49–79.
[25] D. A. Gomes, E. Pimentel and H. Sànchez–Morgado, Time dependent mean–field games in the subquadratic case, 2014, to appear in Comm. Partial Differential Equations.
[26] D. A. Gomes, E. Pimentel and H. Sànchez–Morgado, Time dependent mean–field games - super quadratic Hamiltonians, preprint.
[27] D. A. Gomes, G. E. Pires and H. Sànchez–Morgado, A-priori estimates for stationary mean–field games, Netw. Heterog. Media 7 (2012), no. 2, 303–314.
[28] D. A. Gomes and R. Ribeiro, Mean Field Games with logistic population dynamics, 52nd IEEE Conference on Decision and Control, Florence, December 2013.
[29] D. A. Gomes and J. Saude, Mean Field Games models–a brief survey, Dyn. Games Appl. 4 (2014), no. 2, 110–154.
[30] O. Gueant, Mean Field Games and Applications to Economics, Ph.D. Thesis, Université Paris Dauphine, Paris, 2009.
[31] O. Gueant, A reference case for mean field games models, J. Math. Pures Appl. (9) 92 (2009), no. 3, 276–294.
[32] O. Gueant, J. M. Lasry and P.-L. Lions, Mean Field Games and applications, Paris–Princeton Lectures on Mathematical Finance 2010, 205–266, Lecture Notes in Math., 2003, Springer, Berlin, 2011.
[33] O. Gueant, J. M. Lasry and P.-L. Lions, Mean Field Games and oil production, preprint, 2011.
[34] M. Huang, Large–population LQG games involving a major player: the Nash certainty equivalence principle, SIAM J. Control Optim. 48 (2009/10), no. 5, 3318–3353.
[35] M. Huang, Mean field stochastic games with discrete states and mixed players, Proc. Game Nets, Vancouver, 2012.
[36] M. Huang, P. E. Caines and R. P. Malhamé, Large–population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ–Nash equilibria, IEEE Transactions on Automatic Control 52 (2007), no. 9, 1560–1571.
[37] M. Huang, P. E. Caines and R. P. Malhamé, The Nash Certainty Equivalence Principle and McKean–Vlasov Systems: an Invariance Principle and Entry Adaptation, 46th IEEE Conference on Decision and Control, 121–123, 2007.
[38] M. Huang, P. E. Caines and R. P. Malhamé, An Invariance Principle in Large Population Stochastic Dynamic Games, J. Syst. Sci. Complex. 20 (2007), no. 2, 162–172.

[39] M. Huang, P. E. Caines and R. P. Malhamé, The NCE (mean field) principle with locality dependent cost interactions, IEEE Trans. Automat. Control 55 (2010), no. 12, 2799–2805.
[40] M. Huang, R. P. Malhamé and P. E. Caines, Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle, Commun. Inf. Syst. 6 (2006), no. 3, 221–251.
[41] V. N. Kolokoltsov, J. Li and W. Yang, Mean field games and nonlinear Markov processes, preprint.
[42] A. Lachapelle, J. M. Lasry, C. A. Lehalle and P. L. Lions. Efficiency of the price formation process in presence of high frequency participants: a Mean Field Game analysis, preprint, arXiv:1305.6323v3.
[43] A. Lachapelle, J. Salomon and G. Turinici, Computation of Mean Field equilibria in economics.
[44] J. M. Lasry and P.-L. Lions, Jeux à champ moyen. II. Le cas stationnaire, C. R. Math. Acad. Sci. Paris 343 (2006), no. 9, 619–625.
[45] J. M. Lasry and P.-L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal, C. R. Math. Acad. Sci. Paris 343 (2006), no. 10, 679– 684.
[46] J.-M. Lasry and P.-L. Lions, Mean field games, Jpn. J. Math. 2 (2007), 229–260.
[47] J. M. Lasry and P.-L. Lions, Large investor trading impacts on volatility, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), no. 2, 311–323.
[48] P.-L. Lions, Cours au Collège de France, Mean Field Games, 2007–2011.
[49] R. E. Lucas and B Moll. Knowledge growth and the allocation of time, Journal of Political Economy 122 (2014), no. 1, 1–51.
[50] M. Nourian, P. Caines, R. P. Malhamé and Minyi Huang. Nash, social and centralized solutions to consensus problems via mean field control theory, IEEE Trans. Automat. Control 58 (2013), no.3, 639–653.
[51] F. Santambrogio, A modest proposal for MFG with density constraints, Netw. Heterog. Media 7 (2012), no. 2, 337–347.
[52] H. Tembine, Energy–constrained Mean Field Games in wireless networks, Journal of Strategic Behavior and the Environment, to appear.

Georgia Institute of Technology, Atlanta GA, USA
E-mail address: gangbo@math.gatech.edu

Georgia Institute of Technology, Atlanta GA, USA
E-mail address: swiech@math.gatech.edu