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Hamilton–Jacobi–Bellman Equations for the Optimal Control of the Duncan–Mortensen–Zakai Equation[1]

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We study a class of Hamilton–Jacobi–Bellman (HJB) equations associated to stochastic optimal control of the Duncan–Mortensen–Zakai equation. The equations are investigated in weighted L^2 spaces. We introduce an appropriate notion of weak (viscosity) solution of such equations and prove that the value function is the unique solution of the HJB equation. We apply the results to stochastic optimal control problems with partial observation and correlated noise. © 2000 Academic Press

Key Words: Hamilton–Jacobi–Bellman equations; Duncan–Mortensen–Zakai equation; optimal control of partially observed systems; viscosity solutions.

1. INTRODUCTION

The paper is devoted to the study of Hamilton–Jacobi–Bellman (HJB) equations associated with stochastic optimal control problems for the Duncan–Mortensen–Zakai (DMZ) equation. Such HJB equations have the form

$$\begin{cases} v_t + \inf_{\alpha \in \mathscr{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \langle D^2 v S_\alpha^k x, S_\alpha^k x \rangle - \langle A_\alpha x, Dv \rangle + f(x, \alpha) \right\} = 0 \\ \qquad \text{in} \quad (0, T) \times X \\ v(T, x) = g(x) \qquad \text{in} \quad X, \end{cases} \tag{1}$$

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where X is a real, separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$, and S_α^k, A_α are realizations of respectively first and second order differential operators in X. Cauchy problem (1) has been investigated by P. L. Lions [24, Part II] in the case where X is $L^2(\mathbb{R}^d)$ and S_α^k are just multiplication operators. He proved that the value function of the associated optimal control problem is the unique appropriately defined weak solution of (1). The uniqueness was established by sub- and super-optimality arguments and stochastic analysis. Bellman equations related to control of measure-valued diffusions have also been studied by O. Hijab in [15–17]. However the notion of solution used there was so weak that the author did not obtain any uniqueness results. This paper presents a unified approach to (1) that guarantees existence of unique solutions in a generality that includes equations coming from a large class of reasonable "separated" problems.

The main interest in studying (1) comes from stochastic optimal control of partially observed systems. The DMZ equation is the equation of the unnormalized conditional distribution and is the state equation of a so called "separated" problem (see e.g. [2, 11, 27]). This connection will be discussed in Section 7. Motivated by such problems we want to study the HJB equation (1) in a generality that would include cases coming from problems with correlated noise and having fairly general cost functions (see Section 7 for details). These requirements force the S_α^k to be first order differential operators and may cause functions $f(\cdot, \alpha)$ and g not to be well defined in $L^2(\mathbb{R}^d)$. To handle such difficulties we will investigate (1) in weighted L^2-spaces for which the usual $L^2(\mathbb{R}^d)$ is a special case. We will assume that the operators S_α^k and A_α satisfy a nondegeneracy condition (7), however it seems possible to extend the techniques of the paper to treat the control of degenerate equations of DMZ type that come from partially observed problems.

We define an appropriate notion of weak solution of (1) (called viscosity solution) that has its origin in the definitions that appeared independently in [5] and [8, Part VII], that were later adapted to second order equations in [14], and some ideas from [24, Part II, 30, 31]. The notion of viscosity solution is based on a principle of replacing the possibly nonexisting derivatives of a sub-(super-) solution v by derivatives of test functions φ at points $(\hat{t}, \hat{x})$ such that $v - \varphi$ have maximum (minimum). In our definition of solution we choose an appropriate class of test functions φ and require that the points $\hat{x}$ be in a weighted Sobolev space that will be called X_1 so that all terms in (1) have classical meaning. This helps deal with the technicalities of the proof of uniqueness of solutions however it makes the existence part a little more difficult since first we have to show that we can achieve $\hat{x}$ to be in the desired space when v is the value function.

The major results of the paper are contained in Sections 5 and 6 in which we show that the value function of the associated optimal control problem belongs to an appropriate class of functions and it is the unique viscosity solution of (1) in this class. The proof of comparison that gives uniqueness is fairly general and does not use any references to the control problem. It could be used to treat more general equations, for instance those of Isaacs type. In Section 7 we discuss the connection with partial observation problems and the applicability of our results. Preliminary operator estimates and various estimates for the DMZ equation in weighted spaces are proved in Section 3, while in Section 4 we introduce the optimal control problem, the associated HJB equation, and the definition of viscosity solution.

We refer the reader to [3, 4, 12–14, 19, 24, 30, 31] for related results and a history of second order Hamilton–Jacobi–Bellman-Isaacs equations in Hilbert spaces and to [2, 10, 11, 27] and the references quoted therein for more information on the DMZ equation and stochastic optimal control of partially observed systems.

2. PRELIMINARIES

2.1. *Notation*

We will denote the norm and the inner product in $\mathbb{R}^k$ by $|\cdot|_{\mathbb{R}^k}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ respectively. Moreover for a bounded function $g: \mathbb{R}^k \mapsto \mathbb{R}^d$ (d, $k \in \mathbb{N}$) we set $\|g\|_\infty = \sum_{i=1}^d \sup_{\xi \in \mathbb{R}^k} |g_i(\xi)|_{\mathbb{R}}$.

Let X, Y be Hilbert spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. We denote by $\mathscr{L}(X, Y)$ the Banach space of continuous linear operators $T: X \to Y$ with the operator norm $\|\cdot\|_{\mathscr{L}(X, Y)}$. If $X = Y$ we will denote this space by $\mathscr{L}(X)$. We put

$$\Sigma(X) = \{T \in \mathscr{L}(X), T \text{ self-adjoint}\}.$$

We denote by $C(X, Y)$ (respectively $UC(X, Y)$) the space of all functions $\varphi: X \to Y$ which are continuous (respectively, uniformly continuous) and by $BUC(X, Y)$ the Banach space of all functions $\varphi: X \to Y$ which are bounded and uniformly continuous on X equipped with the usual norm

$$\|\varphi\|_\infty = \sup_{x \in X} \|\varphi(x)\|_Y.$$

By $USC(X)$, $LSC(X)$ we denote respectively the space of upper-semicontinuous functions and the space of lower-semicontinuous functions $\varphi: X \to \mathbb{R}$. For $k \in \mathbb{N}$, we denote by $C^k(X)$ the space of all functions which are continuous on X together with all their Fréchet derivatives up to order k. For given $0 \leqslant t < T$ we will also denote by $C^{1,2}((t, T) \times X)$ the space of all functions $\varphi\colon (t, T) \times X \to \mathbb{R}$ for which φ_t and $D\varphi, D^2\varphi$ (the Fréchet derivatives of φ with respect to $x \in X$) exist and are uniformly continuous on closed and bounded subsets of $(t, T) \times X$.

We say that a function $\sigma\colon [0, +\infty) \to [0, +\infty)$ is a modulus if σ is continuous, nondecreasing, subadditive and $\sigma(0) = 0$. Subadditivity in particular implies that, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\sigma(r) \leqslant \varepsilon + C_\varepsilon r \qquad \text{for every} \quad r \geqslant 0.$$

Moreover a function $\sigma\colon [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a local modulus if σ is continuous, nondecreasing in both variables, subadditive in the first variable, and $\sigma(0, r) = 0$ for every $r \geqslant 0$.

For any $\varphi \in UC(X, Y)$ we denote by σ_φ a continuity modulus of φ i.e. a modulus such that $\|\varphi(x_1) - \varphi(x_2)\|_Y \leqslant \sigma_\varphi(\|x_1 - x_2\|_X)$ for every $x_1, x_2 \in X$. We recall that, if $\varphi \in UC(X, Y)$, then its modulus of continuity always exists and so there exist positive constants C_0, C_1 such that

$$\|\varphi(x)\|_Y \leqslant C_0 + C_1 \|x\|_X, \qquad \text{for every} \quad x \in X.$$

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\colon t \geqslant 0\}, \mathbb{P})$ be a complete probability space with a normal filtration $\{\mathscr{F}_t\colon t \geqslant 0\}$ and let W be an m-dimensional Wiener process adapted to the above filtration. We will call the 5-tuple $(\Omega, \mathscr{F}, \{\mathscr{F}_t\colon t \geqslant 0\}, \mathbb{P}, W)$ a stochastic base.

We will denote by $L^2(\Omega; X)$ the set of all measurable and square integrable random variables $Z\colon \Omega \mapsto X$.

Given $0 \leqslant t \leqslant T$ we denote by $M^0([t, T]; X)$ the set of all X-valued processes measurable on $[t, T]$ and progressively measurable with respect to the filtration $\{\mathscr{F}_s\colon t \leqslant s \leqslant T\}$ and by $M^2([t, T]; X)$ (a subset of $L^2([t, T] \times \Omega; X)$) the space of X-valued processes x such that $x \in M^0([t, T]; X)$ and

$$\mathbb{E}\left(\int_t^T \|x(s)\|_X^2 \, ds\right) < +\infty.$$

2.2. *Sobolev Spaces*

Given $d, k \in \mathbb{N}$ we denote by $H^k(\mathbb{R}^d)$ the Sobolev space of all measurable functions $\mathbb{R}^d \mapsto \mathbb{R}$ with square integrable distributional derivatives up to order k. In particular we set $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. We will write H^k instead of $H^k(\mathbb{R}^d)$. For every $k \in \mathbb{N}$ the space H^k is a Hilbert space with the inner product

$$\langle h_1, h_2 \rangle_k = \sum_{|\alpha| \leqslant k} \int_{\mathbb{R}^d} \partial^\alpha h_1(\xi) \, \partial^\alpha h_2(\xi) \, d\xi.$$

We will write $\|\cdot\|_k$ for the norm in H^k. The topological dual space of H^k will be denoted by $[H^k]'$. As usual one can identify L^2 with its dual by the Riesz identification. This induces an embedding $I\colon L^2 \hookrightarrow [H^k]'$ defined by

$$\langle I(a), b\rangle_{\langle [H^k]', H^k\rangle} = \langle a, b\rangle_0, \qquad a \in H^0, \quad b \in H^k, \tag{2}$$

where $\langle \cdot, \cdot\rangle_{\langle [H^k]', H^k\rangle}$ is the duality pairing between $[H^k]'$ and H^k. Following standard notation we will denote $H^{-k} = [H^k]'$. Except when explicitly stated we will always identify L^2 with its dual. For $a \in H^0$ and $b \in H^k$ we can therefore rewrite (2), as

$$\langle a, b\rangle_{\langle H^{-k}, H^k\rangle} = \langle a, b\rangle_0.$$

Let $B = (-\Delta + I)^{-1}$. It is well known that H^{-k} is the completion of H^0 under the norm

$$\|x\|_{-k} = \|B^{k/2}x\|_0 = \langle B^k x, x\rangle_0$$

and $B^{1/2}$ is an isometry between H^{-2}, H^{-1}, H^0, H^1 and H^{-1}, H^0, H^1, H^2 respectively. Observe also that, for $k \in \mathbb{Z}$, the adjoint of the operator $B^{1/2}\colon H^k \mapsto H^{k+1}$ is $B^{1/2}\colon H^{-k-1} \mapsto H^{-k}$.

2.3. *Weighted Sobolev Spaces*

Let $k = 0, 1, 2$. Given a positive real-valued function $\rho \in C^2(\mathbb{R}^d)$, ($\rho(\xi) > 0$ for every $\xi \in \mathbb{R}^d$) we define the *weighted Sobolev space* $H^k_\rho(\mathbb{R}^d)$ (or simply H^k_ρ) as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$x \mapsto \sum_{|\alpha| \leqslant k} \left(\int_{\mathbb{R}^d} (\partial^\alpha [\rho(\xi)\, x(\xi)])^2\, d\xi \right)^{1/2}.$$

It is well known that H^k_ρ $(k \in \mathbb{N})$ can also be defined as the space of all measurable functions $x\colon \mathbb{R}^d \mapsto \mathbb{R}$ such that $\rho(\cdot) \cdot x(\cdot) \in H^k$ and that $\|x\|_{H^k_\rho} = \|\rho x\|_{H^k}$. We will write $\|\cdot\|_{k,\rho}$ for the norm in H^k_ρ. We will denote by C_ρ the isometry $: H^k_\rho \mapsto H^k$ defined as $(C_\rho x)(\xi) = \rho(\xi)\, x(\xi)$, and by $C_{1/\rho} = C_\rho^{-1}\colon H^k \mapsto H^k_\rho$ its inverse $((C_{1/\rho} x)(\xi) = [\rho(\xi)]^{-1}\, x(\xi))$. We observe that H^k_ρ is a Hilbert space with the inner product $\langle h_1, h_2\rangle_{k,\rho} = \langle C_\rho h_1, C_\rho h_2\rangle_k$. Similarly to the non-weighted case we will denote by $[H^k_\rho]'$ the topological dual space of H^k_ρ and, identifying $L^2_\rho = H^0_\rho$ with its dual, we have $[H^k_\rho] = H^{-k}_\rho$ and

$$H^k_\rho \subset H^0_\rho = [H^0_\rho]' \subset [H^k_\rho]' = H^{-k}_\rho. \tag{3}$$

We will always use this identification, except when explicitly stated.

The adjoint C_ρ^* of C_ρ is an isometry $C_\rho^*\colon [H^k]' \mapsto [H_\rho^k]'$, i.e. $C_\rho^*\colon H^{-k} \mapsto H_\rho^{-k}$ with the above identification so that for $f \in H^{-k}$ we have $\|f\|_{H^{-k}} = \|C_\rho^* f\|_{H_\rho^{-k}}$. Observe that C_ρ^* can be identified with $C_{1/\rho}$.

To simplify notation we will write $X_k = H_\rho^k(\mathbb{R}^d)$.

Let $B_\rho = C_{1/\rho}[(-\Delta + I)^{-1}]\, C_\rho$. Similarly to the case of non-weighted spaces we have that X_{-k} is the completion of X_0 under the norm $\|x\|_{-k,\rho} = \langle B_\rho^k x, x\rangle_{0,\rho} = \langle B^k C_\rho x, C_\rho x\rangle_0$ and $B_\rho^{1/2}$ is an isometry between X_{-2}, X_{-1}, X_0, X_1 and X_{-1}, X_0, X_1, X_2 respectively. The duality pairing between X_{-k} and X_k will be denoted by $\langle \cdot, \cdot \rangle_{\langle X_{-k}, X_k\rangle} = \langle C_\rho \cdot, C_\rho \cdot \rangle_{\langle H^{-k}, H^k\rangle}$. As before we have by (3) that

$$\langle a, b\rangle_{\langle X^{-k}, X^k\rangle} = \langle a, b\rangle_{0,\rho}.$$

In what follows we will consider weight functions ρ of the form:

$$\rho_\beta(\xi) = (1 + |\xi|^2_{\mathbb{R}^d})^{\beta/2}. \tag{4}$$

The proposition below follows by easy calculations.

Proposition 2.1. *Let $\beta > 0$ be fixed. Then, for $k = 0, 1, 2$, we have*

(i) *The function ρ_β is two times continuously differentiable in $\mathbb{R}^d$ and there exists a constant K_β such that*

$$\left\| \frac{\partial \rho}{\rho} \right\|_\infty, \quad \left\| \frac{\partial^2 \rho}{\rho} \right\|_\infty \leqslant K_\beta;$$

(ii) $X_k \subset H^k$;

(iii) *if $\beta > d/2$ then $X_0 \subset L^1(\mathbb{R}^d)$ and $X_k \subset W^{k,1}(\mathbb{R}^d)$.*

3. THE DUNCAN–MORTENSEN–ZAKAI EQUATION

In this section we present various preliminary operator estimates and prove estimates for solutions of the abstract DMZ equation in weighted spaces. These estimates will be needed in our future analysis. The connection between the DMZ equation investigated here and partially observed systems will be discussed in Section 7.

3.1. *Operators and Estimates*

Let $\mathbf{A}$ be a complete, separable metric space. For a given $\alpha \in \mathbf{A}$ we consider the linear differential operator $A_\alpha\colon D(A_\alpha) \subset X_0 \mapsto X_0$ defined as

$$D(A_\alpha) = H^2_\rho(\mathbb{R}^d) = X_2,$$

$$(A_\alpha x)(\xi) = -\sum_{i,\,j=1}^{d} \partial_i [a_{i,\,j}(\xi, \alpha)\, \partial_j x(\xi)] + \sum_{i=1}^{d} \partial_i [b_i(\xi, \alpha)\, x(\xi)] + c(\xi, \alpha)\, x(\xi), \tag{5}$$

and, for given $m \in \mathbb{N}$ and $k = 1, ..., m$, the operators S^k_α: $D(S^k_\alpha) \subset X_0 \mapsto X_0$ defined by

$$D(S^k_\alpha) = H^1_\rho(\mathbb{R}^d) = X_1, \tag{6}$$

$$(S^k_\alpha x)(\xi) = \sum_{i=1}^{d} d_{ik}(\xi, \alpha)\, \partial_i\, x(\xi) + e_k(\xi, \alpha)\, x(\xi); \qquad k = 1, ..., m.$$

We assume:

HYPOTHESIS 3.1. (i) *The coefficients*

$$(a_{ij})_{i,\,j=1,\,...,\,d},\ (b_i)_{i=1,\,...,\,d},\ c,\ (d_{ik})_{i=1,\,...,\,d;\,k=1,\,...,\,m},\ (e_k)_{k=1,\,...,\,m} : \mathbb{R}^d \times \mathbf{A} \mapsto \mathbb{R}$$

are continuous in (ξ, α) *and, as functions of* ξ, *have bounded norms in* $C^2(\mathbb{R}^d)$, *uniformly in* $\alpha \in \mathbf{A}$. *Moreover, there exists a constant* $\lambda > 0$ *such that*

$$\sum_{i,\,j=1}^{d} \left(a_{i,\,j}(\xi, \alpha) - \tfrac{1}{2} \sum_{k=1}^{m} d_{ik}(\xi, \alpha)\, d_{jk}(\xi, \alpha) \right) z_i\, z_j \geqslant \lambda\, |z|^2_{\mathbb{R}^d} \tag{7}$$

for all $\alpha \in \mathbf{A}$ *and* ξ, $z \in \mathbb{R}^d$.

(ii) *The weight* ρ *is of the form* (4).

The lemma below collects some preliminary facts about the operators A_α and S^k_α. It will be left without a proof as the facts are either known or are consequences of straightforward calculations. For two operators T_1, T_2 we write $[T_1, T_2] = T_1 T_2 - T_2 T_1$.

LEMMA 3.2. (i) *The operators* A_α: $X_1 \mapsto X_{-1}$ *are variational operators on the Gelfand triple* $(X_1 \hookrightarrow X_0 \hookrightarrow X_{-1})$, (*and also on* $(X_0 \hookrightarrow X_{-1} \hookrightarrow X_{-2})$, $(X_2 \hookrightarrow X_1 \hookrightarrow X_0)$) *and*

$$\|A_\alpha\|_{\mathscr{L}(X_1,\, X_{-1})} = \|A_\alpha\|_{\mathscr{L}(X_0,\, X_{-2})} = \|A_\alpha\|_{\mathscr{L}(X_2,\, X_0)} \leqslant \|a\|_\infty + \|b\|_\infty + \|c\|_\infty \tag{8}$$

(ii) *The operators* $B_\rho A_\alpha$, $B^{1/2}_\rho S^k_\alpha$: $X_0 \to X_0$ *are bounded and* $\|B_\rho A_\alpha\|_{\mathscr{L}(X_0)}$, $\|B^{1/2}_\rho S^k_\alpha\|_{\mathscr{L}(X_0)} \leqslant C$ *for some constant* C *independent of* $\alpha \in \mathbf{A}$ *and* $1 \leqslant k \leqslant m$.

(iii) *The following commutator equalities hold*:

$$[A_\alpha, C_\rho]\, x = f_1\, \rho x + \sum_{h=1}^{d} f_{2,h} \partial_h [\rho x], \qquad [S_\alpha^k, C_\rho]\, x = f_{3,k}\, \rho x,$$

where

$$f_1(\cdot) = -\sum_{i,j=1}^{d} a_{i,j}(\cdot, \alpha) \left(\frac{\partial_{i,j}^2 \rho}{\rho} - 2\, \frac{\partial_i \rho\, \partial_j \rho}{\rho^2} \right)$$
$$- \sum_{i,j=1}^{d} (\partial_i a_{i,j}(\cdot, \alpha))\, \frac{\partial_j \rho}{\rho} + \sum_{i=1}^{d} b_i(\cdot, \alpha)\, \frac{\partial_i \rho}{\rho},$$

$$f_{2,h}(\cdot) = -\sum_{i=1}^{d} a_{i,h}(\cdot, \alpha)\, \frac{\partial_i \rho}{\rho} - \sum_{j=1}^{d} a_{h,j}(\cdot, \alpha)\, \frac{\partial_j \rho}{\rho}; \qquad h = 1, 2, ..., d,$$

$$f_{3,k}(\cdot) = \sum_{i=1}^{d} d_{ik}(\cdot, \alpha)\, \frac{\partial_i \rho}{\rho}; \qquad k = 1, 2, ..., m.$$

(iv) *The following estimates hold*:

$$\|[A_\alpha, C_\rho]\, x\|_0 \leqslant \|x\|_{1,\rho}\, \|a\|_\infty \left\| \frac{\partial \rho}{\rho} \right\|_\infty + \|x\|_{0,\rho} \left[\|a\|_\infty \left(\left\| \frac{\partial^2 \rho}{\rho} \right\|_\infty + 2 \left\| \frac{\partial \rho}{\rho} \right\|_\infty^2 \right) + (\|\partial a\|_\infty + \|b\|_\infty) \left\| \frac{\partial \rho}{\rho} \right\|_\infty \right], \tag{9}$$

$$\|[S_\alpha^k, C_\rho]\, x\|_0 \leqslant \|x\|_{0,\rho}\, \|d\|_\infty \left\| \frac{\partial \rho}{\rho} \right\|_\infty. \tag{10}$$

We now prove three key coercivity estimates. Their proofs are rather standard in the non-weighted case (see e.g. [23] and the proofs below). The proofs in the weighted case are more technical and use the commutator estimates of Lemma 3.2.

LEMMA 3.3. *Assume that Hypothesis* 3.1 *holds. Then there exists a constant $K > 0$ such that for every $x \in X_1$*

$$\langle A_\alpha x, x \rangle_{\langle X_{-1}, X_1 \rangle} - \frac{1}{2} \sum_{k=1}^{m} \langle S_\alpha^k x, S_\alpha^k x \rangle_{0,\rho} \geqslant \frac{\lambda}{4} \|x\|_{1,\rho}^2 - K \|x\|_{0,\rho}^2, \tag{11}$$

while, for $x \in X_2$,

$$\langle A_\alpha x, B_\rho^{-1} x\rangle_{0,\rho} - \frac{1}{2}\sum_{k=1}^{m} \langle B_\rho^{-1} S_\alpha^k x, S_\alpha^k x\rangle_{\langle X_{-1}, X_1\rangle} \geqslant \frac{\lambda}{4}\|x\|_{2,\rho}^2 - K\|x\|_{1,\rho}^2, \tag{12}$$

and for $x \in X_0$

$$\begin{aligned}\langle A_\alpha x, B_\rho x\rangle_{\langle X_{-2}, X_2\rangle} - \frac{1}{2}\sum_{k=1}^{m} \langle B_\rho S_\alpha^k x, S_\alpha^k x\rangle_{\langle x_1, x_{-1}\rangle}\\ \geqslant \frac{\lambda}{4}\|x\|_{0,\rho}^2 - K\|x\|_{-1,\rho}^2.\end{aligned} \tag{13}$$

Proof. We first sketch the proof in the case $\rho = 1$. Then we will only give the complete proof of (11) and (13) since the proof of (12) is very similar.

Part I. Proof in the case $\rho = 1$.

Step 1. Proof of (11).

Given $x \in H^1$ the estimate

$$\langle A_\alpha x, x\rangle_{\langle H^{-1}, H^1\rangle} - \frac{1}{2}\sum_{k=1}^{m} \langle S_\alpha^k x, S_\alpha^k x\rangle_0 \geqslant \frac{\lambda}{2}\|x\|_1^2 - K\|x\|_0^2 \tag{14}$$

can be rewritten as

$$\left\langle \left[A_\alpha - \frac{1}{2}\sum_{k=1}^{m} [S_\alpha^k]^* S_\alpha^k \right] x, x \right\rangle_{\langle H^{-1}, H^1\rangle} \geqslant \frac{\lambda}{2}\|x\|_1^2 - K\|x\|_0^2, \tag{15}$$

which is a straightforward consequence of (7) and elementary calculations.

Step 2. Proof of (12).

We recall that (see for instance [23, Theorem 1.1]) for every two strongly elliptic operators A_1, A_2 there exist positive constants α and β such that

$$\langle A_1 x, A_2 x\rangle_0 \geqslant \alpha\|x\|_2^2 - \beta\|x\|_1^2, \qquad \forall x \in H^2. \tag{16}$$

We now observe that (recalling that $[A, B]^* = [B^*, A^*]$) we have for $x \in H^2$

$$
\begin{aligned}
&\langle B^{-1}S^k_\alpha x, S^k_\alpha x\rangle_{\langle H^{-1}, H^1\rangle}\\
&\quad=\langle (S^k_\alpha)^* B^{-1}S^k_\alpha x, x\rangle_{\langle H^{-2}, H^2\rangle}\\
&\quad=\langle B^{-1}(S^k_\alpha)^* S^k_\alpha x, x\rangle_{\langle H^{-2}, H^2\rangle}+\langle [(S^k_\alpha)^*, B^{-1}]\, S^k_\alpha x, x\rangle_{\langle H^{-2}, H^2\rangle}\\
&\quad=\langle (S^k_\alpha)^* S^k_\alpha x, B^{-1}x\rangle_0+\langle S^k_\alpha x, [B^{-1}, S^k_\alpha]\, x\rangle_{\langle H^1, H^{-1}\rangle}
\end{aligned}
$$

and that

$$
\begin{aligned}
[B^{-1}, S^k_\alpha]\, x &= (I-\Delta)\, S^k_\alpha x - S^k_\alpha(I-\Delta)\, x = \Delta S^k_\alpha x - S^k_\alpha \Delta x\\
&=\sum_{i=1}^d \partial_{ii}\left[\sum_{j=1}^d d_{jk}\,\partial_j x + e_k x\right]\\
&\qquad-\sum_{j=1}^d d_{jk}\,\partial_j\left[\sum_{i=1}^d \partial_{ii}x\right]-e_k\left[\sum_{i=1}^d \partial_{ii}x\right]\\
&=\sum_{i=1}^d\sum_{j=1}^d (\partial_{ii}d_{jk})\,\partial_j x+2(\partial_i\, d_{jk})(\partial_i\,\partial_j x)\\
&\qquad+\sum_{i=1}^d (\partial_{ii}e_k)\, x+2(\partial_i\, e_k)(\partial_i\, x)
\end{aligned}
$$

so that, thanks to Hypothesis 3.1 (which says that the coefficients d_{jk} and e_k are uniformly bounded in C^2),

$$
\|[B^{-1}, S^k_\alpha]\, x\|_0 \leqslant C\,\|x\|_2.
$$

This gives, using (16) with

$$
A_1 = A_\alpha - \tfrac{1}{2}\sum_{k=1}^m [S^k_\alpha]^*\, S^k_\alpha \qquad\text{and}\qquad A_2 = B^{-1},
$$

that

$$
\begin{aligned}
&\langle A_\alpha x, B^{-1}x\rangle_0 - \tfrac{1}{2}\sum_{k=1}^m \langle B^{-1}S^k_\alpha x, S^k_\alpha x\rangle_{\langle H^{-1}, H^1\rangle}\\
&\quad=\langle A_1 x, A_2 x\rangle - \tfrac{1}{2}\sum_{k=1}^m \langle S^k_\alpha x, [B^{-1}, S^k_\alpha]\, x\rangle_{\langle H^1, H^{-1}\rangle}\\
&\quad\geqslant \alpha\,\|x\|_2^2 - \beta\,\|x\|_1^2 - C\,\|x\|_1\,\|x\|_2
\end{aligned}
$$

and the claim follows upon estimating the last term by sums of squares with appropriate weights.

Step 3. Proof of (13).

Setting for $x \in H^0$, $y = Bx$, we have

$$
\begin{aligned}
&\langle A_\alpha x, Bx \rangle_{\langle H^{-2}, H^2 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle B S_\alpha^k x, S_\alpha^k x \rangle_{\langle H^1, H^{-1} \rangle} \\
&\quad = \langle A_\alpha B^{-1} y, y \rangle_{\langle H^{-2}, H^2 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle B S_\alpha^k B^{-1} y, S_\alpha^k B^{-1} y \rangle_{\langle H^1, H^{-1} \rangle} \\
&\quad = \langle A_\alpha^* y, B^{-1} y \rangle_0 - \tfrac{1}{2} \sum_{k=1}^{m} \langle (S_\alpha^k + B[S_\alpha^k, B^{-1}])\, y, S_\alpha^k B^{-1} y \rangle_{\langle H^1, H^{-1} \rangle} \\
&\quad = \left\langle \left[A_\alpha^* - \tfrac{1}{2} \sum_{k=1}^{m} (S_\alpha^k)^* S_\alpha^k \right] y, B^{-1} y \right\rangle_0 \\
&\qquad - \tfrac{1}{2} \sum_{k=1}^{m} [\, \| B^{1/2} [S_\alpha^k, B^{-1}]\, y \|_0 + \langle [S_\alpha^k, B^{-1}]\, y, S_\alpha^k y \rangle_{\langle H^{-1}, H^1 \rangle}].
\end{aligned}
$$

Calculations similar to those in Step 2 yield $\| B^{1/2} [S_\alpha^k, B^{-1}]\, y \|_0 \leqslant C \, \|y\|_1$. Now, applying (16), the fact that $\|B^{1/2} z\|_0 = \|z\|_{-1}$ for $z \in H^{-1}$, and finally recalling that $y = Bx$ we obtain

$$
\begin{aligned}
&\langle A_\alpha x, Bx \rangle_{\langle H^{-2}, H^2 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle B S_\alpha^k x, S_\alpha^k x \rangle_{\langle H^1, H^{-1} \rangle} \\
&\quad \geqslant \alpha \, \|y\|_2^2 - \beta \, \|y\|_1^2 - C \, \|y\|_2 \, \|y\|_1 - C \, \|y\|_1^2 \\
&\quad = \alpha \, \|x\|_0^2 - \beta \, \|x\|_{-1}^2 - C \, \|x\|_0 \, \|x\|_{-1} - C \, \|x\|_{-1}^2
\end{aligned}
$$

and we conclude as before.

Part II. Proofs in the general case.

Step 1. Proof of (11). For $x \in X_1$ we have

$$
\begin{aligned}
&\langle A_\alpha x, x \rangle_{\langle X_{-1}, X_1 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle S_\alpha^k x, S_\alpha^k x \rangle_{0, \rho} \\
&\quad = \langle C_\rho A_\alpha x, C_\rho x \rangle_{\langle H^{-1}, H^1 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle C_\rho S_\alpha^k x, C_\rho S_\alpha^k x \rangle_0 \\
&\quad = \langle A_\alpha C_\rho x, C_\rho x \rangle_{\langle H^{-1}, H^1 \rangle} - \tfrac{1}{2} \sum_{k=1}^{m} \langle S_\alpha^k C_\rho x, S_\alpha^k C_\rho x \rangle_0 \\
&\qquad - \langle [A_\alpha, C_\rho]\, x, C_\rho x \rangle_{\langle H^{-1}, H^1 \rangle} \\
&\qquad + \tfrac{1}{2} \sum_{k=1}^{m} \langle [S_\alpha^k, C_\rho]\, x, 2 S_\alpha^k C_\rho x - [S_\alpha^k, C_\rho]\, x \rangle_0
\end{aligned}
$$

which, using commutator estimates (9), (10) and (11) for $\rho=1$, gives

$$
\begin{aligned}
&\langle A_\alpha x, x\rangle_{\langle X_{-1}, X_1\rangle} - \frac{1}{2}\sum_{k=1}^m \langle S_\alpha^k x, S_\alpha^k x\rangle_{0,\rho} \\
&\quad \geqslant \frac{\lambda}{2}\|x\|_{1,\rho}^2 - K\|x\|_{0,\rho}^2 \\
&\qquad - \|x\|_{0,\rho}^2 \left[\|a\|_\infty \left(\left\|\frac{\partial^2\rho}{\rho}\right\|_\infty + 2\left\|\frac{\partial\rho}{\rho}\right\|_\infty^2\right)\right. \\
&\qquad \left. + (\|\partial a\|_\infty + \|b\|_\infty)\left\|\frac{\partial\rho}{\rho}\right\|_\infty\right] - \|x\|_{0,\rho}\|x\|_{1,\rho}\|a\|_\infty \left\|\frac{\partial\rho}{\rho}\right\|_\infty \\
&\qquad - \frac{1}{2}\|x\|_{0,\rho}\|d\|_\infty \left\|\frac{\partial\rho}{\rho}\right\|_\infty \\
&\qquad \times \left[(2\|d\|_\infty \|x\|_{1,\rho} + 2\|e\|_\infty\|x\|_{0,\rho}) + \|x\|_{0,\rho}\|d\|_\infty \left\|\frac{\partial\rho}{\rho}\right\|_\infty\right] \\
&\quad \geqslant \frac{\lambda}{2}\|x\|_{1,\rho}^2 - K\|x\|_{0,\rho}^2 - C_1\|x\|_{0,\rho}^2 - C_2\|x\|_{0,\rho}\|x\|_{1,\rho}
\end{aligned}
$$

where in the last inequality we used Hypothesis 3.1(ii) and Proposition 2.1(i). It is now enough to find $M>0$ such that

$$
C_2\|x\|_{0,\rho}\|x\|_{1,\rho} \leqslant \frac{\lambda}{4}\|x\|_{1,\rho}^2 + M\|x\|_{0,\rho}^2.
$$

Step 2: Proof of (13).

We have

$$
\begin{aligned}
&\langle A_\alpha x, B_\rho x\rangle_{\langle X_{-2}, X_2\rangle} - \tfrac{1}{2}\sum_{k=1}^m \langle B_\rho S_\alpha^k x, S_\alpha^k x\rangle_{\langle X_1, X_{-1}\rangle} \\
&\quad = \langle C_\rho A_\alpha x, C_\rho B_\rho x\rangle_{\langle H^{-2}, H^2\rangle} - \tfrac{1}{2}\sum_{k=1}^m \langle C_\rho B_\rho S_\alpha^k x, C_\rho S_\alpha^k x\rangle_{\langle H^1, H^{-1}\rangle} \\
&\quad = \langle C_\rho A_\alpha x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} - \tfrac{1}{2}\sum_{k=1}^m \langle BC_\rho S_\alpha^k x, C_\rho S_\alpha^k x\rangle_{\langle H^1, H^{-1}\rangle} \\
&\quad = \langle A_\alpha C_\rho x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} - \tfrac{1}{2}\sum_{k=1}^m \langle BS_\alpha^k C_\rho x, S_\alpha^k C_\rho x\rangle_{\langle H^1, H^{-1}\rangle} \\
&\qquad - \langle [A_\alpha, C_\rho]\, x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} \\
&\qquad + \tfrac{1}{2}\sum_{k=1}^m \langle B[S_\alpha^k, C_\rho]\, x, 2S_\alpha^k C_\rho x - [S_\alpha^k, C_\rho]\, x\rangle_{\langle H^1, H^{-1}\rangle}.
\end{aligned}
\tag{17}
$$

We now observe that

$$\begin{aligned}\langle [A_\alpha, C_\rho]\, x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} &= \langle f_1(\cdot)\, C_\rho x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} \\ &\quad + \left\langle \sum_{h=1}^{d} f_{2,h}(\cdot)\, \partial_h[C_\rho x], BC_\rho x\right\rangle_{\langle H^{-2}, H^2\rangle} \\ &= \langle f_1(\cdot)\, C_\rho x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle} \\ &\quad + \left\langle \sum_{h=1}^{d} \partial_h[f_{2,h}(\cdot)\, C_\rho x], BC_\rho x\right\rangle_{\langle H^{-2}, H^2\rangle} \\ &\quad - \left\langle \sum_{h=1}^{d} [\partial_h f_{2,h}(\cdot)]\, C_\rho x, BC_\rho x\right\rangle_{\langle H^{-2}, H^2\rangle}\end{aligned}$$

so that, noticing that

$$\begin{aligned}&\left|\left\langle \sum_{h=1}^{d} \partial_h[f_{2,h}(\cdot)\, C_\rho x], BC_\rho x\right\rangle_{\langle H^{-2}, H^2\rangle}\right| \\ &\qquad \leqslant \left\| B^{1/2}\left(\sum_{h=1}^{d} \partial_h[f_{2,h}(\cdot)\, C_\rho x]\right)\right\|_0 \|B^{1/2}C_\rho x\|_0 \\ &\qquad \leqslant \sum_{h=1}^{d} \|f_{2,h}(\cdot)\, C_\rho x\|_0\, \|C_\rho x\|_{-1},\end{aligned}$$

we obtain

$$\begin{aligned}&|-\langle [A_\alpha, C_\rho]\, x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle}| \\ &\qquad \leqslant [\,\|f_1\|_\infty + \|\partial f_2\|_\infty]\, \|x\|_{0,\rho}\, \|x\|_{-2,\rho} + \|f_2\|_\infty\, \|x\|_{0,\rho}\, \|x\|_{-1,\rho}\end{aligned}$$

which, by Lemma 3.2, implies that there exists $C>0$ such that

$$|-\langle [A_\alpha, C_\rho]\, x, BC_\rho x\rangle_{\langle H^{-2}, H^2\rangle}| \leqslant \frac{\lambda}{8}\, \|x\|_{0,\rho}^2 + C\, \|x\|_{-1,\rho}^2, \qquad \forall x \in X_0. \tag{18}$$

On the other hand

$$\begin{aligned}&\sum_{k=1}^{m} \langle B[S_\alpha^k, C_\rho]\, x, 2S_\alpha^k C_\rho x - [S_\alpha^k, C_\rho]\, x\rangle_{\langle H^1, H^{-1}\rangle} \\ &\qquad = 2\sum_{k=1}^{m} \langle B[f_{3,k}(\cdot)\, C_\rho x], S_\alpha^k C_\rho x\rangle_{\langle H^1, H^{-1}\rangle} \\ &\qquad\quad - \sum_{k=1}^{m} \langle B[f_{3,k}(\cdot)\, C_\rho x], f_{3,k}(\cdot)\, C_\rho x\rangle_{\langle H^1, H^{-1}\rangle}\end{aligned}$$

so that

$$\left| \sum_{k=1}^{m} \langle B[S_\alpha^k, C_\rho]\, x, 2S_\alpha^k C_\rho x - [S_\alpha^k, C_\rho]\, x \rangle_{\langle H^1, H^{-1}\rangle} \right|$$

$$\leqslant \sum_{k=1}^{m} [\, \|B^{1/2}[f_{3,k}(\cdot)\, C_\rho x]\|_0 \, \|B^{1/2} S_\alpha^k C_\rho x\|_0 + \|B^{1/2}[f_{3,k}(\cdot)\, C_\rho x]\|_0^2]$$

$$\leqslant \sum_{k=1}^{m} [\, \|f_{3,k}(\cdot)\|_\infty \, \|x\|_{-1,\rho} \, \|B^{1/2} S_\alpha^k\|_{\mathscr{L}(X_0)} \, \|x\|_{0,\rho}$$

$$+ \|f_{3,k}(\cdot)\|_\infty^2 \, \|x\|_{-1,\rho}^2]$$

which, again by Lemma 3.2, yields

$$\left| \sum_{k=1}^{m} \langle B[S_\alpha^k, C_\rho]\, x, 2S_\alpha^k C_\rho x - [S_\alpha^k, C_\rho]\, x \rangle_{\langle H^1, H^{-1}\rangle} \right|$$

$$\leqslant \frac{\lambda}{8} \|x\|_{0,\rho}^2 + C\, \|x\|_{-1,\rho}^2, \qquad \forall x \in X_0. \tag{19}$$

Therefore, using (13) for $\rho = 1$ and putting (18) and (19) into (17), we obtain

$$\langle A_\alpha x, B_\rho x \rangle_{\langle X_{-2}, X_2\rangle} - \frac{1}{2} \sum_{k=1}^{m} \langle B_\rho S_\alpha^k x, S_\alpha^k x \rangle_{\langle X_1, X_{-1}\rangle}$$

$$\geqslant \frac{1}{2} \|x\|_{0,\rho}^2 - K\, \|x\|_{-1,\rho}^2 - \frac{\lambda}{4} \|x\|_{0,\rho}^2 - 2C\, \|x\|_{-1,\rho}^2$$

which gives the claim. ▮

3.2. *Estimates for the Duncan–Mortensen–Zakai Equation*

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geqslant 0\}, \mathbb{P}, W)$ be a stochastic base as introduced in Section 2.1. We fix a finite time horizon $T > 0$ and define for $0 \leqslant t \leqslant \tau \leqslant T$ the set

$$\mathscr{A}_{t,\tau} = \{\alpha : [t, \tau] \times \Omega \mapsto \mathbf{A};\ \text{measurable and progressively measurable with respect to } \{\mathscr{F}_s : s \geqslant 0\}\}. \tag{20}$$

$\mathscr{A}_{t,\tau}$ is the set of admissible control strategies on $[t, \tau]$ when the stochastic base is fixed. This will be the case for the rest of this section. In the control problem in Section 4 we will use relaxed controls allowing the stochastic base to change. Therefore we will pay attention to the fact that our estimates do not depend on the choice of the stochastic base.

For $0 < t < T$, $\alpha \in \mathscr{A}_{t,T}$, and $x \in L^2(\Omega, X_0)$, we consider the following DMZ-type equation in X_0:

$$dY(s) = -A_{\alpha(s)} Y(s)\, ds + \sum_{k=1}^{m} S^k_{\alpha(s)} Y(s)\, dW_k(s),$$
$$t < s < T, \qquad Y(t) = x. \tag{21}$$

We first recall the definition of solution for the above equation (see e.g. [9, Chapter 5, 22, 28])

Definition 3.4. (i) A weak solution of the Eq. (21) is a progressively measurable process $Y \in M^2([t, T]; X_1)$ such that, for every $h \in X_1$ we have

$$\langle Y(s), h \rangle_{0,\rho} = \langle x, h \rangle_{0,\rho} - \int_t^s \langle A_{\alpha(r)} Y(r), h \rangle_{\langle X_{-1}, X_1 \rangle}\, dr$$
$$+ \sum_{k=1}^{m} \int_t^s \langle S^k_{\alpha(r)} Y(r), h \rangle_{0,\rho}\, dW_k(r), \qquad \forall s \in [t, T],\ \mathbb{P} \text{ a.s.} \tag{22}$$

(ii) A strong solution of the Eq. (21) is a progressively measurable process $Y \in M^2([t, T]; X_2)$ such that, (21) is satisfied for almost every $s \in [t, T]$, $\mathbb{P}$ a.s.

Given $0 \leqslant t \leqslant T$, an admissible control strategy $\alpha \in \mathscr{A}_{t,T}$, and an initial datum $x \in L^2(\Omega; X_0)$, a weak (or strong) solution of the state Eq. (21) will be denoted by $Y(\cdot\,; t, x, \alpha)$. We will often denote it simply by $Y(\cdot)$ when there is no possibility of confusion.

Remark 3.5. Recalling the well known equivalence between the notion of adapted and progressively measurable processes on filtered probability spaces (see e.g. [25, p. 68] or [32, p. 17]), we might have asked adapted instead of progressively measurable in the above definition. ∎

The following proposition gathers some results and estimates about solutions of Eq. (21). Many of them are well known (see for instance [22, 28, 36]).

Proposition 3.6. *Assume that Hypothesis* 3.1 *holds.*

(i) *Given* $0 \leqslant t \leqslant T$, *an admissible control strategy* $\alpha \in \mathscr{A}_{t,T}$ *and an initial datum* $x \in L^2(\Omega; X_0)$ *there exists a unique weak solution* $Y(\cdot\,; t, x, \alpha)$ *of the state equation* (21) *satisfying*:

- $Y \in L^2(\Omega; C([t, T], X_0)) \cap M^2([t, T]; X_1)$.
- *For every* $s \in (t, T)$

$$\|Y(s)\|_{0,\rho}^2 = \|x\|_{0,\rho}^2 - 2\int_t^s \langle A_{\alpha(r)} Y(r), Y(r)\rangle_{\langle X_{-1}, X_1\rangle}\, dr + \sum_{k=1}^m \int_t^s \langle S^k_{\alpha(r)} Y(r), Y(r)\rangle_{0,\rho}\, dW_k(r) + \sum_{k=1}^m \int_t^s \langle S^k_{\alpha(r)} Y(r), S^k_{\alpha(r)} Y(r)\rangle_{0,\rho}\, dr, \tag{23}$$

and, in particular

$$\mathbb{E}\,\|Y(s)\|_{0,\rho}^2 = \mathbb{E}\,\|x\|_{0,\rho}^2 - 2\mathbb{E}\int_t^s \langle A_{\alpha(r)} Y(r), Y(r)\rangle_{\langle X_{-1}, X_1\rangle}\, dr + \sum_{k=1}^m \mathbb{E}\int_t^s \langle S^k_{\alpha(r)} Y(r), S^k_{\alpha(r)} Y(r)\rangle_{0,\rho}\, dr. \tag{24}$$

- *The solution* Y *belongs to* $C([t, T], L^2(\Omega; X_0)) \cap L^2([t, T]; L^2(\Omega; X_1))$ *and, more precisely, for a suitable constant* $C > 0$ *independent of the strategy* $\alpha \in \mathcal{A}_{t,T}$

$$\mathbb{E}\,\|Y(s)\|_{0,\rho}^2 \leqslant \mathbb{E}\,\|x\|_{0,\rho}^2 (1 + C(s-t)), \tag{25}$$

$$\mathbb{E}\int_t^T \|Y(s)\|_{1,\rho}^2\, ds \leqslant C\mathbb{E}\,\|x\|_{0,\rho}^2. \tag{26}$$

(ii) *For every initial condition* $x \in L^2(\Omega, X_0)$ *we have, for a suitable constant* $C > 0$ *independent of the strategy* $\alpha \in \mathcal{A}_{t,T}$

$$\mathbb{E}\,\|Y(s)\|_{-1,\rho}^2 \leqslant \mathbb{E}\,\|x\|_{-1,\rho}^2 (1 + C(s-t)), \tag{27}$$

$$\mathbb{E}\int_t^T \|Y(s)\|_{0,\rho}^2\, ds \leqslant C\mathbb{E}\,\|x\|_{-1,\rho}^2. \tag{28}$$

(iii) *For every initial condition* $x \in L^2(\Omega, X_0)$ *we have, for a suitable constant* $C > 0$ *independent of the strategy* $\alpha \in \mathcal{A}_{t,T}$

$$\mathbb{E}\,\|Y(s) - x\|_{-1,\rho}^2 \leqslant C(s-t)\,\mathbb{E}\,\|x\|_{0,\rho}^2, \tag{29}$$

$$\mathbb{E}\int_t^s \|Y(r) - x\|_{0,\rho}^2\, dr \leqslant C(s-t)\,\mathbb{E}\,\|x\|_{0,\rho}^2, \tag{30}$$

and there is a modulus σ, independent of the strategy $\alpha \in \mathscr{A}_{t,T}$ such that

$$\mathbb{E}\,\| Y(s) - x\|^2_{0,\rho} \leqslant \sigma(s-t). \tag{31}$$

(iv) *If $\mathbb{E}\,\|x\|^2_{1,\rho} < \infty$ then Y is a strong solution and for a suitable constant $C > 0$ independent of the strategy $\alpha \in \mathscr{A}_{t,T}$*

$$\mathbb{E}\,\| Y(s)\|^2_{1,\rho} \leqslant \mathbb{E}\,\|x\|^2_{1,\rho}\,(1 + C(s-t)), \tag{32}$$

$$\mathbb{E} \int_t^T \| Y(s)\|^2_{2,\rho}\, ds \leqslant C\mathbb{E}\,\|x\|^2_{1,\rho}, \tag{33}$$

$$\mathbb{E}\,\| Y(s) - x\|^2_{0,\rho} \leqslant C\mathbb{E}\,\|x\|^2_{1,\rho}\,(s-t), \tag{34}$$

$$\mathbb{E} \int_t^s \| Y(r) - x\|^2_{1,\rho}\, dr \leqslant C(s-t)\,\mathbb{E}\,\|x\|^2_{1,\rho}, \tag{35}$$

and finally

$$\mathbb{E}\,\| Y(s) - x\|^2_{1,\rho} \leqslant \sigma(s-t) \tag{36}$$

for some modulus σ independent of the strategy $\alpha \in \mathscr{A}_{t,T}$.

Proof. • Proof of (i). The results can be either found or easily deduced from [22] (see also [36] and [28] for equations without random coefficients).

• Proof of (ii). By the Ito Formula we have, similarly to (24),

$$\begin{aligned}\mathbb{E}\,\| Y(s)\|^2_{-1,\rho} &= \mathbb{E}\,\|B_\rho^{1/2}\, Y(s)\|^2_{0,\rho} \\ &= \mathbb{E}\,\|B_\rho^{1/2} x\|^2_{0,\rho} - 2\mathbb{E} \int_t^s \langle B_\rho^{1/2} A_{\alpha(r)}\, Y(r), B_\rho^{1/2}\, Y(r)\rangle_{0,\rho}\, dr \\ &\quad + \sum_{k=1}^m \mathbb{E} \int_t^s \langle B_\rho^{1/2} S^k_{\alpha(r)}\, Y(r), B_\rho^{1/2} S^k_{\alpha(r)}\, Y(r)\rangle_{0,\rho}\, dr.\end{aligned} \tag{37}$$

Since

$$\langle B_\rho^{1/2} A_{\alpha(r)}\, Y(r), B_\rho^{1/2}\, Y(r)\rangle_{0,\rho} = \langle A_{\alpha(r)}\, Y(r), B_\rho\, Y(r)\rangle_{\langle X_{-2}, X_2\rangle}$$

and

$$\langle B_\rho^{1/2} S^k_{\alpha(r)}\, Y(r), B_{\alpha(r)}^{1/2} S^k_{\alpha(r)}\, Y(s)\rangle_{0,\rho} = \langle B_\rho S^k_{\alpha(r)}\, Y(r), S^k_{\alpha(r)}\, Y(r)\rangle_{\langle X_1, X_{-1}\rangle}$$

we have, thanks to (13),

$$\mathbb{E}\,\|Y(s)\|_{-1,\rho}^2+\frac{\lambda}{2}\int_t^s\|Y(r)\|_{0,\rho}^2\,dr\leqslant\mathbb{E}\,\|x\|_{-1,\rho}^2+2K\int_t^s\|Y(s)\|_{-1,\rho}^2\,dr.$$

Estimates (27)–(28) follow by applying the Gronwall inequality.

- Proof of (iii). For every initial condition $x\in L^2(\Omega, X_0)$ we have

$$\mathbb{E}\,\|Y(s)-x\|_{-1,\rho}^2=\mathbb{E}\,\|Y(s)\|_{-1,\rho}^2+\mathbb{E}\,\|x\|_{-1,\rho}^2-2\mathbb{E}\langle Y(s), B_\rho x\rangle_{0,\rho}$$

which gives, by (37) and the definition of weak solution,

$$\begin{aligned}\mathbb{E}\,\|Y(s)-x\|_{-1,\rho}^2 &= -2\mathbb{E}\int_t^s[\langle A_{\alpha(r)}Y(r), B_\rho Y(r)\rangle_{\langle X_{-1},X_1\rangle}\\ &\quad-\tfrac{1}{2}\langle B_\rho S_{\alpha(r)}^k Y(r), S_{\alpha(r)}^k Y(r)\rangle_{0,\rho}]\,dr\\ &\quad-2\mathbb{E}\int_t^s\langle B_\rho A_{\alpha(r)}Y(r), x\rangle_{0,\rho}.\end{aligned}\tag{38}$$

Therefore, by (13),

$$\begin{aligned}&\mathbb{E}\,\|Y(s)-x\|_{-1,\rho}^2+\frac{\lambda}{2}\int_t^s\|Y(r)\|_{0,\rho}^2\,dr\\ &\quad\leqslant 2K\mathbb{E}\int_t^s\|Y(r)\|_{-1,\rho}^2\,dr+2\mathbb{E}\int_t^s\|x\|_{0,\rho}\,\|B_\rho A_{\alpha(r)}Y(r)\|_{0,\rho}\,dr\end{aligned}$$

Now (see Lemma 3.2(iii)) for a suitable constant $C>0$ we have $\|B_\rho A_{\alpha(r)}Y(r)\|_{0,\rho}\leqslant C\,\|Y(r)\|_{0,\rho}$ so that, by (27) and straightforward calculations,

$$\mathbb{E}\,\|Y(s)-x\|_{-1,\rho}^2+\frac{\lambda}{4}\int_t^s\|Y(r)\|_{0,\rho}^2\,dr\leqslant C(s-t)[\mathbb{E}\,\|x\|_{-1,\rho}^2+\mathbb{E}\,\|x\|_{0,\rho}^2]$$

for some constant $C>0$. This proves (29). Estimate (30) follows upon noticing that

$$\begin{aligned}\mathbb{E}\int_t^s\|Y(r)-x\|_{0,\rho}^2\,dr &\leqslant 2\mathbb{E}\int_t^s\|Y(r)\|_{0,\rho}^2\,dr+2\mathbb{E}\int_t^s\|x\|_{0,\rho}^2\\ &\leqslant C(s-t)\,\mathbb{E}\,\|x\|_{0,\rho}^2.\end{aligned}$$

To prove (31), fix $x \in L^2(\Omega, X_0)$. Since $\mathbb{E}\,\|Y(s)\|^2_{0,\rho}$ is bounded (in s and α), there exists a sequence $s_n \to t$ for $n \to +\infty$ and an element $\bar{Y}$ of $L^2(\Omega, X_0)$ such that, as $n \to +\infty$

$$Y(s_n) \rightharpoonup \bar{Y}, \qquad \text{weakly in } L^2(\Omega, X_0),$$

and hence also weakly in $L^2(\Omega, X_{-1})$. Since by (29) we know that as $n \to +\infty$

$$Y(s_n) \to x, \qquad \text{strongly in } L^2(\Omega, X_{-1})$$

(and uniformly in α), we obtain that $\bar{Y} = x$. So every weakly convergent sequence $\{Y(s_n)\}_{s_n > 0}$ for $s_n \to t$ converges to x. This fact, plus the fact that $\mathbb{E}\,\|Y(s)\|^2_{0,\rho} \to \mathbb{E}\,\|x\|^2_{0,\rho}$ for $s \to t$ provided by (25) implies that $Y(s) \to x$, strongly in $L^2(\Omega, X_0)$ which gives the claim.

- Proof of (iv). Let $\mathbb{E}\,\|x\|^2_{1,\rho} < \infty$. The existence of the strong solution is known (see [22]). By the Ito Formula we have

$$\mathbb{E}\,\|Y(s)\|^2_{1,\rho} = \mathbb{E}\,\|x\|^2_{1,\rho} - 2\mathbb{E} \int_t^s \langle A_{\alpha(r)} Y(r), Y(r) \rangle_{\langle X_0, X_2 \rangle}\, dr$$

$$+ \sum_{k=1}^m \mathbb{E} \int_t^s \langle S^k_{\alpha(r)} Y(r), S^k_{\alpha(r)} Y(r) \rangle_{1,\rho}\, dr,$$

i.e.

$$\mathbb{E}\,\|Y(s)\|^2_{1,\rho} = \mathbb{E}\,\|x\|^2_{1,\rho} - 2\mathbb{E} \int_t^s \langle A_{\alpha(r)} Y(r), B_\rho^{-1} Y(r) \rangle_0\, dr$$

$$+ \sum_{k=1}^m \mathbb{E} \int_t^s \langle B_\rho^{-1} S^k_{\alpha(r)} Y(r), S^k_{\alpha(r)} Y(r) \rangle_0\, dr,$$

which, upon using (12) and applying the same arguments as those in the proof of (ii), proves (32) and (33). The proof of the final three estimates is analogous to the similar ones proved in (iii) and is omitted. ∎

4. OPTIMAL CONTROL PROBLEM FOR THE DUNCAN–MORTENSEN–ZAKAI EQUATION

We consider the following abstract optimal control problem. Given $0 \leqslant t \leqslant T < \infty$ we denote by $\bar{\mathcal{A}}_{t,T}$ the set of admissible (relaxed) controls (see e.g. [32, 35]). The set consists of:

- probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$,
- m-dimensional Brownian motions W, on $[t, T]$.
- measurable processes $\alpha: [t, T] \times \Omega \mapsto \mathbf{A}$ that are $\mathcal{F}_t^W$-adapted where $\mathcal{F}_t^W$ is the filtration generated by W.

We will use the notation $(\Omega, \mathcal{F}, \mathbb{P}, W, \alpha) \in \bar{\mathcal{A}}_{t,T}$. When no ambiguity arises we will leave aside the probability space (regarding it as fixed) and consider admissible controls simply as processes $\alpha \in \mathcal{A}_{t,T}$. From now on we will write $|\cdot|$ for $|\cdot|_{\mathbb{R}}$.

Let now $x \in X_0$, and $(\Omega, \mathcal{F}, \mathbb{P}, W, \alpha) \in \bar{\mathcal{A}}_{t,T}$. We try to minimize the cost functional:

$$J(t, x; \alpha(\cdot)) = \mathbb{E}\left\{\int_t^T f(Y(s; t, x, \alpha), \alpha(s))\, ds + g(Y(T; t, x, \alpha))\right\}, \tag{39}$$

where $Y(\cdot\,; t, x, \alpha)$ is the solution of (21). We assume

HYPOTHESIS 4.1. (i) *f and g are continuous and there exist $C > 0$ and $\gamma < 2$ such that*

$$|f(x, \alpha)|, |g(x)| \leqslant C(1 + \|x\|_{0,\rho}^{\gamma})$$

for every $(x, \alpha) \in X_0 \times \mathbf{A}$;

(ii) *for every $R > 0$ there exists a modulus ω_R such that*

$$\begin{aligned} |f(x, \alpha) - f(y, \alpha)| &\leqslant \omega_R(\|x - y\|_{0,\rho}), \\ |g(x) - g(y)| &\leqslant \omega_R(\|x - y\|_{-1,\rho}) \end{aligned} \tag{40}$$

for every $x, y \in X_0$ such that $\|x\|_{0,\rho}, \|y\|_{0,\rho} \leqslant R$, $\alpha \in \mathbf{A}$.

We will refer to functions satisfying (i) as having less than quadratic growth.

Remark 4.2. We observe that the condition on g in (ii) is satisfied when g is weakly continuous. This will allow us to treat some nontrivial examples of partially observed optimal control problems in Section 7. ∎

The value function is defined as

$$v(t, x) = \inf_{\alpha(\cdot) \in \bar{\mathcal{A}}_{t,T}} J(t, x; \alpha(\cdot)). \tag{41}$$

The corresponding Hamilton–Jacobi–Bellman equation of dynamic programming that should be satisfied by v is:

$$\begin{cases} v_t + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \langle D^2 v S_\alpha^k x, S_\alpha^k x \rangle_{0,\rho} - \langle A_\alpha x, Dv \rangle_{\langle X_{-1}, X_1 \rangle} + f(x, \alpha) \right\} = 0 \\ \qquad \text{in} \quad (0, T) \times X_0, \\ v(T, x) = g(x) \qquad \text{in} \quad X_0. \end{cases} \tag{42}$$

We will use the following definition of solution of (42). It is similar to the one used in [14] that in turn goes back to [5] and [8, Part VII].

Definition 4.3. A function $u \in C((0, T) \times X_0)$ is a viscosity subsolution (respectively, supersolution) if for every function $\varphi \in C^{1,2}((0, T) \times X_{-1})$ and for every function $\delta \in C^1(0, T)$ such that $\delta > 0$ on $[0, T]$, whenever $u - (\varphi + (\delta/2) \|x\|_{0,\rho}^2)$ (respectively $u - (\varphi - (\delta/2) \|x\|_{0,\rho}^2)$) has a global maximum (respectively, minimum) at (t, x) then $x \in X_1$ and

$$0 \leqslant \varphi_t(t, x) + \frac{\delta'(t) \|x\|_{0,\rho}^2}{2} + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \langle (D^2 \varphi(t, x) + \delta(t) I) S_\alpha^k x, S_\alpha^k x \rangle_{0,\rho} \right.$$
$$\left. - \langle A_\alpha x, D\varphi(t, x) + \delta(t) x \rangle_{\langle X_{-1}, X_1 \rangle} + f(x, \alpha) \right\}.$$

(respectively,

$$0 \geqslant \varphi_t(t, x) - \frac{\delta'(t) \|x\|_{0,\rho}^2}{2} + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \langle (D^2 \varphi(t, x) - \delta(t) I) S_\alpha^k x, S_\alpha^k x \rangle_{0,\rho} \right.$$
$$\left. - \langle A_\alpha x, D\varphi(t, x) - \delta(t) x \rangle_{\langle X_{-1}, X_1 \rangle} + f(x, \alpha) \right\}. \Bigg)$$

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The main result of this paper states that the value function v is the unique viscosity solution of the HJB Eq. (42). This will be proved in Sections 5 and 6.

Remark 4.4. If u has less than quadratic growth uniformly for $t \in (0, T)$ and is uppersemicontinuous (respectively, lowersemicontinuous) in $|\cdot|_{\mathbb{R}} \times \|\cdot\|_{-1,\rho}$ norm on bounded subsets of $(0, T) \times X_0$ then the maximum (respectively, the minimum) in the definition of a subsolution (respectively, a supersolution) can be assumed to be strict. To see this, suppose that $u - (\varphi + (\delta/2) \|x\|_{0,\rho}^2)$ has a global maximum at $(\hat{t}, \hat{x})$. Let $\psi \in C^2([0, +\infty))$ be such that $\psi > 0$, $\psi(r) = r^4$ if $r \leqslant 1$ and $\psi(r) = 1$ if $r \geqslant 2$. Then $u - (\varphi + (\delta/2) \|x\|_{0,\rho}^2) - \psi(\|x - \hat{x}\|_{-1,\rho}) - (t - \hat{t})^2$ has a global maximum at $(\hat{t}, \hat{x})$ and for every maximizing sequence (t_n, x_n) we must have $\limsup_{n \to \infty} \|x_n\|_{0,\rho} \leqslant \|\hat{x}\|_{0,\rho}$, and $t_n \to \hat{t}$, $B_\rho x_n \to B_\rho \hat{x}$ as $n \to \infty$. This implies that $x_n \to \hat{x}$.

Remark 4.5. If u is as in the previous remark then for a test function $\varphi \in C^{1,2}((0,T)\times X_{-1})$ we can always assume that $\varphi(t,x)=0$ if $\|x\|_{-1,\rho}$ is sufficiently big and so we can assume that φ and all its derivatives are bounded. Therefore we can assume that

$$\begin{aligned}
&\varphi, \varphi_t \in BUC((0,T)\times X_0),\\
&D\varphi \in BUC((0,T)\times X_0, X_1), \qquad (43)\\
&D^2\varphi \in BUC((0,T)\times X_0, \mathscr{L}(X_{-1}; X_1)).
\end{aligned}$$

We also remark that Ito formula holds for the test functions. The class of test functions φ will be denoted by $\mathscr{T}$.

Remark 4.6. We have defined the family of admissible control strategies in a wider sense of relaxed controls including in the definition of an admissible control also the choice of the stochastic base. This approach is commonly used, in particular to prove the Dynamic Programming Principle and to analyze existence of optimal strategies and optimality conditions (see e.g. [26, 32]). Of course one can consider another setup of a fixed stochastic base, for instance a so called canonical sample space, and then define admissible control strategies as processes adapted to this base. This seems to be a more prevalent setting for partially observed stochastic optimal control problems (see [11, 15, 28]). As far as this paper is concerned the "fixed-space" approach would not change the results and methods of proofs as long as we can prove the Dynamic Programming Principle. We have chosen the relaxed setting since in this case the Dynamic Programming Principle that we strongly use here, is well known and easily quotable.

5. PROPERTIES OF THE VALUE FUNCTION AND THE EXISTENCE RESULT

PROPOSITION 5.1. *Assume that Hypotheses* 3.1 *and* 4.1 *hold true. Then for every* $R>0$ *there exists a modulus* σ_R *such that*

$$|v(t,x)-v(s,y)| \leqslant \sigma_R(|t-s| + \|x-y\|_{-1,\rho}) \qquad (44)$$

for $t, s\in[0,T]$ *and* $\|x\|_{0,\rho}, \|y\|_{0,\rho} \leqslant R$. *Moreover*

$$|v(t,x)| \leqslant C[1+\|x\|^{\gamma}_{0,\rho}] \qquad (45)$$

for a suitable $C>0$.

Proof. The last statement follows easily from Hypothesis 4.1 and (25). We prove the local uniform continuity. Since here no substantial difference arises, we will consider controls simply as elements of $\mathscr{A}_{t,T}$. Throughout the proof C will be a constant that can vary from time to time. By definition

$$v(t,x) = \inf_{\alpha(\cdot)\in\mathscr{A}_{t,T}} \mathbb{E}\left\{\int_t^T f(Y(s;t,x,\alpha),\alpha(s))\,ds + g(Y(T;t,x,\alpha))\right\}$$

so that, taking $\varepsilon > 0$ and $\alpha_{\varepsilon,y} \in \mathscr{A}_{t,T}$ such that $v(t,y) > J(t,y;\alpha_{\varepsilon,y}(\cdot)) - \varepsilon$, we have

$$\begin{aligned} v(t,x) - v(t,y) &\leqslant J(t,x;\alpha_{\varepsilon,y}(\cdot)) - J(t,y;\alpha_{\varepsilon,y}(\cdot)) + \varepsilon \\ &= \mathbb{E}\left\{\int_t^T [f(Y(s;t,x,\alpha_{\varepsilon,y}),\alpha_{\varepsilon,y}(s)) \right. \\ &\qquad \left. - f(Y(s;t,y,\alpha_{\varepsilon,y}),\alpha_{\varepsilon,y}(s))]\,ds\right\} \\ &\qquad + \mathbb{E}\{g(Y(T;t,x,\alpha_{\varepsilon,y})) - g(Y(T;t,y,\alpha_{\varepsilon,y}))\} + \varepsilon \end{aligned}$$

so that, given $R > \|x\|_{0,\rho} \vee \|y\|_{0,\rho}$ we get, writing $Y_x(s)$ for $Y(s;t,x,\alpha_{\varepsilon,y})$ and $Y_y(s)$ for $Y(s;t,y,\alpha_{\varepsilon,y})$,

$$\begin{aligned} v(t,x) - v(t,y) &\leqslant \mathbb{E}\int_t^T \omega_R(\|Y_x(s) - Y_y(s)\|_{0,\rho})\,ds \\ &\qquad + C\int_t^T \mathbb{P}(\|Y_x(s)\|_{0,\rho} \vee \|Y_y(s)\|_{0,\rho} \geqslant R)^{1-\gamma/2} \\ &\qquad \times (\mathbb{E}[1 + \|Y_x(s)\|_{0,\rho}^2 + \|Y_y(s)\|_{0,\rho}^2])^{\gamma/2}\,ds \\ &\qquad + \mathbb{E}\omega_R(\|Y_x(T) - Y_y(T)\|_{-1,\rho}) \\ &\qquad + C\mathbb{P}(\|Y_x(T)\|_{0,\rho} \vee \|Y_y(T)\|_{0,\rho} \geqslant R)^{1-\gamma/2} \\ &\qquad \times (\mathbb{E}[1 + \|Y_x(T)\|_{0,\rho}^2 + \|Y_y(T)\|_{0,\rho}^2])^{\gamma/2} + \varepsilon \end{aligned}$$

Now, by the linearity of the state equation we have that for every $s \in [t,T]$

$$Y(s;t,x,\alpha_{\varepsilon,y}) - Y(s;t,y,\alpha_{\varepsilon,y}) = Y(s;t,x-y,\alpha_{\varepsilon,y})$$

and thus, denoting $Y(s;t,x-y,\alpha_{\varepsilon,y})$ by $Y_{x-y}(s)$ we obtain (by (25) and (27)) that, for $j = -1, 0$,

$$\begin{aligned} \mathbb{E}\omega_R(\|Y_x(s) - Y_y(s)\|_{j,\rho}) &= \mathbb{E}\omega_R(\|Y_{x-y}(s)\|_{j,\rho}) \\ &\leqslant \varepsilon + C_\varepsilon \mathbb{E}\,\|Y_{x-y}(s)\|_{j,\rho} \\ &\leqslant \varepsilon + C_\varepsilon[\mathbb{E}\,\|Y_{s-y}(s)\|_{j,\rho}^2]^{1/2} \\ &\leqslant \varepsilon + C_\varepsilon[\|x-y\|_{j,\rho}^2\,(1 + C(s-t))]^{1/2}. \end{aligned}$$

Moreover by the Chebychev inequality and (25) we get that for sufficiently big R

$$\sup_{s\in[t,T]} \mathbb{P}(\|Y_x(s)\|_{0,\rho} \vee \|Y_y(s)\|_{0,\rho} \geqslant R) \leqslant \frac{C(\|x\|_{0,\rho}^2 + \|y\|_{0,\rho}^2)}{R^2} < \varepsilon$$

which upon using (28) gives

$$\begin{aligned}
&\int_t^T \mathbb{P}(\|Y_x(s)\|_{0,\rho} \vee \|Y_y(s)\|_{0,\rho} \geqslant R)^{1-\gamma/2} \\
&\qquad \times (\mathbb{E}[1 + \|Y_x(s)\|_{0,\rho}^2 + \|Y_y(s)\|_{0,\rho}^2])^{\gamma/2}\, ds \\
&\quad \leqslant C\left[\int_t^T \mathbb{P}(\|Y_x(s)\|_{0,\rho} \vee \|Y_y(s)\|_{0,\rho} \geqslant R)\right]^{1-\gamma/2} \\
&\qquad \times \left[\int_t^T \mathbb{E}[1 + \|Y_x(s)\|_{0,\rho}^2 + \|Y_y(s)\|_{0,\rho}^2]\, ds\right]^{\gamma/2} \\
&\quad \leqslant C\varepsilon^{1-\gamma/2}([1 + \|x\|_{-1,\rho}^2 + \|y\|_{-1,\rho}^2](1 + C(T-t)))^{\gamma/2},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{P}(\|Y_x(T)\|_{0,\rho} \vee \|Y_y(T)\|_{0,\rho} \geqslant R)^{1-\gamma/2} \\
&\qquad \times (\mathbb{E}[1 + \|Y_x(T)\|_{0,\rho}^2 + \|Y_y(T)\|_{0,\rho}^2])^{\gamma/2} \\
&\quad \leqslant \varepsilon^{1-\gamma/2}([1 + \|x\|_{0,\rho}^2 + \|y\|_{0,\rho}^2](1 + C(T-t)))^{\gamma/2}
\end{aligned}$$

so that

$$\begin{aligned}
v(t,x) - v(t,y) \leqslant &\int_t^T \varepsilon + C_\varepsilon[\mathbb{E}\,\|Y_{x-y}(s)\|_{0,\rho}^2]^{1/2}\, ds \\
&+ C\varepsilon^{1-\gamma/2}([1 + \|x\|_{-1,\rho}^2 + \|y\|_{-1,\rho}^2](1 + C(T-t)))^{\gamma/2} \\
&+ \varepsilon + C_\varepsilon[\|x-y\|_{-1,\rho}^2 (1 + C(T-t))]^{1/2} \\
&\times \varepsilon^{1-\gamma/2}([1 + \|x\|_{0,\rho}^2 + \|y\|_{0,\rho}^2](1 + C(T-t)))^{\gamma/2} + \varepsilon.
\end{aligned}$$

Denoting by $\omega_0(\varepsilon)$ a quantity that goes to 0 as ε goes to 0, uniformly for x, y in bounded subsets of X_0, and using (28) we obtain

$$\begin{aligned}
&v(t,x) - v(t,y) \\
&\quad \leqslant \omega_0(\varepsilon) + C_\varepsilon(T-t)^{1/2}\left[\mathbb{E}\int_t^T \|Y(s;t,x-y,\alpha_{\varepsilon,y})\|_{0,\rho}^2\, ds\right]^{1/2} \\
&\qquad + C_\varepsilon(1 + C(T-t))^{1/2}\, \|x-y\|_{-1,\rho} \\
&\quad \leqslant \omega_0(\varepsilon) + C_\varepsilon((T-t)^{1/2}\, C^{1/2} + (1 + C(T-t))^{1/2})\, \|x-y\|_{-1,\rho}
\end{aligned}$$

which yields the required continuity in x.

We now prove the continuity in the time variable. Take $0 \leqslant t_1 \leqslant t_2$, $\varepsilon > 0$, and a control strategy $\alpha_{\varepsilon, t_2} \in \mathscr{A}_{t_2, T}$ such that $v(t_2, x) > J(t_2, x; \alpha_{\varepsilon, t_2}(\cdot)) - \varepsilon$. Set for a given $\bar{\alpha}_0 \in \mathbf{A}$

$$\bar{\alpha}_{\varepsilon, t_2}(s) = \begin{cases} \bar{\alpha}_0 & \text{for} \quad s \in [t_1, t_2) \\ \alpha_{\varepsilon, t_2}(s) & \text{for} \quad s \in [t_2, T] \end{cases}$$

It clearly belongs to $\mathscr{A}_{t_1, T}$ and

$$\begin{aligned} v(t_1, x) - v(t_2, x) \leqslant{} & \varepsilon + \mathbb{E}\{ g(Y(T; t_1, x, \bar{\alpha}_{\varepsilon, t_2})) - g(Y(T; t_2, x, \alpha_{\varepsilon, t_2}))\} \\ & + \mathbb{E}\left\{ \int_{t_1}^{T} f(Y(s; t_1, x, \bar{\alpha}_{\varepsilon, t_2}), \bar{\alpha}_{\varepsilon, t_2}(s))\, ds \right. \\ & \left. - \int_{t_2}^{T} f(Y(s; t_2, x, \alpha_{\varepsilon, t_2}), \alpha_{\varepsilon, t_2}(s))\, ds \right\} \\ ={} & \mathbb{E}\left\{ \int_{t_1}^{t_2} f(Y(s; t_1, x, \bar{\alpha}_0), \bar{\alpha}_0)\, ds \right. \\ & + \int_{t_2}^{T} [f(Y(s; t_1, x, \bar{\alpha}_{\varepsilon, t_2}), \alpha_{\varepsilon, t_2}(s)) \\ & \left. - f(Y(s; t_2, x, \alpha_{\varepsilon, t_2}), \alpha_{\varepsilon, t_2}(s))]\, ds \right\} \\ & + \mathbb{E}\{ g(Y(T; t_1, x, \bar{\alpha}_{\varepsilon, t_2})) - g(Y(T; t_2, x, \alpha_{\varepsilon, t_2}))\} + \varepsilon \end{aligned}$$

so that, writing $Y_1(s)$ for $Y(s; t_1, x, \bar{\alpha}_{\varepsilon, t_2})$ and $Y_2(s)$ for $Y(s; t_2, x, \alpha_{\varepsilon, t_2})$,

$$\begin{aligned} v(t_1, x) - v(t_2, x) \leqslant{} & \varepsilon + \int_{t_1}^{t_2} \mathbb{E}\, |f(Y(s; t_1, x, \bar{\alpha}_0), \bar{\alpha}_0)|\, ds \\ & + \mathbb{E} \int_{t_2}^{T} \omega_R(\| Y_1(s) - Y_2(s)\|_{0, \rho})\, ds \\ & + C \int_{t}^{T} \mathbb{P}(\| Y_1(s)\|_{0, \rho} \vee \| Y_2(s)\|_{0, \rho} \geqslant R)^{1 - \gamma/2} \\ & \times (\mathbb{E}[1 + \| Y_1(s)\|_{0, \rho}^2 + \| Y_2(s)\|_{0, \rho}^2])^{\gamma/2}\, ds \\ & + \mathbb{E}\omega_R(\| Y_1(T) - Y_2(T)\|_{-1, \rho}) \\ & + C\mathbb{P}(\| Y_1(T)\|_{0, \rho} \vee \| Y_2(T)\|_{0, \rho} \geqslant R)^{1 - \gamma/2} \\ & \times (\mathbb{E}[1 + \| Y_1(T)\|_{0, \rho}^2 + \| Y_2(T)\|_{0, \rho}^2])^{\gamma/2}. \end{aligned} \tag{46}$$

Now arguing as previously we estimate the terms in the second and fourth lines of the right hand side, while for the others we observe that

$$\int_{t_1}^{t_2} \mathbb{E}\,|f(Y(s;t_1,x,\bar{\alpha}_0),\bar{\alpha}_0)|\,ds$$
$$\leqslant C(t_2-t_1)[1+\sup_{s\in[t_1,t_2]} \mathbb{E}\,\|Y(s;t_1,x,\bar{\alpha}_0)\|_{0,\rho}^{\gamma}]$$

and, since

$$Y_1(s)-Y_2(s)=Y(s;t_2,Y(t_2;t_1,x,\bar{\alpha}_0)),$$

$$\mathbb{E}\int_{t_2}^{T}\|Y_1(s)-Y_2(s)\|_{0,\rho}^2\,ds$$
$$+\mathbb{E}\,\|Y_1(T)-Y_2(T)\|_{-1,\rho}^2 \leqslant C(t_2-t_1)\,\|x\|_{0,\rho}^2.$$

The rest of the proof follows the arguments used in the proof of the continuity in x. One obtains that there is a modulus σ_R such that

$$v(t_1,x)-v(t_2,x)\leqslant\sigma_R(t_2-t_1).$$

for $\|x\|_{0,\rho}\leqslant R$. The reverse inequality can be obtained by the same method. ∎

Remark 5.2. We observe that assuming instead of (40) that

$$|f(x,\alpha)-f(y,\alpha)|\leqslant\omega_R(\|x-y\|_{1,\rho}),\qquad |g(x)-g(y)|\leqslant\omega_R(\|x-y\|_{0,\rho})$$

we would get (by similar arguments) the uniform continuity of v in x in the X_0 norm. Moreover if ω_R does not depend on R the proof provides the uniform continuity on the whole space. ∎

We will need the dynamic programming principle that is stated below in a simple form. It will not be reproved here even though there seems to be no quotable reference for infinite dimensional problems. However since the value function is continuous and we deal with relaxed controls the proof follows standard arguments, see for instance [32] (see also [21]).

PROPOSITION 5.3. *For every $0\leqslant t\leqslant\tau\leqslant T$ and $x\in X_0$ we have*

$$v(t,x)=\inf_{\alpha(\cdot)\in\bar{\mathscr{A}}_{t,T}}\mathbb{E}\left\{\int_t^T f(Y(s;t,x,\alpha),\alpha(s))\,ds+g(Y(T;t,x,\alpha))\right\}$$
$$=\inf_{\alpha(\cdot)\in\bar{\mathscr{A}}_{t,\tau}}\mathbb{E}\left\{\int_t^{\tau} f(Y(s;t,x,\alpha),\alpha(s))\,ds+v(\tau,Y(\tau;t,x,\alpha))\right\}.$$

THEOREM 5.4. *Assume that Hypotheses* 3.1 *and* 4.1 *are true. Then the value function* v *is the unique viscosity solution of the HJB Eq.* (42) *that satisfies* (44) *and* (45).

Proof. The uniqueness part will follow from Theorem 6.1. Here we will only prove that v is a viscosity solution. The main difficulty of the proof comes from the fact that we have to deal with the unbounded operators S_α^k and A_α. The outline of the proof is the following. First of all we show that the maximum (minimum) points in the definition of sub-(super) solution are in X_1. This part follows the strategy used in [14] and earlier in [5] and [8, Part VII]. Then we use the dynamic programming principle and carefully apply various estimates for solutions of the state equation to pass to the limit and obtain the inequalities in Definition 4.3. We will only show that the value function is a viscosity supersolution. The subsolution part is very similar and in fact easier. We will omit the subscript ρ in the norm and inner product notation throughout the proof. Since here no substantial difference arises, we will consider controls simply as elements of $\mathscr{A}_{t,T}$.

Let $\varphi \in \mathscr{T}$ and $\delta \in C^1(0, T)$ be such that $\delta > 0$ on $[0, T]$. Let $v - (\varphi - (\delta/2)\,\|x\|_0^2)$ have a global minimum at $(t_0, x_0) \in (0, T) \times X_0$.

Step 1. We prove that $x_0 \in X_1$.

For every $(t, x) \in (0, T) \times X_0$

$$v(t, x) - v(t_0, x_0) \geqslant \varphi(t, x) - \varphi(t_0, x_0) - \tfrac{1}{2}[\delta(t)\,\|x\|_0^2 - \delta(t_0)\,\|x_0\|_0^2]. \tag{47}$$

By the dynamic programming principle for every $\varepsilon > 0$ there exists $\alpha_\varepsilon(\cdot) \in \mathscr{A}_{t_0, t_0+\varepsilon}$ such that, writing $Y_\varepsilon(s)$ for $Y(s; t_0, x_0, \alpha_\varepsilon)$, we have

$$v(t_0, x_0) + \varepsilon^2 > \mathbb{E}\left\{\int_{t_0}^{t_0+\varepsilon} f(Y_\varepsilon(s), \alpha_\varepsilon(s))\,ds + v(t_0+\varepsilon, Y_\varepsilon(t_0+\varepsilon))\right\}.$$

Then, by (47),

$$\begin{aligned} &\varepsilon^2 - \mathbb{E}\int_{t_0}^{t_0+\varepsilon} f(Y_\varepsilon(s), \alpha_\varepsilon(s))\,ds \\ &\quad \geqslant \mathbb{E}v(t_0+\varepsilon, Y_\varepsilon(t_0+\varepsilon)) - v(t_0, x_0) \\ &\quad \geqslant \mathbb{E}\varphi(t_0+\varepsilon, Y_\varepsilon(t_0+\varepsilon)) - \varphi(t_0, x_0) \\ &\qquad - \tfrac{1}{2}[\delta(t_0+\varepsilon)\,\mathbb{E}\,\|Y_\varepsilon(t_0+\varepsilon)\|_0^2 - \delta(t_0)\,\|x_0\|_0^2] \end{aligned}$$

and, by (47) and the Ito Formula,

$$
\begin{aligned}
&\varepsilon^2 - \mathbb{E}\int_{t_0}^{t_0+\varepsilon} f(Y_\varepsilon(s), \alpha_\varepsilon(s))\, ds \\
&\quad \geqslant \mathbb{E}\int_{t_0}^{t_0+\varepsilon} \Big[\varphi_t(s, Y_\varepsilon(s)) - \langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), D\varphi(s, Y_\varepsilon(s))\rangle_{\langle X_{-1}, X_1\rangle} \\
&\qquad + \tfrac{1}{2}\sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s))\, S^k_{\alpha_\varepsilon(s)} Y(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s)\rangle_0 \Big]\, ds \\
&\qquad - \tfrac{1}{2}\mathbb{E}\int_{t_0}^{t_0+\varepsilon} \delta'(s)\, \|Y_\varepsilon(s)\|_0^2 \\
&\qquad - \int_{t_0}^{t_0+\varepsilon} \delta(s) \Big[- \langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), Y_\varepsilon(s)\rangle_{\langle X_{-1}, X_1\rangle} \\
&\qquad + \tfrac{1}{2}\sum_{k=1}^{m} \langle S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s)\rangle_0 \Big]\, ds.
\end{aligned}
$$

We now divide both sides of this inequality by ε to obtain

$$
\begin{aligned}
&\varepsilon - \mathbb{E}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} f(Y_\varepsilon(s), \alpha_\varepsilon(s))\, ds \\
&\quad \geqslant \mathbb{E}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \varphi_t(s, Y_\varepsilon(s))\, ds \\
&\qquad - \mathbb{E}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), D\varphi(s, Y_\varepsilon(s))\rangle_{\langle X_{-1}, X_1\rangle}\, ds \\
&\qquad + \frac{1}{2}\,\mathbb{E}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s))\, S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s)\rangle_0\, ds \\
&\qquad + \mathbb{E}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \delta(s) \Big[\langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), Y_\varepsilon(s)\rangle_{\langle X_{-1}, X_1\rangle}\, ds \\
&\qquad - \frac{1}{2}\sum_{k=1}^{m} \langle S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s)\rangle_0 \Big]\, ds \\
&\qquad - \frac{1}{2}\,\mathbb{E}\int_{t_0}^{t_0+\varepsilon} \delta'(s)\, \|Y_\varepsilon(s)\|_0^2\, ds.
\end{aligned} \tag{48}
$$

By Hypothesis 3.1 and (11) we have

$$\mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\delta(s)\Bigg[\langle A_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s),\ Y_\varepsilon(s)\rangle_{\langle X_{-1},\,X_1\rangle}\,ds$$
$$-\frac{1}{2}\sum_{k=1}^{m}\langle S^k_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s),\ S^k_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s)\rangle_0\Bigg]\,ds$$
$$\geqslant \mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\delta(s)\left[\frac{\lambda}{4}\,\|Y_\varepsilon(s)\|_1^2-K\,\|Y_\varepsilon(s)\|_0^2\right]ds.$$

The regularity of φ yields (recall Remark 4.5)

$$\langle A_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s),\ D\varphi(s,\ Y_\varepsilon(s))\rangle_{\langle X_{-1},\,X_1\rangle}$$
$$=\langle B_\rho^{1/2}\,A_{\alpha_\varepsilon(s)},\ B_\rho^{-1/2}\,D\varphi(s,\ Y_\varepsilon(s))\rangle_0\leqslant C\,\|Y_\varepsilon(s)\|_1$$

and

$$\sum_{k=1}^{m}\langle D^2\varphi(s,\ Y_\varepsilon(s))\,S^k_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s),\ S^k_{\alpha_\varepsilon(s)}\,Y_\varepsilon(s)\rangle_{\langle X_1,\,X_{-1}\rangle}\leqslant C\,\|Y_\varepsilon(s)\|_0^2$$

Therefore, using the above inequalities, the assumptions on the rate of growth of f, and again Remark 4.5 (φ_t can be assumed to be bounded), we get from (48)

$$\frac{\lambda}{4\varepsilon}\,\mathbb{E}\int_{t_0}^{t_0+\varepsilon}\delta(s)\,\|Y_\varepsilon(s)\|_1^2\,ds\leqslant C\left[1+\mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\delta(s)\,\|Y_\varepsilon(s)\|_0^2\,ds\right].$$

Take now $\varepsilon=1/n$ and set $Y_n(s)=Y(s;\,t_0,\,x_0,\,\alpha_{1/n})$. The above inequality yields

$$n\int_{t_0}^{t_0+1/n}\mathbb{E}\,\|Y_n(s)\|_1^2\,ds\leqslant C$$

so that, along a sequence $t_n\in(t_0,\,t_0+1/n)$,

$$\mathbb{E}\,\|Y_n(t_n)\|_1^2\leqslant C,$$

and thus along a subsequence, still denoted by t_n, we have

$$Y_n(t_n)\rightharpoonup\bar{Y}$$

weakly in $L^2(\Omega;\,X_1)$ for some $\bar{Y}\in L^2(\Omega;\,X_1)$. This clearly implies also weak convergence in $L^2(\Omega;\,X_0)$. But we know by Proposition 3.6(iii) that $Y_n(t_n)\to x_0$ strongly (and weakly) in $L^2(\Omega;\,X_0)$ since the modulus in (31) is independent of α. This proves that $x_0\in X_1$.

Step 2. The supersolution inequality.

Consider the inequality (48). Applying Chebychev inequality and arguing as in the proof of (44) we observe that for every $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [f(Y_\varepsilon(s), \alpha_\varepsilon(s))\, ds - f(x_0, \alpha_\varepsilon(s))]\, ds \right|$$
$$\leqslant \varepsilon + \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \omega_{R(\varepsilon)}(\| Y_\varepsilon(s) - x_0 \|_0)\, ds$$

which, using Proposition 3.6(iv), gives

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [f(Y_\varepsilon(s), \alpha_\varepsilon(s))\, ds - f(x_0, \alpha_\varepsilon(s))]\, ds \right| \leqslant \omega_0(\varepsilon).$$

Similarly, by the continuity properties of φ_t and δ' (see Remark 4.4), we obtain

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \varphi_t(s, Y_\varepsilon(s))\, ds - \varphi_t(t_0, x_0) \right| \leqslant \omega_0(\varepsilon),$$
$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \| Y_\varepsilon(s) \|_0^2 - \delta'(t_0) \| x_0 \|_0^2 \right| \leqslant \omega_0(\varepsilon).$$

Moreover

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [\langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), D\varphi(s, y_\varepsilon(s)) \rangle_{\langle X_{-1}, X_1 \rangle} \right.$$
$$\left. - \langle A_{\alpha_\varepsilon(s)} x_0, D\varphi(t_0, x_0) \rangle_{\langle X_{-1}, X_1 \rangle}]\, ds \right|$$
$$\leqslant \mathbb{E} \frac{1}{\varepsilon} \left[\left| \int_{t_0}^{t_0+\varepsilon} \langle A_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0), D\varphi(s, Y_\varepsilon(s)) \rangle_{\langle X_{-1}, X_1 \rangle} \right| ds \right.$$
$$\left. + \int_{t_0}^{t} |\langle A_{\alpha_\varepsilon(s)} x_0, D\varphi(s, Y_\varepsilon(s)) - D\varphi(t_0, x_0) \rangle_{\langle X_{-1}, X_1 \rangle}|\, ds \right]$$

so that, observing that

$$\mathbb{E}\, |\langle A_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0), D\varphi(s, Y_\varepsilon(s)) \rangle_{\langle X_{-1}, X_1 \rangle}| \leqslant C \mathbb{E}\, \| Y_\varepsilon(s) - x_0 \|_1$$

and

$$\mathbb{E}\, |\langle A_{\alpha_\varepsilon(s)} x_0, D\varphi(s, Y_\varepsilon(s)) - D\varphi(t_0, x_0) \rangle_{\langle X_{-1}, X_1 \rangle}|$$
$$\leqslant C \| x_0 \|_1\, \mathbb{E} \omega_\varphi(|s - t_0| + \| Y_\varepsilon(s) - x_0 \|_0),$$

we obtain in light of Proposition 3.6(iv) that

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [\langle A_{\alpha_\varepsilon(s)} Y_\varepsilon(s), D\varphi(s, Y_\varepsilon(s)) \rangle_{\langle X_{-1}, X_1 \rangle} - \langle A_{\alpha_\varepsilon(s)} x_0, D\varphi(t_0, x_0) \rangle_{\langle X_{-1}, X_1 \rangle}] \, ds \right| \leqslant \omega_0(\varepsilon).$$

Finally we have

$$\begin{aligned} &\mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[\sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s) \rangle_0 \, ds \right. \\ &\qquad \left. - \sum_{k=1}^{m} \langle D^2\varphi(t_0, x_0) \, S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)} x_0 \rangle_0 \right] ds \\ &\quad = \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s) \rangle_0 \, ds \\ &\qquad + \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0) \rangle_0 \, ds \\ &\qquad + \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \sum_{k=1}^{m} \langle [D^2\varphi(s, Y_\varepsilon(s)) - D^2\varphi(t_0, x_0)] \\ &\qquad \times S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)} x_0 \rangle_0 \, ds. \end{aligned}$$

Now observing that

$$\begin{aligned} &\mathbb{E} \, |\langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s) \rangle_0| \\ &\qquad \leqslant C \, \| Y_\varepsilon(s) - x_0 \|_0 \, \| x_0 \|_0, \\ &\mathbb{E} \, |\langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)}(Y_\varepsilon(s) - x_0) \rangle_0| \\ &\qquad \leqslant C \, \| Y_\varepsilon(s) - x_0 \|_0 \, \| x_0 \|_0, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \, |\langle [D^2\varphi(s, Y_\varepsilon(s)) - D^2\varphi(t_0, x_0)] \, S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)} x_0 \rangle_0| \\ &\qquad \leqslant C \, \| x_0 \|_0^2 \, \mathbb{E} \omega_\varphi(|s - t_0| + \| Y_\varepsilon(s) - x_0 \|_0) \end{aligned}$$

we obtain

$$\left| \mathbb{E} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[\sum_{k=1}^{m} \langle D^2\varphi(s, Y_\varepsilon(s)) \, S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s), S^k_{\alpha_\varepsilon(s)} Y_\varepsilon(s) \rangle_0 \, ds - \sum_{k=1}^{m} \langle D^2\varphi(t_0, x_0) \, S^k_{\alpha_\varepsilon(s)} x_0, S^k_{\alpha_\varepsilon(s)} x_0 \rangle_0 \right] ds \right| \leqslant \omega_0(\varepsilon).$$

We also have analogous estimates for the terms containing δ (the method to produce them is the same as what we have shown above so we omit the calculations). Using all these estimates in (48) we obtain

$$\begin{aligned}
&\varphi_t(t_0, x_0) - \frac{\delta'(t_0)\,\|x_0\|_0^2}{2} \\
&\quad + \frac{1}{2}\,\mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \langle (D^2\varphi(t_0, x_0) - \delta(t_0)\, I)\, S_{\alpha_\varepsilon(s)} x_0, S_{\alpha_\varepsilon(s)} x_0 \rangle_0\, ds \\
&\quad - \mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \langle A_{\alpha_\varepsilon(s)} x_0, D\varphi(t_0, x_0) - \delta(t_0)\, x_0 \rangle_{\langle X_{-1}, X_1\rangle}\, ds \\
&\quad + \mathbb{E}\,\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} f(x_0, \alpha_\varepsilon(s))\, ds \leqslant \omega_0(\varepsilon)
\end{aligned}$$

The claim now follows upon taking first the infimum over $\alpha \in \mathbf{A}$ inside the integral and then letting $\varepsilon \to 0$. This concludes the proof of the supersolution part. ∎

6. UNIQUENESS

In this section we will prove a comparison result for viscosity sub- and supersolutions of (42). As a corollary we will obtain that the value function is the unique solution of (42).

Let $\{e_n\}_{n=1}^{\infty} \subset X_0$ be an orthonormal basis in X_{-1}. Let $X^N = \operatorname{span}\{e_1, ..., e_N\}$. Denote by P_N the orthogonal projection from X_{-1} onto X^N, let $Q_N = I - P_N$ (I is the identity in X_{-1}), and $Y^N = Q_N X_{-1}$. We then have an orthogonal decomposition $X_{-1} = X^N \times Y^N$. For $x \in X_{-1}$ we will write $x = (P_N x, Q_N x)$ and denote $x_N = P_N x$, and $x_N^{\perp} = Q_N x$. It follows from the Closed Graph Theorem that $P_N: X_{-1} \to X_0$ and $Q_N: X_0 \to X_0$ (the restriction of Q_N) are bounded as maps between the indicated spaces. Also, if P_N^*, Q_N^* are the adjoints of P_N, Q_N regarded as maps from X_0 to X_0 then $P_N^* B_\rho P_N = B_\rho P_N$ and $Q_N^* B_\rho Q_N = B_\rho Q_N$. Finally $\|B_\rho^{1/2} Q_N x\|_{0,\rho} \to 0$ as $N \to \infty$ for every $x \in X_{-1}$.

THEOREM 6.1. *Let Hypotheses* 3.1 *and* 4.1 *hold. Let* $u, v\colon (0, T) \times X_0 \to \mathbb{R}$ *be respectively a viscosity subsolution, and a viscosity supersolution of* (42). *Let* u, $-v$ *be bounded on bounded subsets of* $(0, T) \times X_0$, *upper semicontinuous in* $|\cdot|_{\mathbb{R}} \times \|\cdot\|_{-1,\rho}$ *norm on bounded subsets of* $(0, T) \times X_0$, *and such that*

$$\limsup_{\|x\|_{0,\rho} \to \infty} \frac{u(t, x)}{\|x\|_{0,\rho}^2} \leqslant 0, \qquad \limsup_{\|x\|_{0,\rho} \to \infty} \frac{-v(t, x)}{\|x\|_{0,\rho}^2} \leqslant 0. \tag{49}$$

uniformly for $t \in [0, T]$. *Let*

$$\begin{cases} \text{(i)} & \lim_{t\uparrow T}(u(t,x)-g(x))^{+}=0 \\ \text{(ii)} & \lim_{t\uparrow T}(v(t,x)-g(x))^{-}=0 \end{cases} \tag{50}$$

uniformly on bounded subsets of X_0. *Then* $u \leqslant v$.

Proof. As in the previous section we will drop the subscript ρ in the notation for norms, inner products, and operators.

Step 1. Without loss of generality we can assume that u and $-v$ are bounded from above and such that

$$\lim_{\|x\|_0 \to \infty} u(t,x) = -\infty, \qquad \lim_{\|x\|_0 \to \infty} v(t,x) = +\infty. \tag{51}$$

To see this we claim that if K is the constant from (11) then for every $\eta > 0$

$$u_\eta(t,x) = u(t,x) - \eta e^{2K(T-t)}\, \|x\|_0^2, \qquad v_\eta(t,x) = v(t,x) + \eta e^{2K(T-t)}\, \|x\|_0^2$$

are viscosity sub- and supersolutions of (42) and satisfy (50). This follows from (11) since, denoting $\psi(t,x) = \eta^{2K(T-t)}\, \|x\|_0^2$, we have

$$\psi_t + \inf_{\alpha \in \mathbf{A}} \left\{ \tfrac{1}{2} \sum_{k=1}^{m} \langle D^2\psi S_\alpha^k x, S_\alpha^k x\rangle_0 - \langle A_\alpha x, D\psi\rangle_{\langle X_{-1}, X_1\rangle} \right\}$$
$$\leqslant -\eta K e^{2K(T-t)}\, \|x\|_0^2 \leqslant 0.$$

The functions u_η, $-v_\eta$ satisfy (51) in light of (49). Therefore, if we can prove that $u_\eta \leqslant v_\eta$ for every $\eta > 0$ we will recover $u \leqslant v$ by letting $\eta \to 0$.

Step 2. Replacing u and v by

$$u_\mu = u - \frac{\mu}{t}, \qquad v_\mu = v + \frac{\mu}{t}$$

for $\mu > 0$ we have that

$$\lim_{t \to 0} v_\mu(t,x) = -\infty, \qquad \lim_{t \to 0} v_\mu(t,x) = +\infty, \tag{52}$$

uniformly for $x \in X_0$, and u_μ (respectively, v_μ) is a subsolution (respectively, a supersolution) of (42) with the right hand side being μ/T^2 (respectively, $-\mu/T^2$). If we can prove that $u_\mu \leqslant v_\mu$ then we will obtain $u \leqslant v$ by letting $\mu \to 0$.

Step 3. To keep the number of indices down we will write u for u_μ and v for v_μ throughout the rest of the proof. We argue by contradiction. Suppose that $u \nleqslant v$. Let $\epsilon, \delta, \beta > 0$, and let

$$\varphi(t, s, x, y) = u(t, x) - v(s, y) - \frac{\|x-y\|_{-1}^2}{2\epsilon} - \delta(\|x\|_0^2 + \|y\|_0^2) - \frac{(t-s)^2}{2\beta}.$$

Since upper semicontinuity in $\|\cdot\|_{-1}$ on bounded subsets of $(0, T) \times X_0$ implies upper semicontinuity in $\|\cdot\|_{-2}$ (see for instance [8, Part V]), using perturbed optimization results (see for instance [29]) and (51) we have that for every $n \in \mathbb{N}$ there exist $p_n, q_n \in X_0$, $a_n, b_n \in \mathbb{R}$ such that $\|p_n\|_0$, $\|q_n\|_0$, $|a_n|$, $|b_n| \leqslant 1/n$, and

$$\varphi(t, s, x, y) + \langle Bp_n, x\rangle_0 + \langle Bq_n, y\rangle_0 + a_n t + b_n s$$

has a global maximum at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$ which we may assume to be strict (see Remark 4.4). Standard arguments (see for instance [18, 19]) give

$$\limsup_{\beta \to 0} \limsup_{n \to \infty} \frac{(\bar{t}-\bar{s})^2}{2\beta} = 0 \qquad \text{for fixed } \epsilon, \delta, \tag{53}$$

$$\limsup_{\delta \to 0} \limsup_{\beta \to 0} \limsup_{n \to \infty} \delta(\|\bar{x}\|_0^2 + \|\bar{y}\|_0^2) = 0 \qquad \text{for fixed } \epsilon, \tag{54}$$

and

$$\limsup_{\epsilon \to 0} \limsup_{\delta \to 0} \limsup_{\beta \to 0} \limsup_{n \to \infty} \frac{\|\bar{x}-\bar{y}\|_{-1}^2}{2\epsilon} = 0. \tag{55}$$

Therefore, it follows from (50), (51), (52), and (53) that $0 < \bar{t}, \bar{s} < T$, and $\|\bar{x}\|_0$, $\|\bar{y}\|_0$ are bounded independently of $\epsilon, \delta, \beta, n$. We now fix $N \in \mathbb{N}$. Then

$$\|x-y\|_{-1}^2 = \langle BP_N(x-y), P_N(x-y)\rangle_0 + \langle BQ_N(x-y), Q_N(x-y)\rangle_0.$$

Moreover

$$\begin{aligned}
&\langle BQ_N(x-y), Q_N(x-y)\rangle_0 \\
&\qquad \leqslant 2\langle BQ_N(\bar{x}-\bar{y}), x-y\rangle_0 - \langle BQ_N(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle_0 \\
&\qquad\quad + 2\,\|B^{1/2}Q_N(x-\bar{x})\|_0^2 + 2\,\|B^{1/2}Q_N(y-\bar{y})\|_0^2
\end{aligned}$$

with equality if $x=\bar{x}$, $y=\bar{y}$. Therefore, if

$$u_1(t,x)=u(t,x)-\frac{1}{2\epsilon}\left(\langle 2BQ_N(\bar{x}-\bar{y}),x\rangle_0+\langle 2BQ_N(x-\bar{x}),(x-\bar{x})\rangle_0\right.$$

$$\left.+\langle BQ_N(\bar{x}-\bar{y}),(\bar{x}-\bar{y})\rangle_0\right)-\delta\,\|x\|_0^2+\langle Bp_n,x\rangle_0+a_nt$$

and

$$v_1(s,y)=v(s,y)+\frac{1}{2\epsilon}\left(\langle 2BQ_N(\bar{x}-\bar{y}),y\rangle_0+\langle 2BQ_N(y-\bar{y}),(y-\bar{y})\rangle_0\right)$$

$$+\delta\,\|y\|_0^2-\langle Bq_n,y\rangle_0-b_ns$$

we have that

$$u_1(t,x)-v_1(s,y)-\frac{\|P_N(x-y)\|_{-1}^2}{2\epsilon}-\frac{(t-s)^2}{2\beta}$$

has a strict global maximum at $(\bar{t},\bar{s},\bar{x},\bar{y})$. Because of the behavior of u and v at ∞ we see that the functions u_1 and $-v_1$ are upper-semicontinuous in $(0,T)\times X_{-1}$. This means that u_1 and $-v_1$ are equal to $-\infty$ for $x\in X_{-1}\backslash X_0$. Define

$$\tilde{u}_1(t,x_N)=\sup_{x_N^{\perp}\in Y_0^N}u_1(t,x_N,x_N^{\perp}),\qquad \tilde{v}_1(s,y_N)=\inf_{y_N\in Y_0^N}v_1(s,y_N,y_N^{\perp}).$$

Then

$$(\tilde{u}_1)^*\,(t,x)-(\tilde{v}_1)_*\,(s,y)-\frac{\|P_N(x-y)\|_{-1}^2}{2\epsilon}-\frac{(t-s)^2}{2\beta}$$

has a strict global maximum at $(\bar{t},\bar{s},\bar{x}_N,\bar{y}_N)$ in $(0,T)\times(0,T)\times X^N\times X^N$, where $(\tilde{u}_1)^*$ is the upper-semicontinuous envelope of $\tilde{u}_1$ and $(\tilde{v}_1)_*$ is the lower-semicontinuous envelope of $\tilde{v}_1$. We now apply a finite dimensional result (see for instance Theorem 3.2 of [6]) to produce appropriate test functions. For details of this procedure the reader can consult Lemma 6.4 of [19], together with [7, 14, 31]. We obtain that there exist functions φ_i, $\psi_i\in C^{1,2}((0,T)\times X^N)$ (and hence in $C^{1,2}((0,T)\times X_{-1})$ when viewed as cylindrical functions, and satisfying $D\varphi_i\in C((0,T)\times X_0,X_2)$, $D^2\varphi_2\in C((0,T)\times X_0,L(X_{-1},X_1)))$ and points t_i, $s_i\in(0,T)$, x_N^i, $y_N^i\in X^N$ such that

$$t_i\to\bar{t},\qquad s_i\to\bar{s},\qquad x_N^i\to\bar{x}_N,\qquad y_N^i\to\bar{y}_N \tag{56}$$

$$(\tilde{u}_1)^*\,(t_i,x_N^i)\to(\tilde{u}_1)^*\,(\bar{t},\bar{x}_N),\qquad (\tilde{v}_1)_*\,(s_i,y_N^i)\to(\tilde{v}_i)_*\,(\bar{s},\bar{y}_N) \tag{57}$$

as $i\to\infty$, and such that $(\tilde{u})^*-\varphi_i$, and $-(\tilde{v}_1)_*+\psi_i$ have strict, global maxima at (t_i, X^i_N), and (s_i, y^i_N) respectively. Moreover we have

$$(\varphi_i)_t\,(t_i, x^i_N)\to\frac{\bar{t}-\bar{s}}{\beta},\qquad (\psi_i)_t\,(s_i, y^i_N)\to\frac{\bar{t}-\bar{s}}{\beta},\tag{58}$$

$$D\varphi_i(t_i, x^i_N)\to\frac{1}{\varepsilon}\,BP_N(\bar{x}_N-\bar{y}_N)\qquad \text{in } X_2,$$

$$D\psi_i(s_i, y^i_N)\to\frac{1}{\varepsilon}\,BP_N(\bar{x}_N-\bar{y}_N)\qquad \text{in } X_2,\tag{59}$$

$$D^2\varphi_i(t_i, x^i_N)\to L_N,\qquad D^2\psi_i(s_i, y^i_N)\to M_N\quad \text{in } L(X_{-1}, X_1)\tag{60}$$

as $i\to\infty$, where L_N, M_N are bounded independently of i and satisfy

$$\begin{pmatrix} L_N & 0\\ 0 & -M_N\end{pmatrix}\leqslant\frac{1+v}{\varepsilon}\begin{pmatrix} BP_N & -BP_N\\ -BP_N & BP_N\end{pmatrix}\tag{61}$$

for a certain v that will be chosen later. Putting everything back together, and once again applying perturbed optimization results, it follows that for every $i\in\mathbb{N}$ there exist $\hat{p}_i$, $\hat{q}_i\in X$, $\hat{a}_i$, $\hat{b}_i\in\mathbb{R}$ such that $\|\hat{p}_i\|_0$, $\|\hat{q}_i\|_0$, $|\hat{a}_i|$, $|\hat{b}_i|\leqslant 1/i$, and

$$u_1(t, x)-v_1(s, y)-\varphi_i(t, x)+\psi_i(s, y)+\langle B\hat{p}_i, x\rangle_0+\langle B\hat{q}_i, y\rangle_0+\hat{a}_i t+\hat{b}_i s$$

has a strict, global maximum at $(\hat{t}_i, \hat{s}_i, \hat{x}_i, \hat{y}_i)$. It is then rather standard (see [7, 14]) to show that

$$(\hat{t}_i, \hat{s}_i, \hat{x}_i, \hat{y}_i)\to(\bar{t}, \bar{s}, \bar{x}, \bar{y}),\tag{62}$$

and

$$u_1(\hat{t}_i, \hat{x}_i)\to u_i(\bar{t}, \bar{x}),\qquad v_1(\hat{s}_i, \hat{y}_i)\to v_1(\bar{s}, \bar{y})\tag{62}$$

as $i\to\infty$. Using the fact that u is a viscosity subsolution we therefore obtain

$$\begin{aligned}\frac{\mu}{T^2}\leqslant{}& -a_n-a_i+(\varphi_i)_t\,(\hat{t}_i, \hat{x}_i)\\ &+\inf_{\alpha\in\mathbf{A}}\left\{\frac{1}{2}\sum_{k=1}^{m}\left\langle\left(D^2\varphi_i(\hat{t}_i, \hat{x}_i)+\frac{2}{\epsilon}BQ_N+2\delta I\right)S^k_\alpha\hat{x}_i, S^k_\alpha\hat{x}_i\right\rangle_0\right.\\ &-\left\langle A_\alpha\hat{x}_i, D\varphi_i(\hat{t}_i, \hat{x}_i)+\frac{2BQ_N(\hat{x}_i-\bar{x})}{\epsilon}\right.\\ &\left.\left.+2\delta\hat{x}_i-Bp_n-B\hat{p}_i\right\rangle_{\langle X_{-1}, X_1\rangle}+f(\hat{x}_i, \alpha)\right\}.\end{aligned}\tag{64}$$

We now pass to the limit in (64) as $i \to \infty$. To begin we notice that by (11) for every $\alpha \in \mathbf{A}$

$$\sum_{k=1}^{m} \langle S_\alpha^k \hat{x}_i, S_\alpha^k \hat{x}_i \rangle_0 - \langle A_\alpha \hat{x}_i, \hat{x}_i \rangle_{\langle X_{-1}, X_1 \rangle} \leqslant 2\delta K \|\hat{x}_i\|_0^2 \to 2\delta K \|\bar{x}\|_0^2$$

as $i \to \infty$. (In fact one can prove $\hat{x}_i \rightharpoonup \bar{x}$ in X_1). Using this, (58)–(60), Lemma 3.2(iii), and the fact that $\hat{x}_i \to \bar{x}$ as $i \to \infty$ we obtain upon passing to limsup as $i \to \infty$ in (64) that

$$\begin{aligned}
&-a_n + \frac{\bar{t}-\bar{s}}{\beta} + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \left\langle \left(L_N + \frac{2}{\epsilon} BQ_N \right) S_\alpha^k \bar{x}, S_\alpha^k \bar{x} \right\rangle_0 \right. \\
&\qquad \left. - \left\langle A_\alpha \bar{x}, \frac{B(\bar{x}-\bar{y})}{\epsilon} - Bp_n \right\rangle_{\langle X_{-1}, X_1 \rangle} + f(\bar{x}, \alpha) \right\} + 2\delta K \|\bar{x}\|_0^2 \\
&\qquad \geqslant \frac{\mu}{T^2}.
\end{aligned} \tag{65}$$

For the supersolution v we produce similarly

$$\begin{aligned}
&b_n + \frac{\bar{t}-\bar{s}}{\beta} + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \left\langle \left(M_N - \frac{2}{\epsilon} BQ_N \right) S_\alpha^k \bar{y}, S_\alpha^k \bar{y} \right\rangle_0 \right. \\
&\qquad \left. - \left\langle A_\alpha \bar{y}, \frac{B(\bar{x}-\bar{y})}{\epsilon} + Bq_n \right\rangle_{-X_{-1}, X_1} + f(\bar{y}, \alpha) \right\} - 2\delta K \|\bar{y}\|_0^2 \\
&\qquad \leqslant -\frac{\mu}{T^2}.
\end{aligned} \tag{66}$$

By Hypothesis 3.1 the closures of the sets $\{S_\alpha^k \bar{x}\colon \alpha \in \mathbf{A},\ 1 \leqslant k \leqslant m\}$ and $\{S_\alpha^k \bar{y}\colon \alpha \in \mathbf{A},\ 1 \leqslant k \leqslant m\}$ are compact in X_0, and hence in X_{-1}. This yields that

$$\begin{aligned}
&\sup \{ \|BQ_N S_\alpha^k \bar{x}\|_0 \colon \alpha \in \mathbf{A},\ 1 \leqslant k \leqslant m \}, \\
&\sup \{ \|BQ_N S_\alpha^k \bar{y}\|_0 \colon \alpha \in \mathbf{A},\ 1 \leqslant k \leqslant m \} \to 0
\end{aligned} \tag{67}$$

as $N \to \infty$. Moreover, (61) implies that

$$\langle L_N S_\alpha^k \bar{x}, S_\alpha^k \bar{x} \rangle_0 - \langle M_N S_\alpha^k \bar{y}, S_\alpha^k \bar{y} \rangle_0 \leqslant \langle BS_\alpha^k(\bar{x}-\bar{y}), S_\alpha^k(\bar{x}-\bar{y}) \rangle_0. \tag{68}$$

Therefore, subtracting (65) from (66) and using (67) and (68), we have

$$
\begin{aligned}
&a_n + b_n + \inf_{\alpha \in \mathbf{A}} \left\{ -\frac{1+\nu}{2\epsilon} \sum_{k=1}^{m} \langle BS_\alpha^k(\bar{x}-\bar{y}), S_\alpha^k(\bar{x}-\bar{y}) \rangle_0 \right. \\
&\qquad \left. + \frac{1}{\epsilon} \langle A_\alpha(\bar{x}-\bar{y}), B(\bar{x}-\bar{y}) \rangle_{\langle X_{-1}, X_1 \rangle} \right\} \\
&\qquad - \omega_R(\|\bar{x}-\bar{y}\|_0) - 2\delta K(\|\bar{x}\|_0^2 + \|\bar{y}\|_0^2) - \sigma(N, n) \leqslant -\frac{2\mu}{T^2}
\end{aligned} \tag{69}
$$

for some local modulus σ. The number R is chosen so that $\|\bar{x}\|_0$, $\|\bar{y}\|_0 \leqslant R$ independently of n, β, δ, ϵ which is possible in light of Step 1. Now, if ν is small enough it follows from (13) that

$$
\begin{aligned}
&a_n + b_n + \frac{\lambda_1}{\epsilon} \|\bar{x}-\bar{y}\|_0^2 - \frac{K_1}{\epsilon} \|\bar{x}-\bar{y}\|_{-1}^2 \\
&\qquad - \omega_R(\|\bar{x}-\bar{y}\|_0) - 2\delta K(\|\bar{x}\|_0^2 + \|\bar{y}\|_0^2) - \sigma(N, n) \leqslant -\frac{2\mu}{T^2}
\end{aligned} \tag{70}
$$

for some λ_1, $K_1 > 0$. Since ω_R is a modulus we have

$$
\lim_{\epsilon \to 0} \inf_{r \geqslant 0} \left(\frac{\lambda_1}{\epsilon} r^2 - \omega_R(r) \right) = 0. \tag{71}
$$

Therefore we obtain a contradiction in (70) after sending $N \to \infty$, $n \to \infty$, $\beta \to 0$, $\delta \to 0$, $\epsilon \to 0$ in the above order, and using (54), (55), and (71). ∎

Remark 6.2. The condition that u is upper-semicontinuous in $|\cdot|_{\mathbb{R}} \times \|\cdot\|_{-1,\rho}$ on bounded subsets of $(0, T) \times X_0$ is equivalent to the requirement that if $x_n \rightharpoonup x$, $t_n \to t$ and $B_\rho x_n \to B_\rho x$ as $n \to \infty$ then

$$
\limsup_{n \to \infty} u(t_n, x_n) \leqslant u(t, x),
$$

which is the notion of B-upper-semicontinuity used in [8, Part V]. ∎

7. APPLICATION TO PARTIALLY OBSERVED STOCHASTIC OPTIMAL CONTROL PROBLEMS

We devote this section to show how the results of the paper can be applied to the dynamic programming approach to stochastic optimal control problems with partial observation and correlated noises. To be more precise we will consider the so-called "separated" problem (see e.g. [2, 11, 27]). Our setting of the partially observed control system is partly borrowed from [15–17, 27, 34, and 35]. We will recall it briefly here.

7.1. *An Optimal Control Problem with Partial Observation*

Given a stochastic base $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s\in[t,T]}, \mathbb{P})$ we consider a random state process Z_1 in $\mathbb{R}^d$ and a random observation process Y_1 in $\mathbb{R}^m$. (We could also consider the state process to be infinite dimensional, see [1]).

The state-observation equation is

$$\begin{aligned} dZ_1(s) &= b^1(Z_1(s), \alpha(s))\, ds + \sigma^1(Z_1(s), \alpha(s))\, dW^1(s) \\ &\quad + \sigma^2(Z_1(s), \alpha(s))\, dW^2(s); \\ Z_1(t) &= \xi_1 \in L^2(\Omega, \mathbb{R}^d), \end{aligned}$$

$$\begin{aligned} dY_1(s) &= h(Z_1(s))\, ds + dW^2(s); \\ Y_1(t) &= 0, \end{aligned}$$

where W^1 and W^2 are two independent Brownian motions on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. Using the same notation as in Section 3.2 the control set $\mathbf{A}$ is now a closed subset of $\mathbb{R}^n$, and a control strategy is a process $\alpha\colon [t, T]\times\Omega \mapsto \mathbf{A} \subset \mathbb{R}^n$ that is progressively measurable with respect to the filtration $\{\mathscr{F}_s^{Y_1}; s\in[t,T]\}$ generated by the observation process Y_1. The set of such control strategies will be denoted by $\mathscr{A}_{t,T}$. We will later let the probability spaces vary as we have done in Section 4 and we will work with relaxed controls. We assume the following:

HYPOTHESIS 7.1. *The set* $\mathbf{A}$ *is a closed subset of* $\mathbb{R}^n$. *The functions*

$$b^1\colon \mathbb{R}^d\times\mathbf{A} \mapsto \mathbb{R}^d; \qquad h\colon \mathbb{R}^d \mapsto \mathbb{R}^m$$

are uniformly continuous and $b^1(\cdot,\alpha), h$ *have their* $C^2(\mathbb{R}^d)$ *norms bounded, uniformly for* $\alpha\in\mathbf{A}$. *Moreover the functions*

$$\sigma^1\colon \mathbb{R}^d\times\mathbf{A} \mapsto \mathscr{L}(\mathbb{R}^d, \mathbb{R}^d); \qquad \sigma^2\colon \mathbb{R}^d\times\mathbf{A} \mapsto \mathscr{L}(\mathbb{R}^m, \mathbb{R}^d);$$

are uniformly continuous and $\sigma^1(\cdot,\alpha)$, $\sigma^2(\cdot,\alpha)$ *have their* $C^3(\mathbb{R}^d)$ *norms bounded, uniformly for* $\alpha\in\mathbf{A}$, *and*

$$\sigma^1(x,\alpha)[\sigma^1(x,\alpha)]^T \geqslant \lambda I > 0$$

for some $\lambda > 0$ *and all* $x\in\mathbb{R}^d$, $\alpha\in\mathbf{A}$.

This assumption obviously guarantees in particular the existence of a unique strong solution of the above state Eq. (see e.g. [20]).

We now consider the problem of minimizing the cost functional

$$J(t, \xi_1; \alpha) = \mathbb{E}\left\{\int_t^T f_1(Z_1(s), \alpha(s))\, ds + g_1(Z_1(T))\right\}.$$

over all admissible controls where the cost functions

$$f_1 : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}; \qquad g_1 : \mathbb{R}^d \mapsto \mathbb{R}$$

are suitable continuous functions with polynomial growth at infinity.

7.2. *The Separated Problem*

One way of dealing with the partially observed stochastic optimal control problem introduced in the previous section is through the so called "separated" problem (see [2, 11, 27]) that will be outlined below. To be able to do this we need to assume the following hypothesis about the initial condition ξ_1:

HYPOTHESIS 7.2. *The initial condition ξ_1 of the state variable Z_1 is a random variable with density $p_0 \in L^2_\rho(\mathbb{R}^d)$ for some ρ of the form (see (4))*

$$\rho_\beta(\xi) = (1 + |\xi|^2_{\mathbb{R}^d})^{\beta/2}, \qquad \beta > d/2.$$

Recall first that, taking the new probability

$$\overline{\mathbb{P}} = \kappa^{-1}(T)\, \mathbb{P},$$

where

$$\kappa(s) = \exp\left[\int_t^s h(Z_1(r))\, dY_1(r) - \tfrac{1}{2} \int_t^s |h(Z_1(r))|^2\, dr \right]$$

the processes W_1 and Y_1 become two independent Brownian motions on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively.

The "separated" problem replaces the original problem by the problem of controlling the unnormalized conditional probability density $p(\cdot)$: $[t, T] \mapsto L^2_\rho(\mathbb{R}^d) = X_0$ of the state process Z_1 given the observation Y_1. Under the above Hypothesis 7.2 the equation for the unnormalized conditional probability density is well posed in X_0 and is given by the DMZ equation

$$dp(s) = -A_{\alpha(s)}\, p(s)\, ds + \sum_{k=1}^m S^k_{\alpha(s)}\, p(s)\, dY_{1,k}(s), \qquad p(t) = p_0. \tag{72}$$

where

$$(A_\alpha x)(\xi) = -\sum_{i,j=1}^d \partial_i [a_{i,j}(\xi, \alpha)\, \partial_j x(\xi)] + \sum_{i=1}^d \partial_i [b_i(\xi, \alpha)\, x(\xi)], \tag{73}$$

and

$$(S_\alpha^k x)(\xi) = \sum_{i=1}^{d} d_{ik}(\xi, \alpha)\, \partial_i\, x(\xi) + e_k(\xi, \alpha)\, x(\xi); \qquad k = 1, ..., m, \tag{74}$$

where

$$\begin{aligned} a(\xi, \alpha) &= \sigma^1(\xi, \alpha)[\sigma^1(\xi, \alpha)]^T + \sigma^2(\xi, \alpha)[\sigma^2(\xi, \alpha)]^T, \\ b_i(\xi, \alpha) &= b_i^1(\xi, \alpha) - \partial_j\, a_{i,j}(\xi, \alpha); \qquad i = 1, ..., d, \\ d(\xi, \alpha) &= -\sigma^2(\xi, \alpha), \\ e_k(\xi, \alpha) &= h_k(\xi) - \partial_i\, \sigma_{ik}^2(\xi, \alpha); \qquad k = 1, ..., m, \\ D(A_\alpha) &= H_\rho^2(\mathbb{R}^d) = X_2; \qquad D(S_\alpha^k) = H_\rho^1(\mathbb{r}^d) = X_1. \end{aligned}$$

Notice that if Hypothesis 7.1 is satisfied then the above operators A_α and S_α^k satisfy (7).

Remark 7.3. Hypothesis 7.2 can be seen, roughly speaking, as a double requirement that:

- the variable ξ_1 has a density so that the separated problem can be set in a space of functions instead of a space of measures. Such a setting has been considered e.g. in [10] and in [17] when the corresponding HJB equation is studied but no uniqueness result is obtained.
- the density p_0 is polynomially decreasing when $|\xi|_{\mathbb{R}^d} \to +\infty$ with the decay rate of order bigger than $\beta + d/2$. This is of course a further restriction with respect to assuming only $p_0 \in L^1(\mathbb{R}^d)$ but it is verified in many practical cases, for instance when the starting distribution is normal. One can consult e.g. [2, p. 36, 204] for the use of p_0 being Gaussian or [2, pp. 82, 167] for other integrability assumptions on p_0 (see also [28, 34]). If we only assume $p_0 \in L^1(\mathbb{R}^d)$ then the separated problem can still be considered however our results do not apply since we do not study HJB equations in this space. ∎

At this point we have to start explaining why we have chosen to work with the "separated" problem in weighted spaces $L_\rho^2(\mathbb{R}^d)$. We hope the reason for this will become clear soon. Here we just want to say that "separated" problems are usually not well posed in $L^2(\mathbb{R}^d)$. However for a large reasonable class of cost functions and initial densities (see the comments after Theorem 7.6, Example 7.7, and Remark 7.3) they are well posed in a space $L_\rho^2(\mathbb{R}^d)$ for a suitable weight function ρ. This approach also has the advantage that we stay within the framework of Hilbert spaces that is essential for the treatment of the associated HJB equation.

Using the new state Eq. (72) the functional to minimize can now be written in terms of the unnormalized conditional density p as follows:

$$
\begin{aligned}
J(t, \xi_1; \alpha) = J(t, p_0; \alpha) &= \overline{\mathbb{E}} \left\{ \int_t^T \langle f_1(\cdot, \alpha(s)), p(s) \rangle_0 \, ds + \langle g_1(\cdot), p(T) \rangle_0 \right\} \\
&= \overline{\mathbb{E}} \left\{ \int_t^T \langle (1/\rho^2) \, f_1(\cdot, \alpha(s)), p(s) \rangle_{0, \rho} \, ds \right. \\
&\quad \left. + \langle (1/\rho^2) \, g_1(\cdot), p(T) \rangle_{0, \rho} \right\} \\
&= \overline{\mathbb{E}} \left\{ \int_t^T f(p(s), \alpha(s)) \, ds + g(p(T)) \right\},
\end{aligned}
$$

where we set

$$
\begin{aligned}
f(p, \alpha) &= \int_{\mathbb{R}^d} f(\xi, \alpha) \, p(\xi) \, d\xi = \langle f_1(\cdot, \alpha), p \rangle_0 = \left\langle \frac{1}{\rho^2} f_1(\cdot, \alpha), p \right\rangle_{0, \rho}, \\
g(p, \alpha) &= \int_{\mathbb{R}^d} g_1(\xi) \, p(\xi) \, d\xi = \langle g_1(\cdot), p \rangle_0 = \left\langle \frac{1}{\rho^2} g_1(\cdot), p \right\rangle_{0, \rho}.
\end{aligned}
\tag{75}
$$

Employing the setting of relaxed controls the "separated" problem (SP) we consider is the following:

Minimize the functional $J(t, p_0; \alpha)$ *over all relaxed controls* $(\Omega, \mathscr{F}, \mathbb{P}, Y_1, \alpha) \in \overline{\mathscr{A}}_{t, T}$, *where* $\overline{\mathscr{A}}_{t, T}$ *is defined as in Section* 4.

To this problem we can apply the machinery developed in the paper. All we need is another assumption on the cost functions.

Hypothesis 7.4. *The functions*

$$
f_1 \colon \mathbb{R}^d \times \mathbf{A} \mapsto \mathbb{R}; \qquad g_1 \colon \mathbb{R}^d \mapsto \mathbb{R}
$$

are continuous and $(1/\rho) \, f_1(\cdot, \alpha)$, $(1/\rho) \, g_1(\cdot)$ *belong to* $L^2(\mathbb{R}^d)$ *and have uniformly bounded norms for* $\alpha \in \mathbf{A}$.

Under this assumption the following result holds whose proof is rather straightforward in light of Remark 4.2.

Proposition 7.5. *If Hypothesis* 7.4 *is satisfied then the functions*

$$
f \colon X_0 \times \mathbf{A} \mapsto \mathbb{R}; \qquad g \colon X_0 \mapsto \mathbb{R}
$$

defined in (75) *satisfy Hypothesis* 4.1.

The above proposition allows us to apply the results of Sections 5 and 6 to our "separated" problem. The main result is:

THEOREM 7.6. *Assume that Hypotheses* 7.1 *and* 7.4 *hold. Then the value function*

$$v(t, p_0) = \inf_{\alpha \in \bar{\mathscr{A}}_{t,T}} J(t, p_0; \alpha)$$

of the control problem SP is the unique viscosity solution (*in the class of functions satisfying* (44) *and* (45)) *of the HJB equation*

$$\begin{cases} v_t + \inf_{\alpha \in \mathbf{A}} \left\{ \frac{1}{2} \sum_{k=1}^{m} \langle D^2 v S^k_\alpha p, S^k_\alpha p \rangle_{0,\rho} - \langle A_\alpha p, Dv \rangle_{\langle X_{-1}, X_1 \rangle} + f(p, \alpha) \right\} = 0, \\ v(T, p) = g(p). \end{cases}$$

We observe that the main advantage of using weighted spaces in this paper is the fact that, when the initial density is, say, polynomially decreasing at infinity with the decay rate of order at least $\beta + d/2$, $\beta > d/2$ (which does not look like a strong restriction, see Remark 7.3) we can then deal with polynomially growing cost functions with the growth rate less than $\beta - d/2$ (see the example below). This is not possible if we set simply $\rho = 1$ and this fact has been the main reason for the introduction of weighted norms here. In fact every weight ρ satisfying Proposition 2.1 (like some exponential weight) can be used in our setting.

EXAMPLE 7.7. Let p_0 be a gaussian density and let f_1 and g_1 have polynomial growth in ξ, i.e.

$$|f_1(\xi, \alpha)| \leqslant C(1 + |\xi|^{k_1}_{\mathbb{R}^d}), \qquad |g_1(\xi)| \leqslant C(1 + |\xi|^{k_2}_{\mathbb{R}^d}).$$

Our results can be applied to this problem if we choose the space $L^2_\rho(\mathbb{R}^d)$ for $\beta > [k_1 \vee k_2] + d/2$. In the classical quadratic case

$$f_1(\xi, \alpha) = \langle M\xi, \xi \rangle_{\mathbb{R}^d} + \langle N\alpha, \alpha \rangle_{\mathbb{R}^n}, \qquad g_1(\xi) = \langle G\xi, \xi \rangle_{\mathbb{R}^d},$$

where M, N and G are suitable nonnegative definite matrices, these conditions are satisfied with $k_1 = k_2 = 2$ if we choose $\mathbf{A} = B_{\mathbb{R}^n}(0, R)$. Therefore in this case it is enough to take $L^2_\rho(\mathbb{R}^d)$ for $\beta > 2 + d/2$. In particular, when also the state equation is linear in the control α our framework covers e.g. the cases studied in [2, Chapter 4].

Remark 7.8. With minor modifications we could also treat cases of more general coefficients in the state equation, i.e. the non-autonomous case, or the case of h depending on α, under suitable regularity assumptions.

Remark 7.9. Theorem 7.6 lays the groundwork for the future analysis of the "separated" problem SP. One such thing is optimality conditions (a so called verification theorem) that are based on the HJB equation. We are currently working on adapting the approach presented in [32] to our infinite dimensional problem. A related question is the existence of optimal feedback controls if the value function is more regular (see [17] for some results on this). These problems will be investigated in a subsequent work.

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REFERENCES

1. N. U. Ahmed, M. Fuhrman, and J. Zabczyk, On filtering equations for infinite dimensions, *J. Funct. Anal.* **143**(1) (1997), 180–204.
2. A. Bensoussan, "Stochastic Control of Partially Observable Systems," Cambridge University Press, Cambridge, 1992.
3. P. Cannarsa and G. Da Prato, On a functional analysis approach to parabolic equations in infinite dimensions, *J. Funct. Anal.* **118**(1) (1993), 22–42.
4. P. Cannarsa and G. Da Prato, Second order Hamilton–Jacobi equations in infinite dimensions, *SIAM J. Control Optim.* **29**(2) (1991), 474–492.
5. P. Cannarsa and M. E. Tessitore, Infinite dimensional Hamilton–Jacobi equations and Dirichlet boundary control problems in parabolic type, *SIAM J. Control Optim.* **34** (1996), 1831–1847.
6. M. G. Crandall, H. Ishii, and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27**(1) (1992), 1–67.
7. M. G. Grandall, M. Kocan, and A. Święch, On partial sup-convolutions, a lemma of P. L. Lions and viscosity solutions in Hilbert spaces, *Adv. Math. Sci. Appl.* **3** (1993/4), 1–15.
8. M. G. Grandall and P. L. Lions, Hamilton–Jacobi equations in infinite dimensions. Part IV: Hamiltonians with unbounded linear terms, *J. Funct. Anal.* **90** (1990), 237–283. Part V: Unbounded linear terms and *B*-continuous solutions, *J. Funct. Anal.* **97**(2) (1991), 417–465. Part VII: The HJB equation is not always satisfied, *J. Funct. Anal.* **125** (1994), 111–148.
9. G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Encyclopedia Math. Appl., Cambridge University Press, Cambridge, UK, 1992.
10. W. H. Fleming, Nonlinear semigroup for controlled partially observed diffusions, *SIAM J. Control Optim.* **20**(2) (1982), 286–301.
11. W. H. Fleming and E. Pardoux, Optimal control of partially observed diffusions, *SIAM J. Control Optim.* **20**(2) (1982), 261–285.
12. F. Gozzi, Second order Hamilton–Jacobi equation in Hilbert spaces and application to stochastic control, Tesi di perfezionamento, Scuola Normale Superiore, Pisa, 1999.
13. F. Gozzi, Regularity of solutions of a second order Hamilton–Jacobi equation and application to a control problem, *Comm. Partial Differential Equations* **20**(5&6) (1995), 775–826.
14. F. Gozzi, E. Rouy, and A. Święch, Hamilton–Jacobi equations in Hilbert spaces and stochastic boundary control, *SIAM J. Control Optim.* **38**(2) (2000), 400–430.

15. O. Hijab, Partially observed control of Markov processes I, *Stochastics* **28** (1989), 123–144.
16. O. Hijab, Partially observed control of Markov processes II, *Stochastics* **28** (1989), 247–262.
17. O. Hijab, Partially observed control of Markov processes III, *Ann. Probab.* **18**(3) (1990), 1099–1125.
18. H. Ishii, Viscosity solutions for a class of Hamilton–Jacobi equations in Hilbert spaces, *J. Funct. Anal.* **105** (1992), 301–341.
19. H. Ishii, Viscosity solutions of nonlinear second-order partial differential equations in Hilbert spaces, *Comm. Partial Differential Equations* **18** (1993), 601–651.
20. I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calculus," Springer-Verlag, New York, 1991
21. N. V. Krylov, "Controlled Diffusion Processes," Springer-Verlag, New York, 1980.
22. N. V. Krylov and B. L. Rozovskii, Stochastic partial differential equations and diffusion processes, *Uspekhi Mat. Nauk* **37**(6) (1982), 75–95; *Russian Math. Surveys* **37**(6) (1982), 81–105.
23. P.-L. Lions, Une inégalité pour les opérateurs elliptiques du second ordre, *Ann. Mat. Pura Appl.* **127** (1981), 1–11.
24. P.-L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part I: The case of bounded stochastic evolution, *Acta Math.* **161** (1988), 243–278. Part II: Optimal Control of Zakai's equation, Lecture Notes in Mathematics, Vol. 1390 (G. Da Prato and L. Tubaro, Eds.), Springer-Verlag, Berlin, 1989, 147–170. Part III: Uniqueness of viscosity solutions for general second order equations, *J. Funct. Anal.* **86** (1989) 1–18.
25. P. A. Meyer, Martingales and Stochastics Integrals I, *in* "Lecture Notes in Mathematics," Vol. 284, Springer-Verlag, Berlin, 1972.
26. N. Nagase and M. Nisio, Optimal control of stochastic partial differential equations, *SIAM J. Control and Optim.* **28** (1990), 186–213.
27. E. Pardoux, Equations of non-linear filtering and application to stochastic control with partial observation, *in* "Nonlinear Filtering and Stochastic Control (Cortona 1981)," pp. 208–248, Lecture Notes in Mathematics, Vol. 972, Springer, Berlin–New York, 1982.
28. E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, *Stochastics* **3** (1979), 127–167.
29. C. Stegall, Optimization of functions on certain subsets of Banach spaces, *Math. Ann.* **236** (1978), 171–176.
30. A. Święch, Unbounded second order partial differential equations in infinite dimensional Hilbert spaces, Ph.D. thesis, University of California, Santa Barbara, 1993.
31. A. Święch, Unbounded second order partial differential equations in infinite dimensional Hilbert spaces, *Comm. Partial Differential Equations* **19** (1994), 1999–2036.
32. J. Yong and X. Y. Zhou, "Stochastic Control: Hamiltonian Systems and HJB Equations," Springer-Verlag, to appear in 1999.
33. Q. Zhang, Controlled partially observed diffusions, Ph.D. thesis, Brown University, Providence, May 1988.
34. X. Y. Zhou, On the existence of optimal relaxed controls of stochastic partial differential equations, *SIAM J. Control and Optim.* **30**(2) (1992), 247–261.
35. X. Y. Zhou, On the necessary conditions of optimal controls for stochastic partial differential equations, *SIAM J. Control and Optim.* **31** (1993), 1462–1478.
36. X. Y. Zhou, A duality analysis on stochastic partial differential equations, *J. Funct. Anal.* **103** (1992), 275–293.