List-coloring apex-minor-free graphs

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Abstract

A graph H is *t-apex* if $H - X$ is planar for some set $X \subset V(H)$ of size t. For any integer $t \geq 0$ and a fixed t-apex graph H, we give a polynomial-time algorithm to decide whether a $(t+3)$-connected H-minor-free graph is colorable from a given assignment of lists of size $t+4$. The connectivity requirement is the best possible in the sense that for every $t \geq 1$, there exists a t-apex graph H such that testing $(t+4)$-colorability of $(t+2)$-connected H-minor-free graphs is NP-complete. Similarly, the size of the lists cannot be decreased (unless P = NP), since for every $t \geq 1$, testing $(t+3)$-list-colorability of $(t+3)$-connected K_{t+4}-minor-free graphs is NP-complete.

All graphs considered in this paper are finite and simple. Let G be a graph. A function L which assigns a set of colors to each vertex of G is called a *list assignment*. An *L-coloring* ϕ of G is a function such that $\phi(v) \in L(v)$ for each $v \in V(G)$ and such that $\phi(u) \neq \phi(v)$ for each edge $uv \in E(G)$. For an integer k, we say that L is a *k-list assignment* if $|L(v)| = k$ for every $v \in V(G)$, and L is a *$(\geq k)$-list assignment* if $|L(v)| \geq k$ for every $v \in V(G)$.

The concept of list coloring was introduced by Vizing [27] and Erdős et al. [5]. Clearly, list coloring generalizes ordinary proper coloring; a graph has chromatic number at most k if and only if it can be L-colored for the k-list assignment which assigns the same list to each vertex. Consequently, the computational problem of deciding whether a graph can be colored from a given k-list assignment is NP-complete for every $k \geq 3$ [6] (while for $k \leq 2$, it is polynomial-time decidable [5]). Let this problem be denoted by k-LC.

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The complexity of the k-LC problem motivates a study of restrictions which ensure that it becomes polynomial-time decidable. Thomassen [26] proved that every planar graph can be colored from any (≥ 5)-list assignment, thus showing that k-LC is polynomial-time decidable for planar graphs for any $k \geq 5$. On the other hand, k-LC is NP-complete even for planar graphs for $k \in \{3, 4\}$, see [9]. More generally, for any fixed surface Σ, the problem k-LC with $k \geq 5$ is polynomial-time decidable for graphs embedded in Σ [4, 14]. Let us remark that for ordinary coloring, it is not known whether there exists a polynomial-time algorithm deciding whether a graph embedded in a fixed surface other than the sphere is 4-colorable (while all graphs embedded in the sphere are 4-colorable [1, 2, 15]).

In this paper, we study a further generalization of this problem—deciding k-LC for graphs from a fixed proper minor-closed family. Each such family is determined by a finite list of forbidden minors [25], and thus we consider the complexity of k-LC for graphs avoiding a fixed graph H as a minor. A graph H is *t-apex* if $H - X$ is planar for some set $X \subset V(H)$ of size t. Our main result is the following.

Theorem 1. *Let H be a t-apex graph and let $b \geq t+4$ be an integer. There exists a polynomial-time algorithm that, given a $(t+3)$-connected H-minor-free graph G and an assignment L of lists of size at least $t+4$ and at most b to vertices of G, decides whether G is L-colorable.*

Consequently, k-LC is polynomial-time decidable for $(t+3)$-connected t-apex-minor-free graphs for every $k \geq t+4$. Note that for every surface Σ, there exists a 1-apex graph that cannot be embedded in Σ. Hence, Theorem 1 implies that 5-LC is polynomial-time decidable for 4-connected graphs embedded in Σ, which is somewhat weaker than the previously mentioned results [4, 14]. However, the constraints on the number of colors and the connectivity cannot be relaxed in general, unless P = NP.

Theorem 2. *For every integer $t \geq 1$, the problem $(t+3)$-LC is NP-complete for $(t+3)$-connected K_{t+4}-minor-free graphs.*

Theorem 3. *For every integer $t \geq 1$, there exists a t-apex graph H such that it is NP-complete to decide whether an H-minor-free $(t+2)$-connected graph is $(t+4)$-colorable.*

Let us remark that K_{t+4} is a t-apex graph for every $t \geq 0$. Furthermore, forbidding a 0-apex (i.e., planar) graph as a minor ensures bounded tree-width [16], and thus k-LC is polynomial-time decidable for graphs avoiding a 0-apex graph as a minor, for every $k \geq 1$.

In the following section, we design the algorithm of Theorem 1. The algorithm uses a variant of the structure theorem for graphs avoiding a t-apex minor; although it is well known among the graph minor research community, we are not aware of its published proof, and give it in the Appendix for completeness. The hardness results (Theorems 2 and 3) are proved in Section 2.

1 Algorithm

Let G be a graph with a list assignment L and let $X \subseteq V(G)$ be a set of vertices. We let $\Phi(G, L, X)$ denote the set of restrictions of L-colorings of G to X. In other words, $\Phi(G, L, X)$ is the set of L-colorings of X which extend to an L-coloring of G. We say that G is *critical with respect to L* if G is not L-colorable, but every proper subgraph of G is L-colorable. The graph G is *X-critical with respect to L* if $\Phi(G, L, X) \neq \Phi(G', L, X)$ for every proper subgraph G' of G such that $X \subseteq V(G')$. Thus, removing any part of an X-critical graph affects which colorings of X extend to the whole graph. Postle [14] gave the following bound on the size of critical graphs.

Theorem 4 (Postle [14, Lemma 3.6.1])**.** *Let G be a graph embedded in a closed disk, let L be a (≥ 5)-list assignment for G, and let X be the set of vertices of G drawn in the boundary of the disk. If G is X-critical with respect to L, then $|V(G)| \leq 29|X|$.*

Theorem 4 has several surprising corollaries; in particular, it makes it possible to test whether a precoloring of an arbitrary connected subgraph (of unbounded size) extends to a coloring of a graph embedded in a fixed surface from lists of size five [4]. Based on somewhat similar ideas, we use it here to deal with a restricted case of list-coloring graphs from a proper minor-closed class. The algorithm uses the structure theorem of Robertson and Seymour [24], showing that each such graph can be obtained by clique-sums from graphs which are "almost" embedded in a surface of bounded genus, up to vortices and apices.

1.1 Embedded graphs

As the first step, let us consider graphs that can be drawn in a fixed surface. A fundamental consequence of Theorem 4 is that actually all the vertices of G are at distance $O(\log |X|)$ from X.

Lemma 5 (Postle [14, Theorem 3.6.3])**.** *Let G be a graph embedded in a closed disk, let L be a (≥ 5)-list assignment for G, and let X be the set of vertices of G drawn in the boundary of the disk. If G is X-critical with respect to L, then every vertex of G is at distance at most $58\log_2|X|$ from X.*

Hence, removing vertices sufficiently distant from X cannot affect which precolorings of X extend. A *k-nest* in a graph G embedded in a surface, with respect to a set $X \subseteq V(G)$, is a set $\Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_k$ of closed disks in Σ bounded by pairwise vertex-disjoint cycles of G such that no vertex of X is drawn in the interior of Δ_0 and at least one vertex of G is drawn in the interior of Δ_k. The vertices drawn in the interior of Δ_k are called *eggs*.

Lemma 6. *Let G be a graph embedded in a surface Σ, let L be a (≥ 5)-list assignment for G, let X be a subset of $V(G)$ and let $\Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_k$ be a k-nest in G with respect to X. If $k \geq 58\log_2|V(G)|$ and v is an egg, then $\Phi(G, L, X) = \Phi(G - v, L, X)$.*

Proof. Clearly, $\Phi(G - v, L, X) \supseteq \Phi(G, L, X)$, hence it sufices to show that $\Phi(G - v, L, X) \subseteq \Phi(G, L, X)$. Let G' be a minimal subgraph of G such that $X \subseteq V(G')$ and $\Phi(G', L, X) = \Phi(G, L, X)$. Observe that G' is X-critical (with respect to L). Since v is an egg of a k-nest, its distance from X in G is greater than k. On the other hand, all vertices of G' are at distance at most k from X by Lemma 5. Therefore, $v \notin V(G')$. It follows that $G' \subseteq G - v$, and thus $\Phi(G - v, L, X) \subseteq \Phi(G', L, X) = \Phi(G, L, X)$. □

Therefore, we can remove vertices within deeply nested cycles without affecting which colorings of X extend. This is sufficient to restrict tree-width, as we show below. We use the result of Geelen at al. [7] regarding existence of planarly embedded subgrids in grids on surfaces. For integers $a, b \geq 2$, an *$a \times b$ grid* is the Cartesian product of a path with a vertices with a path with b vertices. An embedding of a grid G in a disk is *canonical* if the outer cycle of the grid forms the boundary of the disk.

Lemma 7 (Geelen at al. [7])**.** *Let $g \geq 0$ and $r, s \geq 2$ be integers satisfying $s \leq r/\lceil\sqrt{g+1}\rceil - 1$. If H is an $r \times r$ grid embedded in a surface Σ of Euler genus g, then an $s \times s$ subgrid H' of H is canonically embedded in a closed disk $\Delta \subseteq \Sigma$.*

We also use the following bound on the size of a grid minor in embedded graphs of large tree-width. A *tree decomposition* of a graph G consists of a tree T and a function $\beta : V(T) \to 2^{V(G)}$ such that

- for each edge $uv \in E(G)$, there exists $x \in V(T)$ such that $\{u, v\} \subseteq \beta(x)$, and
- for each $v \in V(G)$, the set $\{x \in V(T) : v \in \beta(x)\}$ induces a non-empty connected subtree of T.

The sets $\beta(x)$ for $x \in V(T)$ are the *bags* of the decomposition. The *width* of the tree decomposition is the maximum of the sizes of its bags minus one. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the minimum width of its tree decomposition.

Lemma 8 (Theorem 4.12 in Demaine et al. [3])**.** *Let $r \geq 2$ and $g \geq 0$ be integers. If G is a graph embedded in a surface of Euler genus g and $tw(G) > 6(g+1)r$, then G contains an $r \times r$ grid as a minor.*

Lemma 9. *Let G be a graph embedded in a surface Σ of Euler genus g, let X be a subset of $V(G)$ and let F be a set of faces of G such that every vertex of X is incident with a face belonging to F. If G contains no k-nest with respect to X, then $tw(G) \leq 12(g+1)\lceil\sqrt{g+|F|+1}\rceil(k+2)$.*

Proof. Suppose that G contains no k-nest with respect to X and that $\mathrm{tw}(G) > 12(g+1)\lceil\sqrt{g+|F|+1}\rceil(k+2)$. Let Σ' be the surface obtained from Σ by adding a crosscap in each of the faces of F. Note that the Euler genus of Σ' is $g' = g + |F|$. Since G contains no k-nest with respect to X, observe that the drawing of G in Σ' contains no k-nest with respect to $\emptyset$.

Let $r = (2k+4)\lceil\sqrt{g'+1}\rceil$. By Lemma 8, G contains an $r \times r$ grid H as a minor. The embedding of G in Σ' specifies an embedding of H in Σ'. By Lemma 7, H contains a $(2k+3) \times (2k+3)$ subgrid embedded in a disk $\Delta \subseteq \Sigma'$. However, such a subgrid contains a k-nest with respect to $\emptyset$, and consequently the embedding of G in Σ' contains a k-nest with respect to $\emptyset$. This is a contradiction. □

Let G be a graph embedded in a surface Σ and let X be a subset of $V(G)$. A graph G' is a *k-nest reduction of G with respect to X* if it is obtained from G by repeatedly finding a k-nest with respect to X and removing its egg, until there is no such k-nest. To test whether a vertex v is an egg of a k-nest with respect to X, we proceed as follows: take all faces incident with v. If their union contains a non-contractible curve, then v is not an egg of a k-nest. Otherwise, the union of their boundaries contains a cycle C_k bounding a disk Δ_k containing v. Next, we similarly consider the union of Δ_k and all the faces incident with vertices of C_k, and either conclude that v is not an egg of a k-nest, or obtain a cycle C_{k-1} bounding a disk

$\Delta_{k-1} \supset \Delta_k$. We proceed in the same way until we obtain the disk Δ_0. Finally, we check whether the interior of Δ_0 contains a vertex of X or not. This can be implemented in linear time. By repeatedly applying this test and removing the eggs, we can obtain a k-nest reduction in quadratic time.

Thus, we have a simple polynomial-time algorithm for deciding colorability of an embedded graph G from lists of size 5: find a k-nest reduction G' of G, where k is given by Lemma 6. By Lemma 9, the resulting graph has tree-width at most $O(\log |V(G)|)$, and thus we can test its colorability using the standard dynamic programming approach in polynomial time (see [10] for details).

1.2 Vortices

Next, we deal with vortices. A *path decomposition* of a graph G is its tree decomposition (T, β) such that T is a path. A *vortex* is a graph G with a path decomposition with bags $B_1, \ldots, B_t$ in order along the path and with distinct vertices $v_1, \ldots, v_t$, where $v_i \in B_i$. The *depth* of the vortex is the order of the largest of the bags of the decomposition. The ordered sequence $v_1, \ldots, v_t$ is called the *boundary* of the vortex. Let Σ be a surface. A graph G is *almost embedded in* Σ*, with vortices* $G_1, \ldots, G_m$, if $G = G_0 \cup G_1 \cup \ldots \cup G_m$ for some graph G_0 such that

- $V(G_i) \cap V(G_j) = \emptyset$ for $1 \le i < j \le m$,
- $V(G_0) \cap V(G_i)$ is exactly the set of boundary vertices of the vortex G_i, for $1 \le i \le m$, and
- there exists an embedding of G_0 in Σ and pairwise disjoint closed disks $\Delta_1, \ldots, \Delta_m \subset \Sigma$ such that for $1 \le i \le m$, the embedding of G_0 intersects Δ_i exactly in the set of boundary vertices of G_i, which are drawn in the boundary of Δ_i in order that matches the order prescribed by the vortex (up to reflection and circular shift).

The tree-width of a graph with vortices depends on their depth as follows.

Lemma 10. *Let* $G = G_0 \cup G_1 \cup \ldots \cup G_m$ *for some graph* G_0 *and vortices* $G_1, \ldots, G_m$ *of depth at most* d*. Suppose that*

- $V(G_i) \cap V(G_j) = \emptyset$ *for* $1 \le i < j \le m$*, and*
- $V(G_0) \cap V(G_i)$ *is exactly the set of boundary vertices of the vortex* G_i*, for* $1 \le i \le m$*, and*

- *the boundary vertices of the vortex G_i in order form a path in G_0, for $1 \leq i \leq m$.*

Then, $tw(G) \leq d(tw(G_0)+1)-1$.

Proof. Consider a tree decomposition (T, β) of G_0 such that each bag of this decomposition has order at most $\mathrm{tw}(G_0)+1$. For each boundary vertex v of a vortex, let X_v be the corresponding bag in the path decomposition of the vortex. For all other vertices, let $X_v = \{v\}$. For each $x \in V(T)$, let $\beta'(x) = \bigcup_{v \in \beta(x)} X_v$.

Consider an edge $uv \in E(G)$. If $uv \in E(G_0)$, then there exists $x \in V(T)$ with $\{u, v\} \subseteq \beta(x) \subseteq \beta'(x)$. If $uv \notin E(G_0)$, then uv is an edge of one of the vortices, and thus there exists a vertex $w \in V(G_0)$ such that $\{u, v\} \subseteq X_w$. Since (T, β) is a tree decomposition of G_0, there exists $x \in V(T)$ such that $w \in \beta(x)$, and thus $\{u, v\} \subseteq \beta'(x)$.

Next, consider a vertex $v \in V(G)$. If $v \in V(G_0)$, then let $Z_0 = \{x \in V(T) : v \in \beta(x)\}$, otherwise let $Z_0 = \emptyset$. Since (T, β) is a tree decomposition of G_0, Z_0 induces a connected subtree of T. If v belongs to a vortex, say to G_1, then let Y be the set of boundary vertices of G_1 whose bags in the path decomposition of G_1 contain v, and let $Z_1 = \{x \in V(T) : \beta(x) \cap Y \neq \emptyset\}$; otherwise, let $Z_1 = \emptyset$. The elements of Y form a contiguous interval in the sequence of boundary vertices of G_1, and thus they form a path in G_0. Since this path is a connected subgraph of G_0 and (T, β) is a tree decomposition of G_0, we conclude that Z_1 induces a connected subtree of T. Observe that at least one of Z_0 and Z_1 is non-empty, and if they are both non-empty, then they are not disjoint. Consequently, $\{x \in V(T) : v \in \beta'(x)\} = Z_0 \cup Z_1$ induces a non-empty connected subtree of T.

It follows that (T, β') is a tree decomposition of G. Since every bag of (T, β') has order at most $d(\mathrm{tw}(G_0)+1)$, the claim of the lemma follows. □

1.3 Structure theorem

A *clique-sum* of two graphs G_1 and G_2 is a graph obtained from them by choosing cliques of the same size in G_1 and G_2, identifying the two cliques, and possibly removing some edges of the resulting clique. The usual form of the structure theorem for graphs avoiding a fixed minor is as follows [24].

Theorem 11. *For any graph H, there exist integers $m, d, a \geq 0$ with the following property. If G is H-minor-free, then G is a clique-sum of graphs $G_1, \ldots, G_s$ such that for each $1 \leq i \leq s$, there exists a surface Σ_i and a set $A_i \subseteq V(G_i)$ satisfying the following:*

- $|A_i| \leq a$,
- H *cannot be drawn in* Σ_i, *and*
- $G_i - A_i$ *can be almost embedded in* Σ_i *with at most* m *vortices of depth at most* d.

The graphs $G_1, \ldots, G_s$ are called the *pieces* of the decomposition. Let us remark that it is possible that Σ_i is null for some $i \in \{1, \ldots, s\}$, and thus $A_i = V(G_i)$. We need a strengthening of this characterization that restricts the apex vertices, as well as the properties of the embedding. For a graph H and a surface Σ, let $a(H, \Sigma)$ denote the smallest size of a subset B of vertices of H such that $H - B$ can be embedded in Σ.

Theorem 12. *For any graph* H, *there exist integers* m, d *and* a *with the following property. If* G *is* H*-minor-free, then* G *is a clique-sum of graphs* $G_1, \ldots, G_s$ *such that for* $1 \leq i \leq s$, *there exists a surface* Σ_i *and a set* $A_i \subseteq V(G_i)$ *satisfying the following:*

- $|A_i| \leq a$,
- H *cannot be drawn in* Σ_i,
- $G_i - A_i$ *can be almost embedded in* Σ_i *with at most* m *vortices of depth at most* d,
- *every triangle in the embedding bounds a 2-cell face, and*
- *all but at most* $a(H, \Sigma_i) - 1$ *vertices of* A_i *are only adjacent in* G_i *to vertices contained either in* A_i *or in the vortices.*

That such a strengthening is possible is well known among the graph minor research community, but as far as we are aware, it has never been published in this form. For this reason, we provide a proof in the Appendix. Let us also remark that the decomposition of Theorem 12 can be found in polynomial time in the same way as the decomposition of Theorem 11 (see [11, 8] for details), as all the steps of the proof outlined in the Appendix can be caried out in polynomial time.

Suppose that H is a t-apex graph and that G is a $(t+3)$-connected H-minor-free graph G. For $1 \leq i \leq s$, let G_i, A_i and Σ_i be as in Theorem 12. Let G'_i be the part of $G_i - A_i$ embedded in Σ_i (i.e., excluding the non-boundary vertices of the vortices). We say that G'_i is the *embedded part* of G_i. If G'_i has at most four vertices, then we say that the piece G_i is *degenerate*.

Let A'_i be the subset of A_i consisting of the vertices that have neighbors in $G_i - A_i$ that do not belong to any vortex. We have $|A'_i| \le a(H, \Sigma_i) - 1 \le t-1$. Suppose that G_i is not degenerate and that K is a clique in G_i through that G_i is summed with other pieces of the decomposition, and that K contains a vertex that is neither in A_i nor in the vortices. It follows that $V(K) \subseteq V(G'_i) \cup A'_i$. Note that $V(K)$ forms a cut in G, and thus $|V(K)| \ge t+3$. Therefore, K contains a subclique K' of size at least four that contains no vertex of A'_i. Since $K' \subseteq G'_i$ and every triangle in G'_i bounds a 2-cell face, it follows that $|V(K')| = 4$ and that $G'_i = K'$. However, this contradicts the assumption that G_i is not degenerate. Therefore, we conclude that for each non-degenerate piece G_i, all the clique-sums are over cliques contained in the union of A_i and the vortices.

Lemma 13. *Let H be a t-apex graph, let G be a $(t+3)$-connected H-minor-free graph and let L be a $(\ge t+4)$-list assignment for G. Let $G_1, \ldots, G_s$ be the pieces of a decomposition of G as in Theorem 12. For $1 \le i \le s$, if G_i is degenerate, then let G'_i and G''_i be null. Otherwise, let X_i be the set of boundary vertices of the vortices of G_i and let G'_i be the embedded part of G_i. Let G''_i be a k-nest reduction of G'_i with respect to X_i, where $k = \lceil 58 \log_2 |V(G)| \rceil$. Let $G' = G - \bigcup_{i=1}^{s} (V(G'_i) \setminus V(G''_i))$. Then G is L-colorable if and only if G' is L-colorable.*

Proof. For $1 \le i \le s$, let A_i and Σ_i be as in Theorem 12. Let $A'_i \subseteq A_i$ be the set of vertices that have neighbors in G_i that belong neither to A nor to a vortex. Let us recall that $|A'_i| \le t-1$. As we observed, if G_i is not degenerate, then all the clique-sums in the decomposition of G involving G_i are over cliques contained in the union of A_i and the vortices of G_i. Therefore, we can assume that G'_i is a subgraph of G. Let $G^\star_i$ be the subgraph of G consisting of G'_i, the vertices A'_i and all edges between G'_i and A'_i.

Suppose that v is an egg of a k-nest in G'_i with respect to X_i. Consider any L-coloring ψ of the vertices of A'_i, and let L' be the list assignment for G'_i defined by $L'(w) = L(w) \setminus \{\psi(u) : u \in A'_i, uw \in E(G)\}$. Note that $|L'(w)| \ge (t+4)-(t-1) = 5$. By Lemma 6, we have $\Phi(G'_i, L', X_i) = \Phi(G'_i - v, L', X_i)$. Since this holds for every ψ, we conclude that $\Phi(G^\star_i, L, X_i \cup A'_i) = \Phi(G^\star_i - v, L, X_i \cup A'_i)$. Since $X_i \cup A'_i$ separates $G^\star_i - (X_i \cup A'_i)$ from the rest of G, it follows that removing v does not affect the L-colorability of G. Repeating this idea for all removed eggs in all the pieces, we conclude that G is L-colorable if and only if G' is L-colorable. □

1.4 The algorithm

Proof of Theorem 1. Let $k = \lceil 58 \log_2 |V(G)| \rceil$. For $1 \le i \le s$, let G_i, A_i and Σ_i be as in Theorem 12. In polynomial time, we can find a reduction G' of G as in Lemma 13.

Let us consider some $i \in \{1, \ldots, s\}$, and let $G'_i = G_i - (V(G) \setminus V(G'))$. Note that the graph $G'_i - A_i$ is almost embedded with at most m vortices of depth at most d in Σ_i, and the embedded part has no k-nest with respect to the boundaries of the vortices (note that this is obvious if G_i is degenerate). Let G''_i be obtained from $G'_i - A_i$ by adding edges that trace the boundaries of all the vortices in Σ_i. Note that the embedded part G'''_i of G''_i has no $(k+1)$-nest with respect to the boundaries of the vortices. By Lemma 9, we have $\mathrm{tw}(G'''_i) \le 12(g+1)\lceil\sqrt{g+m+1}\rceil(k+3)$, where g is the Euler genus of H. By Lemma 10, we have $\mathrm{tw}(G''_i) \le d(12(g+1)\lceil\sqrt{g+m+1}\rceil(k+3)+1)-1$. Note that $\mathrm{tw}(G'_i) \le a + \mathrm{tw}(G''_i)$, and thus $\mathrm{tw}(G'_i) = O(\log |V(G)|)$.

The graph G' is a clique-sum of $G'_1, G'_2, \ldots, G'_s$, and thus $\mathrm{tw}(G') = O(\log |V(G)|)$. Since the sizes of the lists are bounded by the constant b, the algorithm of Jansen and Scheffler [10] enables us to test L-colorablity of G' in time $2^{O(\mathrm{tw}(G'))}|V(G)|$, which is polynomial in $|V(G)|$. By Lemma 13, G is L-colorable if and only if G' is L-colorable. □

Let us remark that without the upper bound b on the sizes of the lists, we would only get an algorithm with time complexity $|V(G)|^{O(\log |V(G)|)}$.

2 Complexity

Let us start with a simple observation.

Lemma 14. *Let G_1 and G_2 be graphs and let G'_i be the graph obtained from G_i by adding a vertex u_i adjacent to all other vertices, for $i \in \{1, 2\}$. Then G_1 is a minor of G_2 if and only if G'_1 is a minor of G'_2.*

Let us now prove the hardness results justifying the choice of the assumptions in Theorem 1.

Proof of Theorem 2. By Lemma 14, it suffices to show that 5-LC is NP-complete for K_5-minor-free 4-connected graphs. Note that a 4-connected graph is K_5-minor-free if and only if it is planar [28]. We construct a reduction from 3-colorability of connected planar graphs, which is known to be NP-complete [6].

Let G be a connected planar graph. Let G_1 be obtained from G by replacing each edge uv by a subgraph depicted in Figure 1. Observe that

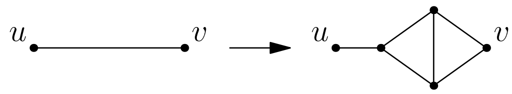

Figure 1: A quasiedge replacement.

G is 3-colorable if and only if G_1 is 3-colorable. Furthermore, G_1 does not contain separating triangles, and every vertex of G_1 is incident with a face of length greater than three.

Gutner [9] constructed a plane graph H without separating triangles that is critical with respect to a 4-list assignment L. We can assume that L does not use colors 1, 2 and 3. Let x be an arbitrary vertex incident with the outer face of H and let L' be the list assignment obtained from L by removing any three colors from the list of x and adding colors 1, 2 and 3 instead. Since H is critical with respect to L, it follows that every L'-coloring ψ of H satisfies $\psi(x) \in \{1, 2, 3\}$, and furthermore for every $i \in \{1, 2, 3\}$, there exists an L'-coloring ψ_i of H such that $\psi_i(x) = i$.

Let G_2 be the graph obtained from G_1 as follows. For each vertex $v \in V(G_1)$, add a copy H_v of H and identify its vertex x with v. The graph H_v is drawn in the face of G_1 incident with v of length at least four, so that G_2 has no separating triangles. Let L_2 be the list assignment for G obtained as the union of the list assignments L' for the copies of H appearing in G_2. Note that G_2 is L_2-colorable if and only if G_1 is 3-colorable.

Finally, let G_3 be obtained from G_2 as follows. For each face f of G_2, consider its boundary walk $v_1v_2\ldots v_m$. Add to f a wheel with rim $w_1w_2\ldots w_m$ and add edges v_iw_i and v_iw_{i+1} for $1 \le i \le m$, where $w_{m+1} = w_1$. Let L_3 be the list assignment obtained from L_2 by giving each vertex of the newly added wheels a list of size four disjoint from the lists of all other vertices of G_3. Clearly, G_3 is L_3-colorable if and only if G_2 is L_2-colorable. Furthermore, G_3 is a triangulation without separating triangles, and thus it is 4-connected.

This gives a polynomial-time algorithm that, given a connected planar graph G, constructs a 4-connected planar graph G_3 and 4-list assignment L_3 such that G is 3-colorable if and only if G_3 is L_3-colorable. Therefore, 4-list colorability of 4-connected planar graphs is NP-complete. □

Next, let us argue that the connectivity assumption is necessary, even if we consider ordinary coloring instead of list coloring. Let $k \ge 3$ be an integer, let G be a graph, let X be a triple of vertices of G and let S be a set of k-colorings of X closed under permutations of colors. Let L be

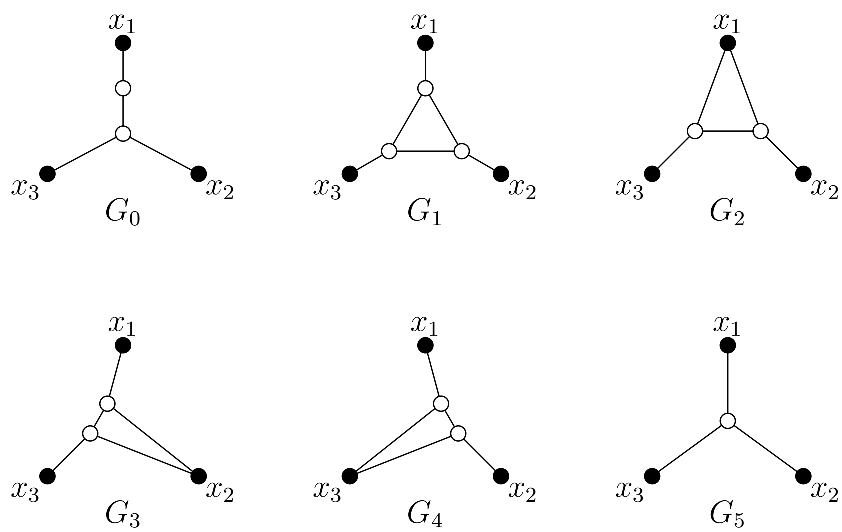

Figure 2: Gadgets from Lemma 15

the list assignment such that $L(v) = \{1, \ldots, k\}$ for every $v \in V(G)$. If $\Phi(G, L, X) = S$, then we say that G is an *S-gadget on X for k-coloring.*

Lemma 15. *For every integer $k \geq 3$ and every set $S \subseteq \{1, \ldots, k\}^3$ closed under permutations of colors, there exists a $\min(k-2, 3)$-connected S-gadget for k-coloring.*

Proof. Let $S_0 = \{(i, j, m) : 1 \leq i, j, m \leq k\}$, $S_1 = S_0 \setminus \{(i, i, i) : 1 \leq i \leq k\}$, $S_2 = S_0 \setminus \{(i, j, j) : 1 \leq i, j \leq k, i \neq j\}$, $S_3 = S_0 \setminus \{(j, i, j) : 1 \leq i, j \leq k, i \neq j\}$, $S_4 = S_0 \setminus \{(j, j, i) : 1 \leq i, j \leq k, i \neq j\}$ and $S_5 = S_0 \setminus \{(i, j, m) : 1 \leq i, j, m \leq k, i \neq j \neq m \neq i\}$. For $0 \leq i \leq 5$, let G_i be the graph obtained from the graph G'_i depicted in Figure 2 by adding $k - 3$ universal vertices. Let $X = (x_1, x_2, x_3)$, and observe that for $0 \leq i \leq 5$, the graph G_i is a $(k - 2)$-connected S_i-gadget on X for k-coloring.

Observe that either $S = S_0$ or there exists non-empty $I \subseteq \{1, \ldots, 5\}$ such that $S = \bigcap_{i \in I} S_i$. In the former case, G_0 is an S-gadget. In the latter case, $\bigcup_{i \in I} G_i$ is an S-gadget. □

Proof of Theorem 3. By Lemma 14, it suffices to show this claim for $t = 1$, i.e., that 5-colorability of 3-connected graphs avoiding some 1-apex minor is NP-complete. We give a reduction from planar 3-SAT.

Let $X = (x_1, x_2, x_3)$. Let A be the set of 5-colorings of X such that either all vertices have the same color, or they have three different colors.

Let B be the set of 5-colorings of X such that the color of x_1 is different from the color of x_2 if and only if x_1 and x_3 have the same color. Let C be the set of 5-colorings of X such that if x_1 and x_3 have the same color, then x_2 also has this color. Let D be the set of 5-colorings of X such that not all its vertices have the same color. Let Δ_A, Δ_B, Δ_C and Δ_D be 3-connected A-, B-, C- and D-gadgets, respectively, on X for 5-coloring, which exist by Lemma 15.

Given a planar instance ϕ of 3-SAT, we construct a graph G_ϕ which is 5-colorable if and only if the instance is satisfiable, in the following way. Let Z_ϕ be the incidence graph of ϕ drawn in plane. By modifying the formula ϕ if necessary, we can assume that Z_ϕ is 2-connected.

First, for each variable x that appears in k clauses of ϕ, let G_x be the graph consisting of vertices c_x, x_0, x_1, $\ldots$, x_k and k A-gadgets on (c_x, x_0, x_1), (c_x, x_1, x_2), $\ldots$, (c_x, x_{k-1}, x_k), respectively. Note that in every 5-coloring ψ of G_x, either $\psi(x_{i-1}) = \psi(x_i)$ for $1 \le i \le k$, or $\psi(x_{i-1}) \ne \psi(x_i)$ for $1 \le i \le k$. Furthermore, if ψ is a 5-coloring of $\{x_0, x_1, \ldots, x_k\}$ satisfying one of the conditions and additionally $\psi(x_i) \ne 5$ for $0 \le i \le k$, then ψ extends to a 5-coloring of G_x.

Next, we construct a graph H_x from G_x as follows. Let us number the appearances of x in ϕ from 1 to k according to their order in the drawing of Z_ϕ around the vertex corresponding to x. Consider the i-th appearance and add vertices x'_i and x''_i and a copy of an A-gadget on (x_{i-1}, x_i, x'_i). If the i-th appearance of x is negated, then add a copy of a B-gadget on (x'_i, x_i, x''_i), otherwise add a copy of an A-gadget on (x'_i, x_i, x''_i).

Finally, we process each clause c of ϕ. Suppose that c is the conjunction of variables x, y and z or their negations (in order according to their drawing around the vertex corresponding to c in Z_ϕ) and the appearance of x, y and z in c is the i_x-th, i_y-th and i_z-th one, respectively. We identify x''_{i_x} with y'_{i_y}, add a C-gadget on $(x'_{i_x}, x''_{i_x}, y''_{i_y})$, add two new vertices w_c and w'_c, add A-gadgets on $(x'_{i_x}, y''_{i_y}, w_c)$, (x'_{i_x}, w_c, w'_c) and (w'_c, w_c, z'_{i_z}), and add a D-gadget on $(w'_c, z'_{i_z}, z''_{i_z})$.

Let G_ϕ be the resulting graph, whose construction can clearly be performed in polynomial time. Note that every 5-coloring ψ of G_ϕ gives a satisfying assignment to ϕ (in that x is true if and only if $\psi(c_x) \ne \psi(x_0)$). Conversely, given a satisfying assignment to ϕ, we can find a 5-coloring ψ of G_ϕ by setting $\psi(x_j) = 1$ for each false variable x and $0 \le j \le k$, $\psi(x_j) = 3 + (j \bmod 2)$ for each true variable x and $0 \le j \le k$, and extending the coloring to the rest of G_ϕ in the obvious way.

By the planarity of ϕ, observe that G_ϕ is obtained from a plane graph by clique-sums with A-, B-, C- or D-gadgets on triangles. We conclude that if H is a non-planar 4-connected graph with

$$|V(H)| > \max(|V(\Delta_A)|, |V(\Delta_B)|, |V(\Delta_C)|, |V(\Delta_D)|),$$

then G_ϕ is H-minor-free. Furthermore, observe that since Z_ϕ is 2-connected, the graph G_ϕ is 3-connected.

Since planar 3-SAT is NP-complete [13], and since there exist arbitrarily large 4-connected 1-apex graphs, the claim of Theorem 3 follows. □

Acknowledgments

We would like to thank Luke Postle for fruitful discussions regarding the problem.

Appendix

In this section, we assume that the reader is familiar with the graph minor theory, and in particular, knows and understands the following concepts and definitions (see Robertson and Seymour [18, 19, 20, 24, 23]): a *(respectful) tangle (controlling a minor)*, the metric derived from a respectful tangle, the function (*slope*) ins defined by a respectful tangle, free sets with respect to a tangle, *p-vortex*, *($\mathcal{T}$-central) segregation (of type (p,k))*, an *arrangement* of a segregation, *true z-redundant ($\mathcal{T}$-central) portrayal with warp $\le p$*.

Let $\mathcal{T}$ be a tangle in a graph G and suppose that S is a $\mathcal{T}$-central segregation of G of type (p,k) with an arrangement in a surface Σ. For $s \in S$, let ∂s denote the boundary vertices of s. We call the elements of S with boundary of size at most three *cells* and the remaining (at most k) elements *p-vortices*. We say that S is *linked* if every cell $s \in S$ is connected, has non-empty boundary, and the clique with vertex set ∂s is a rooted minor of s. Note that every segregation (with at least one p-vortex) can be transformed to a linked one by splitting cells on (≤ 1)-cuts, and by adding cells with $\partial s = \emptyset$ to one of the p-vortices of S. Let $T(S)$ be the multigraph such that $V(T(S)) = \bigcup_{s\in S} \partial s$ and $E(T(S)) = \{uv : s \in S, |\partial s| \le 3, u, v \in \partial s, u \ne v\}$ (if two vertices belong to several cells, they are joined by the corresponding number of edges). Note that $T(S)$ has an embedding in Σ such that the boundary vertices of each p-vortex appear in order in the boundary of some face of $T(S)$; we call such a face a *vortex face*. Furthermore, for each cell $s \in S$ with $|\partial s| = 3$, the triangle corresponding to s bounds a face. Note that if S is linked, then $T(S)$ is a minor of G. Each tangle $\mathcal{T}'$ in a minor of

G determines an *induced* tangle $\mathcal{T}''$ of the same order in G (see [18], (6.1)). If a tangle $\mathcal{T}$ in G satisfies $\mathcal{T}'' \subseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *conformal* with $\mathcal{T}$.

As a starting point, we use the local version of Theorem 11 in the following form, which is essentially (13.4) of [23].

Theorem 16. *For any graph H, there exist integers p and q such that for any non-decreasing positive function σ of one variable, there exist integers $\theta > z \geq 0$ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no H-minor of G. Then there exists $A \subseteq V(G)$ with $|A| \leq z$ and a true $\sigma(|A|)$-redundant $(\mathcal{T} - A)$-central portrayal of $G - A$ with warp $\leq p$ and at most q cuffs in a surface in which H cannot be embedded.*

Let us remark that there are several differences between the statements of Theorem 16 and of (13.4) in [23].

- In (13.4), there is a different order of the quantifiers meaning that p and q depend also on σ and not only on H. However, an inspection of their choices in the first paragraph of the proof of (13.4) shows that they are independent on σ, and thus the order of quantifiers in Theorem 16 is correct.

- Furthermore, the function σ in (13.4) may have two additional parameters—p and the surface of the portrayal. Here, we chose a simpler formulation without this dependence. However, since p only depends on H and there are only finitely many choices of the surface (also depending only on H), we can maximize σ over the possible choices of the parameters, showing that our formulation is not significantly weaker.

Let us reformulate Theorem 16 in terms of arrangements of segregations.

Corollary 17. *For any graph H, there exist integers $k, p \geq 0$ such that for any non-decreasing positive function ϕ of one variable, there exist integers $\theta > \alpha \geq 0$ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no H-minor of G. Then there exists $A \subseteq V(G)$ with $|A| \leq \alpha$ and a $(\mathcal{T} - A)$-central linked segregation S of $G - A$ of type (p, k), which has an arrangement in a surface in which H cannot be embedded. Furthermore, the embedding of $T(S)$ is 2-cell, $T(S)$ contains a respectful tangle $\mathcal{T}'$ of order at least $\phi(|A|)$ conformal with $\mathcal{T} - A$, and if f_1 and f_2 are vortex faces of $T(S)$ corresponding to distinct p-vortices and d' is the metric defined by $\mathcal{T}'$, then $d'(f_1, f_2) \geq \phi(|A|)$.*

Proof. Choose integers p_0 and q so that Theorem 16 for H is satisfied with p, q replaced by p_0, q. Let $k = \max(q, 1)$ and $p = 2p_0 + 2$. Let $\sigma(x) = \max(x, \phi(x)) + 4p + 5$ and let θ and z be as in Theorem 16. Let $\alpha = z$.

We apply Theorem 16 to G, obtaining a set $A \subseteq V(G)$ and a true $\sigma(|A|)$-redundant $(\mathcal{T} - A)$-central portrayal π of $G - A$ in a surface Σ' in which H cannot be embedded. Let Σ be the surface without boundary obtained from Σ' by, for each cuff Θ of Σ', adding an open disk with boundary Θ disjoint with Σ'. We turn the portrayal π into an arrangement of a segregation S of $G - A$ of type (p, k) in Σ by replacing each cuff Θ by a p-vortex consisting of the union of all the graphs in the border cells of Θ. Let us note that S is $(\mathcal{T} - A)$-central by (2.1) of [22] and (4.3) of [23], since $\mathcal{T} - A$ has order at least $4p + 2$. Note that by (9.1) and (9.2) of [23], $T(S)$ can be made linked by, for each cell s, moving all components of s that do not contain ∂s to a p-vortex (creating one if necessary).

The embedding of $T(S)$ in Σ is 2-cell by (8.1) of [23] (the vortex faces of $T(S)$ are 2-cell by (8.3)). Let us define a tangle $\mathcal{T}'$ in $T(S)$ of order $\phi(|A|)$ as follows. By (6.1) and (6.5) of [19], it suffices to define an even slope ins of order $\phi(|A|)$ in the radial drawing of $T(S)$. Let c be a simple closed $T(S)$-normal curve intersecting $T(S)$ in less than $\phi(|A|)$ vertices which corresponds to a cycle in the radial drawing of $T(S)$. Note that since π is $\sigma(|A|)$-redundant, (6.3) of [23] implies that c intersects at most one vortex face. If c intersects no vortex face, then by (6.4) of [23], there exists a disk $\Delta \subseteq \Sigma'$ bounded by c such that the part of the portrayal π inside c is small in the tangle $\mathcal{T} - A$. In this case, we set $\mathrm{ins}(c) = \Delta$. Suppose now that c intersects a vortex face f, and let v_1 and v_2 be the vertices in the intersection of c and the boundary of f. Let F be the p-vortex of S corresponding to f and let Θ be the corresponding cuff of Σ'. For $i \in \{1, 2\}$, if $v_i \in V(F)$, then let $\Delta_i = \emptyset$. If $v_i \notin V(F)$, then let Δ_i be a closed disk in $\Sigma' \cap \overline{f}$ intersecting the boundary of f in two vertices—v_i and another vertex $v'_i \in V(F)$—such that c and Δ_i intersect in a simple curve contained in the boundary of Δ_i. Furthermore, choose Δ_1 and Δ_2 so that they are disjoint. Let c' be the closed curve given as the symmetric difference of c and the boundary of $\Delta_1 \cup \Delta_2$. By (6.3) of [23] applied to the I-arc $c' \cap \Sigma'$, there exists a disk Δ' whose boundary is contained in $\Theta \cup (c' \cap \Sigma')$ such that the part of the portrayal π inside c is small in the tangle $\mathcal{T} - A$. Let Δ'' be the closure of the symmetric diffecence of Δ' and $\Delta_1 \cup \Delta_2$. Let Δ be the closed disk contained in $\Delta'' \cup (\Sigma \setminus \Sigma')$ bounded by c. We set $\mathrm{ins}(c) = \Delta$. Note that the choice of Δ_i (and even v'_i) is not necessarily unique, but it is easy to see that all possible choices give the same value of $\mathrm{ins}(c)$. Observe that ins is an even slope, and the corresponding tangle $\mathcal{T}'$ in $T(S)$ is conformal with $\mathcal{T} - A$. Furthermore, (6.3) and (6.4) of [23] imply that if f_1 and f_2 are vortex faces of $T(S)$ corresponding to distinct p-vortices and d' is the metric defined by $\mathcal{T}'$, then $d'(f_1, f_2) \geq \phi(|A|)$. □

We also use several further results from the graph minor series. For p-vortices, we use the following characterization, which is essentially (8.1) of [17].

Lemma 18. *If F is a p-vortex with boundary $v_1, \ldots, v_m$, then G has a path decomposition with bags $X_1, \ldots, X_m$ in order, such that $v_i \in X_i$ for $1 \le i \le m$ and $|X_i \cap X_j| \le p$ for $1 \le i < j \le m$.*

We call a path decomposition satisfying the conditions of Lemma 18 a *standard decomposition of a p-vortex*. For each p-vortex F, we fix such a decomposition. For a vertex $v_i \in \partial F$, we let $X(v_i) = \{v_1\} \cup (X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1})$, where $X_0 = X_{m+1} = \emptyset$. The following result, which is (3.2) of [20], describes sufficient conditions for the existence of a rooted minor in a graph embedded in a surface.

Theorem 19. *For every surface Σ without boundary and integers k and z, there exists an integer θ such that the following holds. Let G have a 2-cell embedding in Σ, let $\mathcal{T}$ be a respectful tangle in Σ of order at least θ and let d be the metric defined by $\mathcal{T}$. Let $f_1, \ldots, f_k$ be faces of G such that $d(f_i, f_j) \ge \theta$ for $1 \le i < j \le k$. Let Z be a set of at most z vertices such that each $v \in Z$ is incident with one of $f_1, \ldots, f_k$. Assume that $Z \cap f_i$ is free for $1 \le i \le k$ and let $\Delta_1, \ldots, \Delta_k \subset \Sigma$ be pairwise disjoint closed disks such that $G \cap \Delta_i = Z \cap f_i$. If M is a forest with $Z \subseteq V(M)$ embedded in Σ so that $M \cap \Delta_i = Z \cap f_i$ for $1 \le i \le k$, then there exists a forest $M' \subseteq G$ such that $M' \cap (\Delta_1 \cup \ldots \cup \Delta_k) = Z$ and two vertices of Z belong to the same component of M' if and only if they belong to the same component of M.*

We will also need an auxiliary result concerning the metric derived from a respectful tangle (see (9.2) in [21]). Let G be a graph with a 2-cell embedding in a surface Σ and let $\mathcal{T}$ be a respectful tangle in G with metric d. If a is a vertex or a face of G, then a *t-zone around a* is an open disk $\Lambda \subset \Sigma$ bounded by a cycle $C \subseteq G$ such that $a \subseteq \Lambda$ and $d(a, a') \le t$ for all atoms a' of G contained in $\overline{\Lambda}$.

Lemma 20. *Let G be a graph with a 2-cell embedding in Σ, let $\mathcal{T}$ be a respectful tangle in G of order θ and let d be the metric derived from $\mathcal{T}$. Let a be a vertex or a face of G, and let $2 \le t \le \theta - 3$. Then there exists a $(t+2)$-zone Λ around a such that every atom a' of G with $d(a, a') < t$ satisfies $a' \subseteq \Lambda$.*

Clearing a zone (i.e., removing everything contained inside it from the graph) does not affect the order of the tangle or the distances significantly, as stated in the following lemma. For its proof, see (7.10) of [20].

Lemma 21. *Let G be a graph with a 2-cell embedding in Σ, let $\mathcal{T}$ be a respectful tangle in G of order $\theta \geq 4t+3$ and let d be the metric derived from $\mathcal{T}$. Let Λ be a t-zone around some vertex or face of G and let G' be the graph obtained from G by clearing Λ. Then, there exists a unique respectful tangle $\mathcal{T}'$ in G' of order $\theta - 4t - 2$ defining a metric d' such that whenever a', b' are atoms of G' and a, b atoms of G with $a \subseteq a'$ and $b \subseteq b'$, then $d(a,b) - 4t - 2 \leq d(a', b') \leq d(a,b)$. Furthermore, $\mathcal{T}'$ is conformal with $\mathcal{T}$.*

The following technical lemma is standard (we include its proof for completeness).

Lemma 22. *For all integers $t, n > 0$ and every non-decreasing positive function f, there exists an integer T such that the following holds. Let Z and U be sets of points of a metric space with metric d, such that $|Z| \leq n$ and for every $u \in U$ there exists $z \in Z$ with $d(u,z) < t$. Then, there exists a subset $Z' \subseteq Z$ and an integer $t' \leq T$ such that*

- *for every $u \in U$, there exists $z \in Z'$ with $d(u,z) < t'$, and*
- *for distinct $z_1, z_2 \in Z'$, $d(z_1, z_2) \geq f(t')$.*

Furthermore, if the elements of a set $Z'' \subseteq Z$ are at distance at least T from each other, then we can choose Z' so that $Z'' \subseteq Z'$.

Proof. Let $t_0 = t$ and for $1 \leq i \leq n-1$, let $t_i = t_{i-1} + f(t_{i-1})$; and set $T = t_{n-1}$. We construct a sequence of sets $Z = Z_0 \supset Z_1 \supset \ldots \supset Z_{n'}$ with $n' < n$ such that for every $u \in U$ and $i \leq n'$, there exists $z \in Z_i$ with $d(u,z) < t_i$, as follows: suppose that we already found Z_i. If $d(z_1, z_2) \geq f(t_i)$ for every distinct $z_1, z_2 \in Z_i$, then set $n' = i$ and stop. Otherwise, there exist distinct $z_1, z_2 \in Z_i$ such that $d(z_1, z_2) < f(t_i)$ and $z_2 \notin Z''$; in this case, set $Z_{i+1} = Z_i \setminus \{z_2\}$. Clearly, the set $Z' = Z_{n'}$ has the required properties. □

We use the following construction to add more vertices to p-vortices. Let $\mathcal{T}$ be a tangle in a graph G and let S be a $\mathcal{T}$-central linked segregation of G of type (p,k) with an arrangement in a surface Σ such that the embedding of $T(S)$ is 2-cell. Let $\mathcal{T}'$ be a respectful tangle of order θ in $T(S)$ conformal with $\mathcal{T}$ and let d' be the associated metric. Let F be a p-vortex of S and let f be the corresponding face of $T(S)$. Let $t \geq 2$ be an integer such that $\theta > 8t+10$ and $d'(f_1, f_2) > 8t+10$ for any distinct vortex faces f_1 and f_2 of $T(S)$. If there exists a simple closed $T(S)$-normal curve c intersecting $T(S)$ in at most t vertices such that $f \subseteq \mathrm{ins}(c)$, then choose c so that $\mathrm{ins}(c)$ is maximal and let $\Delta_0 = \mathrm{ins}(c)$; otherwise, let Δ_0 be the disk of the arrangement of S

representing F. Let R be the set of vertices $w \in T(S)$ such that there exists a simple $T(S)$-normal curve joining w to $V(T(S)) \cap \mathrm{bd}(\Delta_0)$ intersecting G_0 in less than t points. By Lemma 20, there exists a $(2t+2)$-zone $\Lambda \subset \Sigma$ in $T(S)$ such that $R \subset \Lambda$. Therefore, the face g of $T(S) - R$ that contains Δ_0 is a subset of Λ. Observe that there exists a cycle C in the boundary of g such that the open disk $\Lambda' \subseteq \Lambda$ bounded by C contains g. Let S_1 be the set of cells $s \in S$ such that the clique induced by ∂s in $T(S)$ is drawn in the closure of Λ', and let $S_2 = S \setminus (S_1 \cup \{F\})$. Let $F' = F \cup \bigcup_{s \in S_1} s$, $\partial F' = V(C)$ and let $S' = \{F'\} \cup S_2$. Note that S' is a linked segregation of G with an arrangement in Σ (the disk representing F' is contained in the closure of Λ' and intersects the boundary of Λ' exactly in $\partial F'$). We say that S' and F' are the *t-extension of F in S*. Note that $T(S')$ is a subgraph of $T(S)$.

Lemma 23. *Let $\mathcal{T}$ be a tangle of order at least ϕ in a graph G and let S be a $\mathcal{T}$-central segregation of G of type (p,k) with an arrangement in a surface Σ such that the embedding of $T(S)$ is 2-cell. Let $\mathcal{T}'$ be a respectful tangle of order ϕ in $T(S)$ conformal with $\mathcal{T}$ and let d' be the associated metric. Let F be a p-vortex of S and let f be the corresponding face of $T(S)$. Let $t \geq 2$ be an integer such that $\phi \geq 15t + 20p + 12$ and $\mathrm{d}'(f_1, f_2) > 8t + 10$ for any distinct vortex faces f_1 and f_2 of $T(S)$. If S' and F' is the t-extension of F in S, then the following claims hold.*

1. *S' is a segregation of type $(3t + 4p, k)$.*

2. *The embedding of $T(S')$ is 2-cell and $T(S')$ has a respectful tangle $\mathcal{T}''$ of order $\phi - 8t - 10 > 2(3t + 4p) + 1$ conformal with $\mathcal{T}'$ (and thus also with $\mathcal{T}$).*

3. *If d'' is the metric asociated with $\mathcal{T}''$, then $\mathrm{d}''(f_1, f_2) \geq \mathrm{d}'(f_1, f_2) - 8t - 10$ for any distinct vortex faces f_1 and f_2 of $T(S)$.*

4. *S' is $\mathcal{T}$-central.*

5. *If $s \in S$ is a cell such that $\mathrm{d}'(f, v) \leq t$ for every $v \in \partial s$, then $s \subseteq F'$.*

Proof. For the first claim, it suffices to prove that F' is a $(3t + 4p)$-vortex. Let Δ be the disk representing F in the arangement of S, let Δ' be the disk representing F' in the arrangement of S', and let Δ_0 be the disk from the construction of F'. Consider any partition of $\partial F'$ (ordered along the boundary of Δ') to two arcs A and B. Let a_1 and a_2 be the endpoints of A, and for $i \in \{1, 2\}$, let c_i be a $T(S)$-normal simple curve drawn in Δ' intersecting $T(S)$ in at most t vertices and joining a_i with a vertex $v_i \in$

bd(Δ_0). If $\Delta_0 = \Delta$, then let $Z = ((c_1 \cup c_2) \cap T(S)) \cup X(v_1) \cup X(v_2)$, otherwise let $Z = (c_1 \cup c_2 \cup \mathrm{bd}(\Delta_0)) \cap T(S)$. Observe that Z separates A from B in F' and that $|Z| < 2t + \max(4p, t)$. Since the choice of A and B was arbitrary, this shows that F' contains no transaction of order $3t + 4p$, as required.

The second and third claims follow from Lemma 21, since the disk Λ' from the construction of F' is a $(2t + 2)$-zone.

To show the fourth claim, consider first a separation (A, B) of G of order at most $2(3t + 4p) + 1$ with $B \subseteq s$ for some $s \in S'$. If $s \neq F'$, then we have $(B, A) \in \mathcal{T}$, since the segregation S is $\mathcal{T}$-central and the order of (A, B) is less than ϕ. Suppose now that $s = F'$. Note that $T(S')$ is a minor of G with a model that uses no edges of F'. Let (A', B') be the separation of $T(S')$ corresponding to (A, B) as in the definition of the induced tangle. Since $B \subseteq F'$ and the model of S' uses no edges of F', we have $E(B') = \emptyset$, and since the order of (A', B') is smaller than the order of $\mathcal{T}''$, we have $(B', A') \in \mathcal{T}''$. The tangle $\mathcal{T}''$ is conformal with $\mathcal{T}$, and thus $(B, A) \in \mathcal{T}$. Since this holds for every such separation (A, B) of G and $\phi \geq 5(3t+4p)+2$, it follows that S' is $\mathcal{T}$-central by (2.1) of [22].

By the construction of F', to prove the last claim it suffices to show that every edge e of the clique on ∂s is drawn in the closure of Λ'. Let v be a vertex incident with e. Since $d'(f, v) \leq t$, there exists a tie (see [19], section 9) c intersecting $T(S)$ in at most t vertices such that $f, v \subset \mathrm{ins}(c)$. The tie c either intersects f or contains a simple closed $T(S)$-normal curve c' with $f \in \mathrm{ins}(c')$, and thus the disk Δ_0 from the construction of F' intersects c. Consequently, all vertices of $T(S) \cap c$ belong to R, and thus either $v \in R$ or there exists a connected component K of $T(S) - R$ surrounded by g. Since this holds for both endpoints of e, we have $e \subset \Lambda'$ as required. □

Let S be a linked segregation, and let v be a vertex belonging to some cell $s \in S$. Since S is linked, the cell s contains as a minor a clique with vertex set ∂s, i.e., there exist vertex-disjoint trees $\{T_u \subset s : u \in \partial s\}$ such that $u \in V(T_u)$ for each $u \in \partial s$ and s contains an edge between $V(T_u)$ and $V(T_w)$ for any distinct $u, w \in \partial s$. Since s is connected, we can assume that $v \in V(T_u)$ for some $u \in \partial s$. We define $T(v) = u$. Note that if $v \in \partial(s)$, then $T(v) = v$.

Next, we aim to prove a local version of Theorem 12 analogous to Corollary 17. First, let us show that we can restrict attachments of the apex vertices.

Lemma 24. *For every graph H, surface Σ in that H cannot be embedded and for all integers p_0, k_0 and a_0, there exist integers p, k and ϕ with the* *following property. Let G be a graph with a tangle $\mathcal{T}$ and let A be a subset of $V(G)$ with $|A| \leq a_0$. Suppose that $\mathcal{T}$ has order at least $\phi + |A|$ and let S be a $(\mathcal{T} - A)$-central linked segregation of $G - A$ of type (p_0, k_0) with an arrangement in Σ. Furthermore, assume that the embedding of $T(S)$ is 2-cell and has a respectful tangle $\mathcal{T}_0$ of order at least ϕ conformal with $\mathcal{T} - A$, and if f_1 and f_2 are vortex faces of $T(S)$ corresponding to distinct p_0-vortices and d_0 is the metric defined by $\mathcal{T}_0$, then $d_0(f_1, f_2) \geq \phi$. If H is not a minor of G, then there exists a $(\mathcal{T} - A)$-central segregation S' of $G - A$ of type (p, k) with an arrangement in Σ such that all but at most $a(H, \Sigma) - 1$ vertices of A are only adjacent to vertices contained either in A or in the p-vortices of S'. Furthermore, every triangle in $T(S')$ bounds a 2-cell face.*

Proof. By subdividing the edges of H, we can assume that H contains a set B of vertices of size $a(H, \Sigma)$ such that $H - B$ has an embedding in Σ and H contains no path of length at most two between vertices of B.

Let θ_1 be the constant of Theorem 19 for Σ, $k = |V(H)| + |E(H)|$ and $z = 2|E(H)| + |V(H)|$. Let $\theta_2 = (\theta_1 + 7)(2 + |E(H)|)$. Let $f_2(t) = 7t + 20p + 16 + (8t + 26)(a_0|V(H)| + k_0)$ and let T_2 be the bound from Lemma 22 for the function f_2, with $n = a_0|V(H)| + k_0$ and $t = 2\theta_2$. Let $\phi = \max(2\theta_2, f_2(T_2))$. Let $p = 3T_2 + 4p_0 + 6$ and $k = a_0|V(H)| + k_0$.

Let $A_1 \subseteq A$ consist of all vertices $v \in A_1$ such that there exists a set N_v of $|V(H)|$ neighbors belonging to the cells of S and satisfying $d_0(T(w_1), T(w_2)) \geq 2\theta_2$ for all distinct $w_1, w_2 \in N_v$.

Suppose that $|A_1| \geq a(H, \Sigma)$. For each $b \in B$, choose a distinct vertex $v_b \in A_1$, and choose greedily a set $M_b \subseteq N_{v_b}$ of $\deg_H(b)$ vertices such that $d_0(T(w_1), T(w_2)) \geq \theta_2$ for all $w_1 \in M_{b_1}$ and $w_2 \in M_{b_2}$ for distinct $b_1, b_2 \in B$ (this is possible, since the total number of chosen vertices is smaller than $|V(H)|$). Let G_0 be the graph obtained from $T(S)$ by adding vertices $\{v_b : b \in B\}$ and edges $v_b T(w)$ for each $b \in B$ and $w \in M_b$. Observe that G_0 is a minor of G. Since the embedding of $T(S)$ is 2-cell, it is connected, and thus $\{z\}$ is free with respect to $\mathcal{T}_0$ for each $z \in V(T(S))$. Let $Z = \{T(w) : b \in B, w \in M_b\}$. Let e be an edge of $T(S)$ incident with some vertex $z \in Z$. By (8.12) of [19], there exists another edge $f \in T(S)$ such that $d_0(e, f) \geq \phi$. Let P be a path joining e with f. As in (4.3) of [20], we conclude that there exists a set R of $|E(H)|$ edges of P such that the distance between each two edges in R is at least θ_1, the distance of eacg edge of R from e is at least $\theta_1 + 2$, but at most $\theta_2 - \theta_1 - 2$ (consequently, its distance from any vertex of Z is at least θ_1), and the endvertices of each edge of R form a free set with respect to $\mathcal{T}_0$. By applying a homeomorphism, we obtain an embedding of $H - B$ in Σ such that for each edge $e_1 \in E(H - B)$, there exists exactly one edge $e_2 \in R$ such that the curve representing e_2 is a subset of the curve representing e_1, and such that for each $b \in B$, the points representing the neighbors of b in $H - B$ coincide with the points representing $\{T(w) : w \in M_b)$. By Theorem 19, we conclude that $T(S)$ contains a minor of $H - B$ with edges represented by the edges in R and with the subgraphs representing the neighbors of B in $H - B$ containing the corresponding vertices of Z. Consequently, H is a minor of G_0, and thus also a minor of G. This is a contradiction.

Therefore, $|A_1| \leq a(H, \Sigma) - 1$. For each $v \in A \setminus A_1$, there exists a set K_v of size less than $|V(H)|$ such that for each neigbor $w \in V(G - A)$ of v, either w belongs to one of the p_0-vortices of S, or K_v contains a vertex at distance less than $2\theta_2$ from T_w. Let U be the set of vertices $T(w)$ for all vertices w that belong to a cell of S and have a neighbor in $A \setminus A_1$. Let $K_0 = \bigcup_{v \in A \setminus A_1} K_v$ and note that $|K_0| < a_0|V(H)|$. Let K consist of K_0 and of the vortex faces of $T(S)$. By Lemma 22, there exists $K_1 \subseteq K$ such that each vertex of U is at distance at most $t \leq T_2$ from K_1, the distance between any two elements of K_1 is at least $f_2(t)$, and all the vortex faces belong to K_1.

For each vertex $v \in K_1$, add a 0-vortex to S consisting only of v (clearly, it is possible to add disks representing these 0-vortices to the arrangement of S). Next, apply the operation of $(t+2)$-extension to every p_0- or 0-vortex corresponding to an element of K_1. By Lemma 23, the resulting segregation S' is $(\mathcal{T} - A)$-central and has type (p, k), and furthermore, every vertex of $A \setminus A_1$ has only neighbors in A and in the p-vortices.

Additionally, $T(S')$ has a respectful tangle of order greater than three, and thus every triangle C in $T(S)$ bounds a disk $\mathrm{ins}(C)$. If $\mathrm{ins}(C)$ is not a face, then we can replace all elements of S' whose boundary is contained in $\mathrm{ins}(C)$ either by a new 1-vortex or a new cell with boundary $V(C)$, depending on whether $\mathrm{ins}(C)$ contains one of the p-vortices or not. By repeating this operation, we can assume that every triangle in $T(S')$ bounds a 2-cell face. □

Lemma 25. *For any graph H, there exist integers $k, p, a, \theta \geq 0$ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G. If H is not a minor of G, then there exists $A \subseteq V(G)$ with $|A| \leq a$ and a $(\mathcal{T} - A)$-central segregation S of $G - A$ of type (p, k) with an arrangement in a surface Σ in which H cannot be embedded, such that all but at most $a(H, \Sigma) - 1$ vertices of A are only adjacent to vertices contained either in A or in the p-vortices of S. Furthermore, every triangle in $T(S)$ bounds a 2-cell face.*

Proof. Let p_0 and k_0 be the constants of Corollary 17 for H. Let p_1, k_1 and ϕ_1 be positive non-decreasing functions such that $p_1(a_0)$, $k_1(a_0)$ and $\phi_1(a_0)$ are greater or equal to the corresponding constants given by Lemma 24 applied to H, p_0, k_0, a_0 and for any surface Σ in that H cannot be embedded. Let α and θ be the constants given by Corollary 17 for H and the function ϕ_1. Let $k = k_1(\alpha)$, $p = p_1(\alpha)$ and $a = \alpha$. Lemma 25 then follows by applying Lemma 24 to the segregation obtained by Corollary 17. □

The main result now follows similarly to the proof of (1.3) based on (3.1) in [24]. The essential claim is the following.

Lemma 26. *For any graph H, there exist integers $m, d, a, \theta \geq 0$ with the following property. Let G be a graph containing a set $Y \subseteq V(G)$ of size $3\theta - 2$ such that there exists no separation (C, D) of G of order less than θ such that $|V(C) \cap (V(D) \cup Y)| < 3\theta - 2$ and $|V(D) \cap (V(C) \cup Y)| < 3\theta - 2$. If H is not a minor of G, then there exist graphs $G_1, \ldots, G_k \subseteq G$ and a graph G_0 satisfying the following.*

- *$G \subseteq G_0 \cup G_1 \cup \ldots \cup G_k$.*
- *$V(G_i) \cap V(G_j) \subseteq V(G_0)$ for $1 \leq i \leq j \leq k$.*
- *$V(G_i) \cap V(G_0)$ induces a clique in G_0 and $|V(G_i) \cap V(G_0)| < 2\theta$, for $1 \leq i \leq k$.*
- *$Y \subseteq V(G_0)$ and Y induces a clique in G_0.*
- *for some surface Σ in that H cannot be drawn and a set $A \subseteq V(G_0)$ of size at most a,*
  - *$G_0 - A$ can be almost embedded in Σ with at most m vortices of depth at most d,*
  - *every triangle in the embedding bounds a 2-cell face, and*
  - *all but at most $a(H, \Sigma) - 1$ vertices of A are only adjacent to vertices contained either in A or in the vortices.*

Proof. Let θ_0, k_0, p_0, a_0 be the constants of Lemma 25 applied to H. Let $d = 2p_0 + 1$, $\theta = \max(\theta_0, a_0 + d + 1)$, $m = k_0 + 3\theta - 2$ and $a = a_0 + 3\theta - 2$.

Consider any separation (C, D) of G of order $r < \theta$. If $|V(C) \cap Y| \geq \theta$, then $|V(D) \cap (V(C) \cup Y)| = |(V(D) \setminus V(C)) \cap Y| + r = |Y| - |V(C) \cap Y| + r < 3\theta - 2$. Symmetrically, if $|V(D) \cap Y| \geq \theta$, then $|V(C) \cap (V(D) \cup Y)| < 3\theta - 2$. By the assumptions of the theorem, we have either $|V(C) \cap Y| \leq \theta - 1$ or

$|V(D) \cap Y| \leq \theta - 1$. Note that exactly one of the inequalities holds, since $|Y| > 2\theta - 2$. Let $\mathcal{T}$ consist of the separations of G of order less than θ such that $|V(C) \cap Y| \leq \theta - 1$. Then $\mathcal{T}$ is a tangle of order θ in G (see (11.2) of [18]).

We apply Lemma 25 to the tangle $\mathcal{T}$. Let $A_0 \subseteq V(G)$ be the set and S the $(\mathcal{T} - A_0)$-central segregation of $G - A_0$ of type (p_0, k_0) and let Σ be the surface in that S is arranged. Let $G_1, \ldots, G_k$ consist of the following graphs:

- for each cell $s \in S$, the subgraph of G induced by $V(s) \cup A_s$, where A_s is the set of vertices of A_0 that have a neighbor in $s - \partial s$.

- for each p_0-vortex $s \in S$ and for each bag B of the standard path decomposition of s, the subgraph of G induced by $B \cup A_0$.

Let G_0'' be the graph consisting of $T(S)$ and the cliques with vertex set $X(v)$ for each p_0-vortex $s \in S$ and each $v \in \partial s$. Note that G_0'' is embedded in Σ with at most k_0 vortices of depth d, and that $V(G_0'') \cap V(G_i)$ induces a clique in G_0'' for $1 \leq i \leq k$. Furthermore, every triangle in the embedded part of G_0'' bounds a 2-cell face.

Let G_0' be the graph obtained from G_0'' by adding a clique with vertex set A_0, for each cell $s \in S$ adding all edges between vertices of A_s and of ∂s, and for each vertex v of a vortex of G_0'', adding all edges between v and A. We have $G \subseteq G_0' \cup G_1 \cup \ldots \cup G_k$, $V(G_0') \cap V(G_i)$ induces a clique in G_0' for $1 \leq i \leq k$, and $G_0'' = G_0' - A_0$.

Let G_0 be the graph obtained from G_0' by adding a clique with vertex set Y and for each $i \in \{1, \ldots, k\}$ and each vertex $y \in Y$ adjacent to some vertex in $V(G_i) \setminus V(G_0')$, adding all edges between y and $V(G_i) \cap V(G_0')$. Let us now argue that $G_0, G_1, \ldots, G_k$ satisfy the conditions of the lemma, in order:

- This holds, since $G_0 \supseteq G_0'$ and $G \subseteq G_0' \cup G_1 \cup \ldots \cup G_k$.

- Consider $v \in V(G_i) \cap V(G_j)$. If $v \in A_0$, then $v \in V(G_0') \subseteq V(G_0)$. If $v \notin A_0$, then $v \in V(G_0'') \subseteq V(G_0)$, by the definition of a segregation and the properties of path decompositions of p_0-vortices.

- $V(G_i) \cap V(G_0)$ induces a clique in G_0 by the construction of G_0'', G_0' and G_0. Since the segregation S is $(\mathcal{T} - A_0)$-central and its order is greater than d, there exists a separation $(G_i - A_0, D'') \in \mathcal{T} - A_0$ of $G - A_0$ with $V((G_i - A_0) \cap D'') = V(G_i \cap G_0'')$. Consequently, there exists a separation $(G_i, D') \in \mathcal{T}$ of G with $V(G_i \cap D') = V(G_i \cap G_0')$. By the definition of $\mathcal{T}$, we have $|V(G_i) \cap Y| < \theta$. Consequently, $|V(G_i \cap G_0)| \leq |V(G_i) \cap Y| + |V(G_i \cap G'_0)| < 2\theta$.

- This is clear by the construction of G_0.
- We set $A = Y \cup A_0$. Consider the embedding of $G_0 - A = G''_0 - Y$ in Σ with at most k_0 vortices of depth d. Observe that for each $y \in Y$, there exists a face f_y in the embedded part of $G''_0 - Y$ such that all neigbors of y in the embedded part are incident with this face. Let $F = \{f_y : y \in Y\}$. For each face in F, introduce a new vortex to $G'' - Y$ consisting only of the vertices incident with F (and no edges). Such a vortex has depth $1 \leq d$.
  - By the construction, $G_0 - A$ is embedded in Σ with at most $k_0 + |Y| \leq m$ vortices of depth d.
  - This holds in G''_0 and cannot be violated by the removal of vertices of Y.
  - All but at most $a(H, \Sigma) - 1$ vertices of A_0 are only adjacent to vertices contained either in A_0 or in the p_0-vortices of S. In the construction of G'_0, we take care not to introduce a new edges to the embedded part from such apex vertices. In the last step, we introduce new vortices containing the neighbors of the vertices of Y. Consequently, all but at most $a(H, \Sigma) - 1$ vertices of A only have neigbhors in A or in the vortices.

□

The rest of the proof is straightforward.

Proof of Theorem 12. Let m, d, a_0 and θ be the constants of Lemma 26 applied to H, and let $a = \max(a, 4\theta - 2)$. We prove a stronger claim: for any set $Y \subseteq V(G)$ of size at most $3\theta - 2$, there exists a decomposition as in Theorem 12 such that $Y \subseteq V(G_1)$ and Y induces a clique in G_1.

If $|V(G)| < 3\theta - 2$, then we set $k = 1$, let G_1 be the clique with vertex set $V(G)$ and let $A_1 = V(G)$. Therefore, suppose that $|V(G)| \geq 3\theta - 2$, and thus we can add vertices to Y so that $|Y| = 3\theta - 2$.

If there exists a separation (C, D) of G such that $|V(C) \cap (V(D) \cup Y)| < 3\theta - 2$ and $|V(D) \cap (V(C) \cup Y)| < 3\theta - 2$, then we apply the stronger claim inductively to C with the set $V(C) \cap (V(D) \cup Y)$ and to D with the set $V(D) \cap (V(C) \cup Y)$ (note that $C \neq G \neq D$, since neither C nor D contains all vertices of Y). Then G is a clique-sum of the resulting pieces and of the piece G_1 consisting of a clique with vertex set $(V(C) \cap V(D)) \cup Y$. Note that $|V(G_1)| = |Y| + |V(C) \cap V(D)| \leq 4\theta - 2$, and thus we can set $A_1 = V(G_1)$.

Finally, if there exists no such separation, then we apply Lemma 26, obtaining graphs $G'_0, G'_1, \ldots, G'_k$. We apply the stronger claim inductively to $G'_1, \ldots, G'_k$, and obtain G as a clique-sum of the resulting pieces and of $G_1 = G'_0$. □

References

[1] K. Appel and W. Haken, *Every planar map is four colorable, Part I: discharging*, Illinois J. of Math., 21 (1977), pp. 429–490.

[2] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable, Part II: reducibility*, Illinois J. of Math., 21 (1977), pp. 491–567.

[3] E. Demaine, F. Fomin, M. Hajiaghayi, and D. Thilikos, *Subexponential parameterized algorithms on bounded-genus graphs and h-minor-free graphs*, J. ACM, 52 (2005), pp. 866–893.

[4] Z. Dvorak and K. ichi Kawarabayashi, *List-coloring embedded graphs*, in Khanna [12], pp. 1004–1012.

[5] P. Erdős, A. L. Rubin, and H. Taylor, *Choosability in graphs*, Congr. Numer., 26 (1980), pp. 125–157.

[6] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, WH Freeman & Co. New York, NY, USA, 1979.

[7] J. F. Geelen, R. B. Richter, and G. Salazar, *Embedding grids in surfaces*, European J. Combin., 25 (2004), pp. 785–792.

[8] M. Grohe, K. ichi Kawarabayashi, and B. A. Reed, *A simple algorithm for the graph minor decomposition - logic meets structural graph theory*, in Khanna [12], pp. 414–431.

[9] S. Gutner, *The complexity of planar graph choosability*, Discrete Math., 159 (1996), pp. 119–130.

[10] K. Jansen and P. Scheffler, *Generalized coloring for tree-like graphs*, Discrete Appl. Math., 75 (1997), pp. 135–155.

[11] K. Kawarabayashi and P. Wollan, *A simpler algorithm and shorter proof for the graph minor decomposition*, in STOC, L. Fortnow and S. P. Vadhan, eds., ACM, 2011, pp. 451–458.

[12] S. Khanna, ed., *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*, SIAM, 2013.

[13] D. Lichtenstein, *Planar formulae and their uses*, SIAM J. Comput., 11 (1982), pp. 329–343.

[14] L. Postle, *5-List-Coloring Graphs on Surfaces*, PhD thesis, Georgia Institute of Technology, 2012.

[15] N. Robertson, D. P. Sanders, P. Seymour, and R. Thomas, *The four colour theorem*, J. Combin. Theory, Ser. B, 70 (1997), pp. 2–44.

[16] N. Robertson and P. D. Seymour, *Graph Minors. V. Excluding a planar graph*, J. Combin. Theory, Ser. B, 41 (1986), pp. 92–114.

[17] ——, *Graph Minors. IX. Disjoint crossed paths*, J. Combin. Theory, Ser. B, 49 (1990), pp. 40–77.

[18] ——, *Graph Minors. X. Obstructions to tree-decomposition*, J. Combin. Theory, Ser. B, 52 (1991), pp. 153–190.

[19] ——, *Graph Minors. XI. Circuits on a surface*, J. Combin. Theory, Ser. B, 60 (1994), pp. 72–106.

[20] ——, *Graph Minors. XII. Distance on a surface*, J. Combin. Theory, Ser. B, 64 (1995), pp. 240–272.

[21] ——, *Graph Minors. XIV. Extending an embedding*, J. Combin. Theory, Ser. B, 65 (1995), pp. 23–50.

[22] ——, *Graph Minors. XV. Giant steps*, J. Combin. Theory, Ser. B, 68 (1996), pp. 112–148.

[23] ——, *Graph Minors: XVII. Taming a vortex*, J. Combin. Theory, Ser. B, 77 (1999), pp. 162–210.

[24] ——, *Graph Minors. XVI. Excluding a non-planar graph*, J. Combin. Theory, Ser. B, 89 (2003), pp. 43–76.

[25] ——, *Graph Minors. XX. Wagner's conjecture*, J. Combin. Theory, Ser. B, 92 (2004), pp. 325–357.

[26] C. Thomassen, *Every planar graph is 5-choosable*, J. Combin. Theory, Ser. B, 62 (1994), pp. 180–181.

[27] V. G. Vizing, *Vertex colorings with given colors (in russian)*, Metody Diskret. Analiz, Novosibirsk, 29 (1976), pp. 3–10.

[28] K. Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Mathematische Annalen, 114 (1937), pp. 570–590.

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