MINORS OF TWO-CONNECTED GRAPHS OF LARGE PATH-WIDTH[1]

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Abstract

Let P be a graph with a vertex v such that $P\backslash v$ is a forest, and let Q be an outerplanar graph. We prove that there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least p has a minor isomorphic to P or Q. This result answers a question of Seymour and implies a conjecture of Marshall and Wood.

1 Introduction

All *graphs* in this paper are finite and simple; that is, they have no loops or parallel edges. *Paths* and *cycles* have no "repeated" vertices or edges. A graph H is a *minor* of a graph G if we can obtain H by contracting edges of a subgraph of G. An H *minor* is a minor isomorphic to H. A tree-decomposition of a graph G is a pair (T, X), where T is a tree and X is a family $(X_t : t \in V(T))$ such that:

(W1) $\bigcup_{t\in\ V((T)} X_t = V(G)$, and for every edge of G with ends u and v there exists $t \in V(T)$ such that $u, v \in X_t$, and

(W2) if $t_1, t_2, t_3 \in V(T)$ and t_2 lies on the path in T between t_1 and t_3, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$.

The *width* of a tree-decomposition (T, X) is $\max\{|X_t| - 1 : t \in V(T)\}$. The *tree-width* of a graph G is the smallest width among all tree-decompositions of G. A *path-decomposition* of G is a tree-decomposition (T, X) of G, where T is a path. We will often denote a path-decomposition as $(X_1, X_2, \dots, X_n)$, rather than having the constituent sets indexed by the vertices of a path. The *path-width* of G is the smallest width among all path-decompositions of G. Robertson and Seymour [11] proved the following:

Theorem 1.1. *For every planar graph H there exists an integer $n = n(H)$ such that every graph of tree-width at least n has an H minor.*

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Robertson and Seymour [10] also proved an analogous result for path-width:

Theorem 1.2. *For every forest F, there exists an integer $p = p(F)$ such that every graph of path-width at least p has an F minor.*

Bienstock, Robertson, Seymour and the second author [1] gave a simpler proof of Theorem 1.2 and improved the value of p to $|V(F)| - 1$, which is best possible, because K_k has path-width $k - 1$ and does not have any forest minor on $k + 1$ vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [5].

While Geelen, Gerards and Whittle [7] generalized Theorem 1.1 to representable matroids, it is not *a priori* clear what a version of Theorem 1.2 for matroids should be, because excluding a forest in matroid setting is equivalent to imposing a bound on the number of elements and has no relevance to path-width. To overcome this, Seymour [4, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. Our main result answers Seymour's question in the affirmative:

Theorem 1.3. *Let P be a graph with a vertex v such that $P \backslash v$ is a forest, and let Q be an outerplanar graph. Then there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least p has a P or Q minor.*

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3, given a graph G, we may assume that G is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of G. Then the new graph is 2-connected, and by Theorem 1.3, it has a P or Q minor. By choosing suitable P and Q, we can get an F minor in G.

Marshall and Wood [8] define $g(H)$ as the minimum number for which there exists a positive integer $p(H)$ such that every $g(H)$-connected graph with no H minor has path-width at most $p(H)$. Then Theorem 1.2 implies that $g(H) = 0$ iff H is a forest. There is no graph H with $g(H) = 1$, because path-width of a graph G is the maximum of the path-widths of its connected components. Let A be the graph that consists of a cycle $a_1a_2a_3a_4a_5a_6a_1$ and extra edges a_1a_3, a_3a_5, a_5a_1. Let $C_{3,2}$ be the graph consisting of two disjoint triangles. In Section 2 we prove a conjecture of Marshall and Wood [8]:

Theorem 1.4. *A graph H has no $K_4, K_{2,3}$, $C_{3,2}$ or A minor if and only if $g(H) \leq 2$.*

In Section 3 we describe a special tree-decomposition, whose existence we establish in [3]. In Section 4 we introduce "cascades", our main tool, and prove that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an "injective" cascade of large height. In Section 5 we prove that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal. In Section 6 we analyze those minimal linkages and prove that there are essentially only two types of linkage. This is where we use the properties of tree-decompositions from Section 3. Finally, in Section 7 we convert the two types of linkage into the two families of graphs from Theorem 1.3.

2 Proof of Theorem 1.4

In this section we prove that Theorem 1.4 is implied by Theorem 1.3.

Definition Let $h \geq 0$ be an integer. By a *binary tree of height* h we mean a tree with a unique vertex r of degree two and all other vertices of degree one or three such that every vertex of degree one is at distance exactly h from r. Such a tree is unique up to isomorphism and so we will speak of the binary tree of height h. We denote the binary tree of height h by CT_h and we call r the *root* of CT_h. Each vertex in CT_h with distance k from r has *height* k. We call the vertices at distance h from r the *leaves of* CT_h. If t belongs to the unique path in CT_h from r to a vertex $t' \in V(T_h)$, then we say that t' is a *descendant* of t and that t is an *ancestor* of t'. If, moreover, t and t' are adjacent, then we say that t is the *parent* of t' and that t' is a *child* of t.

Let $\mathcal{P}_k$ be the graph consisting of CT_k and a separate vertex that is adjacent to every leaf of CT_k.

Lemma 2.1. *If a graph H has no $K_4, C_{3,2}$, or A minor, then H has a vertex v such that $H \backslash v$ is a forest.*

Proof. We proceed by induction on $|VH)|$. The lemma clearly holds when $|V(H)| = 0$, and so we may assume that H has at least one vertex and that the lemma holds for graphs on fewer than $|V(H)|$ vertices. If H has a vertex of degree at most one, then the lemma follows by induction by deleting such vertex. We may therefore assume that H has minimum degree at least two.

If H has a cutvertex, say v, then v is as desired, for if C is a cycle in $H \backslash v$, then $H \backslash V(C)$ also contains a cycle (because H has minimum degree at least two), and hence H has a $C_{3,2}$ minor, a contradiction. We may therefore assume that H is 2-connected.

We may assume that H is not a cycle, and hence it has an ear-decomposition $H = H_0 \cup H_1 \cup \cdots \cup H_k$, where $k \geq 1$, H_0 is a cycle and for $i = 1, 2, \ldots, k$ the graph H_i is a path with ends $u_i, v_i \in V(H_0 \cup H_1 \cup \cdots \cup H_{i-1})$ and otherwise disjoint from $H_0 \cup H_1 \cup \cdots \cup H_{i-1}$. If $u_1 \in \{u_i, v_i\}$ for all $i \in \{2, 3, \ldots, k\}$, then u_1 satisfies the conclusion of the lemma, and similarly for v_1. We may therefore assume that there exist $i, j \in \{2, 3, \ldots, k\}$ such that $u_1 \notin \{u_i, v_i\}$ and $v_1 \notin \{u_j, v_j\}$. It follows that H has a $K_4, C_{3,2}$, or A minor, a contradiction. □

Lemma 2.2. *If a graph H has a vertex v such that $H \backslash v$ is a forest. then there exists an integer k such that H is isomorphic to a minor of $\mathcal{P}_k$.*

Proof. Let v be such that $T := H \backslash v$ is a forest. We may assume, by replacing H by a graph with an H minor, that T is isomorphic to CT_t for some t, and that v is adjacent to every vertex of T. It follows that H is isomorphic to a minor of $\mathcal{P}_{2t}$, as desired. □

Definition Let $\mathcal{Q}_1$ be K_3. An arbitrary edge of $\mathcal{Q}_1$ will be designated as *base edge*. For $i \geq 2$ the graph $\mathcal{Q}_i$ is constructed as follows: Now assume that $\mathcal{Q}_{i-1}$ has already been defined, and let Q_1 and Q_2 be two disjoint copies of $\mathcal{Q}_{i-1}$ with base edges u_1v_1 and u_2v_2, respectively. Let T be a copy of K_3 with vertex-set $\{w_1, w_2, w\}$ disjoint from Q_1 and Q_2. The graph $\mathcal{Q}_i$ is obtained from $Q_1 \cup Q_2 \cup T$ by identifying u_1 with w_1, u_2 with w_2, and v_1 and v_2 with w. The edge w_1w_2 will be the *base edge* of $\mathcal{Q}_i$.

A graph is *outerplanar* if it has a drawing in the plane (without crossings) such that every vertex is incident with the unbounded face. A graph is a *near-triangulation* if it is drawn in the plane in such a way that every face except possibly the unbounded one is bounded by a triangle.

Let H and G be graphs. If G has an H minor, then to every vertex u of H there corresponds a connected subgraph of G, called the *node of* u.

Lemma 2.3. *Let H be a 2-connected outerplanar near-triangulation with k triangles. Then H is isomorphic to a minor of $\mathcal{Q}_k$. Furthermore, the minor inclusion can be chosen in such a way that for every edge $a_1a_2 \in E(H)$ incident with the unbounded face and for every $i \in \{1, 2\}$, the vertex w_i belongs to the node of a_i, where w_1w_2 is the base edge of $\mathcal{Q}_k$.*

Proof. We proceed by induction on k. The lemma clearly holds when $k = 1$, and so we may assume that H has at least two triangles and that the lemma holds for graphs with fewer than k triangles. The edge a_1a_2 belongs to a unique triangle, say a_1a_2c. The triangle a_1a_2c divides H into two near-triangulations H_1 and H_2, where the edge a_ic is incident with the unbounded face of H_i. Let $Q_1, Q_2, u_1, v_1, u_2, v_2, w_1, w_2$ be as in the definition of $\mathcal{Q}_k$. By the induction hypothesis the graph H_i is isomorphic to a minor of Q_i in such a way that the vertex u_i belongs to the node of a_i and the vertex v_i belongs to the node of c. It follows that H is isomorphic to $\mathcal{Q}_k$ in such a way that w_i belongs to the node of a_i. □

Lemma 2.4. *Let H be a graph that has no K_4 or $K_{2,3}$ minor. Then there exists an integer k such that H is isomorphic to a minor of $\mathcal{Q}_k$.*

Proof. It is well-known [6, Exercise 23] that the hypotheses of the lemma imply that H is outerplanar. We may assume, by replacing H by a graph with an H minor, that H is a 2-connected outerplanar near-triangulation. The lemma now follows from Lemma 2.3. □

Corollary 2.5. *Let H be a graph that has no K_4, $K_{2,3}$, $C_{3,2}$, or A minor. Then there exists an integer k such that H is isomorphic to a minor of $\mathcal{P}_k$ and H is isomorphic to a minor of $\mathcal{Q}_k$.*

Proof. This follows from Lemmas 2.1, 2.2 and 2.4. □

Proof of Theorem 1.4, assuming Theorem 1.3. To prove the"if" part notice that $\mathcal{P}_k$ and $\mathcal{Q}_k$ are 2-connected and have large path-width when k is large, because $\mathcal{Q}_k$ has a CT_{k-1} minor. There is no vertex v in A such that $A \backslash v$ is acyclic. So, A and $C_{3,2}$ are not minors of $\mathcal{P}_k$ for any k. The graph $\mathcal{Q}_k$ is outerplanar, so K_4 and $K_{2,3}$ are not minors of $\mathcal{Q}_k$ for any positive integer k. This means $g(H) \geq 3$ for $H \in \{K_4, K_{2,3}, C_{3,2}, A\}$. This proves the "if" part.

To prove the "only if" part, if H has no $K_4, K_{2,3}, C_{3,2}$ or A minor, then by Corollary 2.5 H is a minor of both $\mathcal{P}_k$ and $\mathcal{Q}_k$ for some k. Then $g(H) \leq 2$ by Theorem 1.3. □

3 A Special Tree-decomposition

In this section we review properties of tree-decompositions established in [3, 9, 12]. The proof of the following easy lemma can be found, for instance, in [12].

Lemma 3.1. *Let (T, Y) be a tree-decomposition of a graph G, and let H be a connected subgraph of G such that $V(H) \cap Y_{t_1} \neq \emptyset \neq V(H) \cap Y_{t_2}$, where $t_1, t_2 \in V(T)$. Then $V(H) \cap Y_t \neq \emptyset$ for every $t \in V(T)$ on the path between t_1 and t_2 in T.*

A tree-decomposition (T, Y) of a graph G is said to be *linked* if

(W3) for every two vertices t_1, t_2 of T and every positive integer k, either there are k disjoint paths in G between Y_{t_1} and Y_{t_2}, or there is a vertex t of T on the path between t_1 and t_2 such that $|Y_t| < k$.

It is worth noting that, by Lemma 3.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [12].

Lemma 3.2. *If a graph G admits a tree-decomposition of width at most w, where w is some integer, then G admits a linked tree-decomposition of width at most w.*

Let (T, Y) be a tree-decomposition of a graph G, let $t_0 \in V(T)$, and let B be a component of $T \backslash t_0$. We say that a vertex $v \in Y_{t_0}$ is *B-tied* if $v \in Y_t$ for some $t \in V(B)$. We say that a path P in G is *B confined* if $|V(P)| \geq 3$ and every internal vertex of P belongs to $\bigcup_{t \in V(B)} Y_t - Y_{t_0}$. We wish to consider the following three properties of (T, Y):

(W4) if t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$,

(W5) if $t_0 \in V(T)$ and B is a component of $T \backslash t_0$, then $\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset$,

(W6) if $t_0 \in V(T)$, B is a component of $T \backslash t_0$, and u, v are B-tied vertices in Y_{t_0}, then there is a B-confined path in G between u and v.

The following strengthening of Lemma 3.2 is proved in [9].

Lemma 3.3. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)-(W6).

We need one more condition, which we now introduce. Let T be a tree. If $t, t' \in V(T)$, then by $T[t, t']$ we denote the set of vertices belonging to the unique path in T from t to t'. A *triad in T* is a triple t_1, t_2, t_3 of vertices of T such that there exists a vertex t of T, called the *center*, such that t_1, t_2, t_3 belong to different components of $T \backslash t$. Let (T, W) be a tree-decomposition of a graph G, and let t_1, t_2, t_3 be a triad in T. The *torso of (T, W) at t_1, t_2, t_3* is the subgraph of G induced by the set $\bigcup W_t$, the union taken over all vertices $t \in V(T)$ such that either $t \in \{t_1, t_2, t_3\}$, or for all $i \in \{1, 2, 3\}$, t belongs to the component of $T \backslash t_i$ containing the center of t_1, t_2, t_3. We say that the triad t_1, t_2, t_3 is W*-separable* if, letting $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$, the graph obtained from the torso of (T, W) at t_1, t_2, t_3 by deleting X can be partitioned into three disjoint non-null graphs H_1, H_2, H_3 in such a way that for all distinct $i, j \in \{1, 2, 3\}$ and all $t \in T[t_j, t_0]$, $|V(H_i) \cap W_t| \geq |V(H_i) \cap W_{t_j}| = |W_{t_j} - X|/2 \geq 1$. (Let us remark that this condition implies that $|W_{t_1}| = |W_{t_2}| = |W_{t_3}|$ and $V(H_i) \cap W_{t_i} = \emptyset$ for $i = 1, 2, 3$.) The last property of a tree-decomposition (T, W) that we wish to consider is

(W7) if t_1, t_2, t_3 is a W–separable triad in T with center t, then there exists an integer $i \in \{1, 2, 3\}$ with $W_{t_i} \cap W_t - (W_{t_1} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset$.

The following is proven in [3].

Theorem 3.4. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)-(W7)*.*

This theorem is used to prove Theorem 1.3 in Section 7.

4 Cascades

In this section we introduce "cascades", our main tool. The main result of this section, Lemma 4.6, states that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an "injective" cascade of large height

Lemma 4.1. *Let p, w be two positive integers and let G be a graph of tree-width strictly less than w and path-width at least p. Then for every tree-decomposition (T, X) of G of width strictly less than w, the path-width of T is at least $\lfloor p/w \rfloor$.*

Proof. We will prove the contrapositive. Assume there exists a tree-decomposition (T, X) of G of width $< w$ such that the path-width of T is less than $\lfloor p/w \rfloor$. Because the path-width of T is less than $\lfloor p/w \rfloor$, there exists a path-decomposition $(Y_1, Y_2, ..., Y_s)$ of T with $|Y_i| \leq \lfloor p/w \rfloor$ for all i. We will construct a path-decomposition $(Z_1, Z_2, ..., Z_s)$ for G of width less than p. Set $Z_i = \bigcup_{y \in Y_i} X_y$ for every $i \in \{1, 2, ..., s\}$. For every vertex $v \in V(G)$, v belongs to at least one set X_t for some $t \in V(T)$. The vertex t of the tree T must be in Y_l for some $l \in \{1, 2, ..., s\}$, so $v \in X_t \subseteq Z_l$. Therefore, $\bigcup Z_i = V(G)$. Similarly, for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$. Therefore, $u, v \in Z_l$ for some $l \in \{1, 2, ..., s\}$.

Now, if a vertex $v \in V(G)$ belongs to both Z_a and Z_b for some $a, b \in \{1, 2, ..., s\}, a < b$, we will show that $v \in Z_c$ for all c such that $a < c < b$. Let c be an arbitrary integer satisfying $a < c < b$. The fact that $v \in Z_a$ implies $v \in X_{y_1}$ for some $y_1 \in Y_a$. Similarly, $v \in X_{y_2}$ for some $y_2 \in Y_b$. Let H be the set of vertices of T on the path from y_1 to y_2. Since $y_1 \in Y_a$ and $y_2 \in Y_b$, $H \cap Y_a \neq \emptyset \neq H \cap Y_b$. Hence, by Lemma 3.1 with $H = T$ and (T, Y) the path-decomposition $(Y_1, Y_2, ..., Y_s)$, we have $H \cap Y_c \neq \emptyset$. Let $t \in H \cap Y_c$, then $v \in X_t \subseteq Z_c$. So $(Z_1, Z_2, ..., Z_s)$ is a path-decomposition of G. Since the width of (T, X) is less than w, we have $|X_y| \leq w$ for every $y \in Y_i$, where $i \in \{1, 2, ..., s\}$. Therefore, $|Z_i| \leq w.\lfloor p/w \rfloor \leq p$ for every $i \in \{1, 2, ..., s\}$. Therefore, the width of $(Z_1, Z_2, ..., Z_s)$ is less than p, so the path-width of G is less than p, as desired. □

Let T, T' be trees. A *homeomorphic embedding of* T *into* T' is a mapping $\eta : V(T) \to V(T')$ such that

- η is an injection, and
- if tt_1, tt_2 are edges of T with a common end, and P_i is the unique path in T' with ends $\eta(t)$ and $\eta(t_i)$, then P_1 and P_2 are edge-disjoint.

We will write $\eta : T \hookrightarrow T'$ to denote that η is a homeomorphic embedding of T into T'. Since CT_a has maximum degree at most three, the following lemma follows from [8, Lemma 6].

Lemma 4.2. *Let* T *be a forest of path-width at least* $a \geq 1$*. Then there exists a homeomorphic embedding* $CT_{a-1} \hookrightarrow T$*.*

For every integer $h \geq 1$ we will need a specific type of tree, which we will denote by T_h. The tree T_h is obtained from CT_h by subdividing every edge not incident with a vertex of degree one exactly once, and adding a new vertex r' of degree one adjacent to the root r of CT_h. The vertices of T_h of degree three will be called *major*, and all the other vertices will be called *minor*. We say that r is the *major root* of T_h and that r' is the *minor root* of T_h. Each major vertex at distance $2k$ from r has *height* k, and each minor vertex at distance $2k$ from r' has *height* k.

If t belongs to the unique path in T_h from r' to a vertex $t' \in V(T_h)$, then we say that t' is a *descendant* of t and that t is an *ancestor* of t'. If, moreover, t and t' are adjacent, then we say that t is the *parent* of t' and that t' is a *child* of t. Thus every major vertex t has exactly three minor neighbors. Exactly one of those neighbors is an ancestor of t. The other two neighbors are descendants of t. We will assume that one of the two descendant neighbors is designated as the *left neighbor* and the other as the *right neighbor*. Let t_0, t_1, t_2 be the parent, left neighbor and right neighbor of t, respectively. We say that the ordered triple (t_0, t_1, t_2) is the *trinity at* t. In case we want to emphasize that the trinity is at t, we use the notation $(t_0(t), t_1(t), t_2(t))$.

Let $\eta : T \hookrightarrow T'$. We define $sp(\eta)$, the *span of* η, to be the set of vertices $t \in V(T')$ that lie on the path from $\eta(t_1)$ to $\eta(t_2)$ for some vertices $t_1, t_2 \in V(T)$.

Let $s > 0$ be an integer and let (T, X) be a tree-decomposition of a graph G. By a *cascade of height* h *and size* s *in* (T, X) we mean a homeomorphic embedding $\eta : T_h \hookrightarrow T$ such that $|X_{\eta(t)}| = s$ for every minor vertex $t \in V(T_h)$ and $|X_t| \geq s$ for every t in the span of η.

Lemma 4.3. *For any positive integer* h *and nonnegative integers* a, k*, the following holds. Let* $m = (a+2)h + a$*. Let* (T, X) *be a tree-decomposition of a graph* G *and let* $\phi : CT_m \hookrightarrow T$ *be a homeomorphic embedding such that* $|X_t| \geq k$ *for all* $t \in sp(\phi)$*. If for every* $t \in V(CT_m)$ *at height* $l \leq m - a$ *there exist a descendant* t' *of* t *at height* $l + a$ *and a vertex* $r \in T[\phi(t), \phi(t')]$ *such that* $|X_r| = k$*, then there exists a cascade* η *of height* h *and size* k *in* (T, X)*.*

Proof. By hypothesis there exist a vertex $x_0 \in V(CT_m)$ at height a and a vertex $u_0 \in V(T)$ on the path from the image under ϕ of the root of CT_m to $\phi(x_0)$ such that $|X_{u_0}| = k$. Let x be a child of x_0, and let x_1 and x_2 be the children of x. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $2a + 2$ that is a descendant of x_i and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let r be the major root of T_1, and let (t_0, t_1, t_2) be its trinity. We define $\eta_1 : T_1 \hookrightarrow T$ by $\eta_1(t_i) = u_i$ for $i = 0, 1, 2$ and $\eta_1(r) = \phi(x)$. Then η_1 is a cascade of height one and size k in (T, X). If $h = 1$, then η_1 is as desired, and so we may assume that $h > 1$.

Assume now that for some positive integer $l < h$ we have constructed a cascade $\eta_l : T_l \hookrightarrow T$ of height l and size k in (T, X) such that for every leaf t_0 of T_l other than the minor root there exists a vertex $x_0 \in V(CT_m)$ at height $(a + 2)l + a$ such that the image under η_l of every vertex on the path in T_l from the minor root to t_0 belongs to the path in T from the image under ϕ of the root of CT_m to $\phi(x_0)$. Our objective is to extend η_l to a cascade η_{l+1} of height $l + 1$ and size k in (T, X) with the same property. To that end let $\eta_{l+1}(t) = \eta_l(t)$ for all $t \in V(T_l)$, let t_0 be a leaf of T_l other than the minor root and let x_0 be as earlier in the paragraph. Let x be a child of x_0, and let x_1 and x_2 be the children of x. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $(a+2)(l+1)+a$ that is a descendant of x_i and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let r be the child of t_0 in T_{l+1}, and let (t_0, t_1, t_2) be its trinity. We define $\eta_{l+1}(t_i) = u_i$ for $i = 1, 2$ and $\eta_{l+1}(r) = \phi(x)$. This completes the definition of η_{l+1}.

Now η_h is as desired. □

Lemma 4.4. *For any two positive integers h and w, there exists a positive integer $p = p(h, w)$ such that if G is a graph of path-width at least p, then in any tree-decomposition of G of width less than w, there exists a cascade of height h.*

Proof. Let $a_{w+1} = 0$, and for $k = w, w - 1, \ldots, 0$ let $a_k = (a_{k+1} + 2)h + a_{k+1}$, and let $p = w(a_0 + 1)$. We claim that p satisfies the conclusion of the lemma. To see that let (T, X) be a tree-decomposition of G of width less than w. Let $k \in \{0, 1, \ldots, w + 1\}$ be the maximum integer such that there exists a homeomorphic embedding $\phi : CT_{a_k} \hookrightarrow T$ satisfying $|X_t| \geq k$ for all $t \in sp(\phi)$. Such an integer exists, because $k = 0$ satisfies those requirements by Lemmas 4.1 and 4.2, and it satisfies $k \leq w$, because the width of (T, X) is less than w. The maximality of k implies that for the integers h, k and a_{k+1} the hypothesis of Lemma 4.3 is satisfied. Thus the lemma follows from Lemma 4.3. □

Let (T, X) be a tree-decomposition of a graph G, and let $\eta : T_h \hookrightarrow T$ be a cascade of height h and size s in (T, X). We say that η is *injective* if there exists $I \subseteq V(G)$ such that $|I| < s$ and $X_{\eta(t)} \cap X_{\eta(t')} = I$ for every two distinct vertices $t, t' \in V(T_h)$. We call this set I the *common intersection set* of η.

Lemma 4.5. *Let a, b, s, w be positive integers and let k be a nonnegative integer. Let (T, X) be a tree-decomposition of a graph G of width strictly less than w. Let $h =(2(a + 2)w + 2)b$. If there is a cascade η of height h and size $s + k$ in (T, X) such that $|\bigcap_{t \in V(T_h)} X_{\eta(t)}| \geq k$, then either there is a cascade η' of height a and size $s + k$ in (T, X) such that $|\bigcap_{t \in V(T_a)} X_{\eta'(t)}| \geq k + 1$ or there is an injective cascade η' of height b, size $s + k$ and common intersection set of size k in (T, X).*

Proof. We may assume that

(∗) there does not exist a cascade η' of height a and size $s+k$ in (T,X) such that $|\bigcap_{t\in V(T_a)} X_{\eta'(t)}| \geq k+1$.

Let $F = \bigcap_{t\in V(T_h)} X_{\eta(t)}$. By (∗), $|F| = k$. We claim the following.

Claim 4.5.1. *For every vertex $t \in V(T_h)$ at height $l \leq h-a-2$ and every $u \in X_{\eta(t)} - F$ there exists a descendant $t' \in V(T_h)$ of t at height at most $l+a+2$ such that $u \notin X_{\eta(t')}$.*

To prove the claim let $u \in X_{\eta(t)} - F$. By (∗) in the subtree of T_h consisting of t and its descendants there is a vertex t' of height at most $l+a+2$ such that $u \notin X_{\eta(t')}$. This proves the claim.

We use the previous claim to deduce the following generalization.

Claim 4.5.2. *For every vertex $t \in V(T_h)$ at height $l \leq h-(a+2)w$ there exists a descendant $t' \in V(T)$ of t at height at most $l+(a+2)w$ such that $X_{\eta(t)} \cap X_{\eta(t')} = F$.*

To prove the claim let $X_{\eta(t)} \backslash F = \{u_1, u_2, \ldots, u_p\}$, where $p \leq w$. By Claim 4.5.1 there exists a descendant $t_1 \in V(T)$ of t at height at most $l+a+2$ such that $u_1 \notin X_{\eta(t')}$. By another application of Claim 4.5.1 there exists a descendant $t_2 \in V(T)$ of t_1 at height at most $l+2(a+2)$ such that $u_2 \notin X_{\eta(t')}$. By (W2) $u_1 \notin X_{\eta(t')}$. By continuing to argue in the same way we finally arrive at a vertex t_p that is a descendant of t at height at most $l+(a+2)p$ such that $X_{\eta(t)} \cap X_{\eta(t_p)} = F$. Thus t_p is as desired. This proves the claim.

Let $x_0 \in V(T_h)$ be the minor root of T_h. By Claim 4.5.2 and (W2) there exists a major vertex $x \in V(T)$ at height at most $(a+2)w+1$ such that $X_{\eta(x_0)} \cap X_{\eta(x)} = F$. Let y_1 and y_2 be the children of x. By Claim 4.5.2 and (W2) there exists, for $i = 1,2$, a minor vertex $x_i \in V(T_h)$ at height at most $2(a+2)w+2$ that is a descendant of y_i and such that $X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let r be the major root of T_1, and let (t_0,t_1,t_2) be its trinity. We define $\eta_1 : T_1 \hookrightarrow T$ by $\eta_1(t_i) = \eta(x_i)$ for $i = 0,1,2$ and $\eta_1(r) = \eta(x)$. Then η_1 is an injective cascade of height one and size $s+k$ in (T,X) with common intersection set F. If $b = 1$, then η_1 is as desired, and so we may assume that $b > 1$.

Assume now that for some positive integer $l < b$ we have constructed an injective cascade $\eta_l : T_l \hookrightarrow T$ of height l and size $s+k$ with common intersection set F in (T,X) such that for every leaf t_0 of T_l other than the minor root there exists a vertex $x_0 \in V(T_h)$ at height $(2(a+2)w+2)l$ such that the image under η_l of every vertex on the path in T_l from the minor root to t_0 belongs to the path in T from the image under η of the root of T_h to $\eta(x_0)$. Our objective is to extend η_l to an injective cascade η_{l+1} of height $l+1$, size $s+k$, and common intersection set F in (T,X) with the same property. To that end let $\eta_{l+1}(t) = \eta_l(t)$ for all $t \in V(T_l)$, let t_0 be a leaf of T_l other than the minor root, and let x_0 be as earlier in the paragraph. By Claim 4.5.2 and (W2) there exists a descendant x of x_0 at height at most $(2(a+2)w+2)l+(a+2)w+1$ such that x is major and $X_{\eta_l(t_0)} \cap X_{\eta(x)} = F$. Let y_1 and y_2 be the children of x. By Claim 4.5.2 and (W2) there exists, for $i = 1,2$, a minor vertex $x_i \in V(T_h)$ at height at most $(2(a+2)w+2)(l+1)$ that is a descendant of y_i and such that $X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let r be the child of t_0 in T_{l+1}, and let (t_0, t_1, t_2) be its trinity. We define $\eta_{l+1}(t_i) = \eta(x_i)$ for $i = 1, 2$ and $\eta_{l+1}(r) = \eta(x)$. This completes the definition of η_{l+1}.

Now η_b is as desired. □

Lemma 4.6. *For any two positive integers h and w, there exists a positive integer $p = p(h, w)$ such that if G is a graph of tree-width less than w and path-width at least p, then in any tree-decomposition (T, X) of G that has width less than w and satisfies (W4), there is an injective cascade of height h.*

Proof. Let $a_w = 0$, and for $k = w-1, \ldots, 0$ let $a_k = (2(a_{k+1}+2)w+2)h$. Let p be the integer in Lemma 4.4 for input integers a_0 and w. We claim that p satisfies the conclusion of the lemma. To see that let (T, X) be a tree-decomposition of G of width less than w satisfying (W4). By Lemma 4.4, there exists a cascade η of height a_0 in (T, X). Let $k \in \{0, 1, \ldots, w\}$ be the maximum integer such that there exists a cascade $\eta' : T_{a_k} \hookrightarrow T$ satisfying $|\bigcap_{t \in V(T_{a_k})} X_{\eta'(t)}| \geq k$. Such an integer exists, because $k = 0$ satisfies those requirements and $k < w$ because of (W4) and because the width of (T, X) is less than w. The maximality of k implies that there does not exist a cascade $\eta'' : T_{a_{k+1}} \hookrightarrow T$ satisfying $|\bigcap_{t \in V(T_{a_{k+1}})} X_{\eta''(t)}| \geq k+1$. Thus the lemma follows from Lemma 4.5. □

5 Ordered Cascades

The main result of this section, Theorem 5.5, states that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal.

Let (T, X) be a tree-decomposition of a graph G, and let η be an injective cascade in (T, X) with common intersection set I. Assume the size of η is $|I| + s$. Then we say η is *ordered* if for every minor vertex $t \in V(T_h)$ there exists a bijection $\xi_t : \{1, 2, \ldots, s\} \to X_{\eta(t)} - I$ such that for every major vertex t_0 with trinity (t_1, t_2, t_3), there exist s disjoint paths $P_1, P_2, \ldots, P_s$ in $G \backslash I$ such that the path P_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, and there exist s disjoint paths $Q_1, Q_2, \ldots, Q_s$ in $G \backslash I$ such that the path Q_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$. In that case we say that η is an *ordered cascade with orderings ξ_t*. We say that the set of paths $P_1, P_2, \ldots, P_s$ is a *left t_0-linkage with respect to η*, and that the set of paths $Q_1, Q_2, \ldots, Q_s$ is a *right t_0-linkage with respect to η*.

We will need to fix a left and a right t_0-linkage for every major vertex $t_0 \in V(T_h)$; when we do so we will indicate that by saying that η is an *ordered cascade in (T, X) with orderings ξ_t and specified linkages*, and we will refer to the specified linkages as the *left specified t_0-linkage* and the *right specified t_0-linkage*. We will denote the left specified t_0-linkage by $P_1(t_0), P_2(t_0), \ldots, P_s(t_0)$ and the right specified t_0-linkage by $Q_1(t_0), Q_2(t_0), \ldots, Q_s(t_0)$. We say that the specified t_0-linkages are *minimal* if for every set of disjoint paths $P_1, P_2, \ldots, P_s$ in $G \backslash I$ from $X_{\eta(t_1)} - I$ to $X_{\eta(t_2)} - I$ such that $\xi_{t_1}(i)$ is an end of P_i (let the other end be p_i) and every set of disjoint paths $Q_1, Q_2, \ldots, Q_s$ in $G \backslash I$ from $X_{\eta(t_1)} - I$ to $X_{\eta(t_3)} - I$ such that $\xi_{t_1}(i)$ is an end of Q_i (let the other end be q_i) we have

$$\left| E\left(\bigcup (x_i P_i p_i \cup x_i Q_i q_i) \right) \right| \geq \left| E\left(\bigcup (y_i P_i(t_0) \xi_{t_2}(i) \cup y_i Q_i(t_0) \xi_{t_3}(i)) \right) \right|, \qquad (1)$$

where the unions are taken over $i \in \{1, 2, ..., s\}$, x_i is the first vertex from $\xi_{t_1}(i)$ that P_i departs from Q_i, and y_i is the first vertex from $\xi_{t_1}(i)$ that $P_i(t_0)$ departs from $Q_i(t_0)$.

Lemma 5.1. *Let h and s be two positive integers, and let $\eta : T_h \hookrightarrow T$ be an injective cascade of height h and size s in a linked tree-decomposition (T, X) of a graph G. Then the cascade η can be turned into an ordered cascade with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.*

Proof. Let $s' := s - |I|$. To show that η can be made ordered let r be the minor root of T_h, let $\xi_r : \{1, 2, \ldots, s'\} \to X_{\eta(r)} - I$ be arbitrary, assume that for some integer $l \in \{0, 1, \ldots, h-1\}$ we have already constructed $\xi_t : \{1, 2, \ldots, s'\} \to X_{\eta(t)} - I$ for all minor vertices $t \in V(T_h)$ at height at most l, let $t \in V(T_h)$ be a minor vertex at height exactly l, let t_0 be its child, and let (t, t_1, t_2) be the trinity at t_0. By condition (W3) there exist s' disjoint paths $P_1, P_2, \ldots, P_{s'}$ in $G \backslash I$ from $X_{\eta(t)} - I$ to $X_{\eta(t_1)} - I$ and s' disjoint paths $Q_1, Q_2, \ldots, Q_{s'}$ in $G \backslash I$ from $X_{\eta(t)} - I$ to $X_{\eta(t_2)} - I$. We may assume that $\xi_t(i)$ is an end of P_i and Q_i and we define $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$ to be their other ends, respectively. We may also assume that these paths satisfy the minimality condition (1). It follows that η is an ordered cascade with orderings ξ_t and specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. □

Let h, h' be integers. We say that a homeomorphic embedding $\gamma : T_{h'} \hookrightarrow T_h$ is *monotone* if

- t is a major vertex of $T_{h'}$ with trinity (t_1, t_2, t_3), then $\gamma(t_2)$ is the left neighbor of $\gamma(t)$ and $\gamma(t_3)$ is the right neighbor of $\gamma(t)$, and
- the image under γ of the minor root of $T_{h'}$ is the minor root of T_h.

Lemma 5.2. *For every two integers $a \geq 1$ and $k \geq 1$ there exists an integer $h = h(a, k)$ such that the following holds. Color the major vertices of T_h using k colors. Then there exists a monotone homeomorphic embedding $\eta : T_a \hookrightarrow T_h$ such that the major vertices of T_a map to major vertices of the same color in T_h.*

Proof. Let c be one of the colors. We will prove by induction on k and subject to that by induction on b that there is a function $h = g(a, b, k)$ such that there is either a monotone homeomorphic embedding $\eta : T_a \hookrightarrow T_h$ such that the major vertices of T_a map to major vertices of the same color in T_h, or a monotone homeomorphic embedding $\eta : T_b \hookrightarrow T_h$ such that the major vertices of T_b map to major vertices of color c in T_h. In fact, we will show that $g(a, b, 1) = a$, $g(a, 1, k+1) \leq g(a, a, k)$ and $g(a, b+1, k+1) \leq g(a, b, k+1) + g(a, a, k)$.

The assertion holds for $k = 1$ by letting $h = a$ and letting η be the identity mapping. Assume the statement is true for some $k \geq 1$, let the major vertices of T_h be colored using $k + 1$ colors, and let c be one of the colors. If $b = 1$, then if T_h has a major vertex colored c, then the second alternative holds; otherwise at most k colors are used and the assertion follows by induction on k.

We may therefore assume that the assertion holds for some integer $b \geq 1$ and we must prove it for $b + 1$. To that end we may assume that T_h has a major vertex t_0 colored c

at height at most $g(a,a,k)$, for otherwise the assertion follows by induction on k. Let the trinity at t_0 be (t_1,t_2,t_3). For $i=2,3$ let R_i be the subtree of T_h with minor root t_i. If for some $i \in \{2,3\}$ there exists a monotone homeomorphic embedding $T_a \hookrightarrow R_i$ such that the major vertices of T_a map to major vertices of the same color in T_h, then the statement holds. We may therefore assume that for $i \in \{2,3\}$ there exists a monotone homeomorphic embedding $\eta_i : T_b^i \hookrightarrow R_i$ such that the major vertices of T_b^i map to major vertices of color c, the major root of T_{b+1} is r_0, the trinity at r_0 is (r_1,r_2,r_3) and T_b^i is the subtree of $T_{b+1} - \{r_0,r_1\}$ with minor root r_i. Let $\eta : T_{b+1} \hookrightarrow T_h$ be defined by $\eta(t) = \eta_i(t)$ for $t \in V(T_b^i)$, $\eta(r_0) = t_0$ and $\eta(r_1)$ is defined to be the minor root of T_h. Then $\eta : T_{b+1} \hookrightarrow T_h$ is as desired. This proves the existence of the function $g(a,b,k)$.

Now $h(a,k) = g(a,a,k)$ is as desired. □

Let G be a graph, let $v \in V(G)$ and for $i=1,2,3$ let P_i be a path in G with ends v and v_i such that the paths P_1,P_2,P_3 are pairwise disjoint, except for v. Assume that at least two of the paths P_i have length at least one. We say that $P_1 \cup P_2 \cup P_3$ is a *tripod* with *center* v and *feet* v_1,v_2,v_3.

Let (T,X) be a tree-decomposition of a graph G, and let $\eta : T_h \hookrightarrow T$ be an injective cascade in (T,X) with common intersection set I. Let $t_0 \in V(T_h)$ be a major vertex, and let (t_1,t_2,t_3) be the trinity at t_0. We define the *η-torso at t_0* as the subgraph of G induced by $\bigcup X_t - I$, where the union is taken over all t in $V(T)$ such that the unique path in T from t to $\eta(t_0)$ does not contain $\eta(t_1)$,$\eta(t_2)$, or $\eta(t_3)$ as an internal vertex.

Let $s > 0$ be an integer. Let (T,X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T,X) of size $|I|+s$ and with orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1,t_2,t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i,j \in \{1,2,\dots,s\}$ be distinct. We say that t_0 *has property A_{ij} in η* if there exist disjoint tripods L_i,L_j in G' such that for each $m \in \{i,j\}$ the tripod L_m has feet $\xi_{t_1}(m), \xi_{t_2}(m_2), \xi_{t_3}(m_3)$ for some $m_2,m_3 \in \{i,j\}$.

We say that t_0 *has property B_{ij} in η* if there exist vertices $v_{x,y}$ for all $x \in \{i,j\}, y \in \{1,2,3\}$, and tripods L_i,L_j in G' with centers c_i,c_j such that

- for each $y \in \{1,2,3\}$, $\{v_{i,y}, v_{j,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j)\}$
- for each $m \in \{i,j\}$, L_m has feet $v_{m,1}, v_{m,2}, v_{m,3}$
- $L_i \cap L_j = c_i L_i v_{i,3} \cap c_j L_j v_{j,2}$ and it is a path that does not contain c_i, c_j.

We say that t_0 *has property C_{ij} in η* if there exist three pairwise disjoint paths R_i, R_j, R_{ij} and a path R in G' such that the ends of R_i are $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, the ends of R_j are $\xi_{t_1}(j)$ and $\xi_{t_3}(j)$, the ends of R_{ij} are $\xi_{t_2}(j)$ and $\xi_{t_3}(i)$, and R is internally disjoint from R_i, R_j, R_{ij} and connects two of these three paths. We will denote these paths as $R_i(t_0), R_j(t_0), R_{ij}(t_0), R(t_0)$ when we want to emphasize they are in the torso at the major vertex t_0.

We say that the path P_i of a left or right t_0-linkage is *confined* if it is a subgraph of the η-torso at t_0.

Now let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T,X) with orderings ξ_t and specified linkages. Let $t_0 \in V(T_h)$ be a major vertex with trinity (t_1,t_2,t_3), and let $P_1,P_2,\dots,P_s$

be the left specified t_0-linkage. We define A_{t_0} to be the set of integers $i \in \{1, 2, \dots, s\}$ such that the path P_i is confined, and we define B_{t_0} in the same way but using the right specified t_0-linkage instead. Define C_{t_0} as the set of all triples (i, l, m) such that $i \in \{1, 2, \dots, s\}$, the path P_i is not confined and when following P_i from $\xi_{t_1}(i)$, it exits the η-torso at t_0 for the first time at $\xi_{t_3}(l)$ and re-enters the η-torso at t_0 for the last time at $\xi_{t_3}(m)$. Let D_{t_0} be defined similarly, but using the right t_0-linkage instead. We call the sets A_{t_0}, B_{t_0}, C_{t_0} and D_{t_0} the *confinement sets for η at t_0 with respect to the specified linkages.*

Let A_{t_0} and B_{t_0} be the confinement sets for η at t_0. We say that t_0 *has property C in η* if s is even, A_{t_0} and B_{t_0} are disjoint and both have size $s/2$, and there exist disjoint paths $R_1, R_2, \dots, R_{3s/2}$ in G' in such a way that

- each R_i is a subpath of both the left specified t_0-linkage and the right specified t_0-linkage,
- for $i \in A_{t_0}$, the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$,
- for $i \in B_{t_0}$ the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$, and
- for $i = s+1, s+2, \dots, 3s/2$ the path R_i has one end $\xi_{t_2}(k)$ and the other and $\xi_{t_3}(l)$ for some $k \in B_{t_0}$ and $l \in A_{t_0}$.

Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be a cascade in (T, X) and let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding. Then the composite mapping $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ is a cascade in (T, X) of height h', and we will call it a *subcascade of η*.

Lemma 5.3. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t, specified linkages and common intersection set I, let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding, and let $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ be a subcascade of η of height h'. Then for every major vertex $t_0 \in V(T_{h'})$*

(i) *η' is an ordered cascade with orderings $\xi_{\gamma(t)}$ and common intersection set I,*

(ii) *if the vertex $\gamma(t_0)$ has property A_{ij} (B_{ij}, C_{ij}, resp.) in η, then t_0 has property A_{ij} (B_{ij}, C_{ij}, resp.) in η'.*

Furthermore, the specified linkages for η' may be chosen in such a way that

(iii) $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)})$,

(iv) *the vertex t_0 has property C in η' if and only if $\gamma(t_0)$ has property C in η, and*

(v) *if the specified linkages for η are minimal, then the specified linkages for η' are minimal.*

Proof. For each major vertex $t \in V(T_{h'})$ or $t \in V(T_h)$ we denote its trinity by $(t_1(t), t_2(t), t_3(t))$. Assume t_0 is a major vertex of $T_{h'}$. Let $v_0 = \gamma(t_1(t_0)), v_1, \ldots, v_k = t_1(\gamma(t_0))$ be the minor vertices on $T_h[v_0, v_k]$. Let U be the union of the left (or right) linkage from $X_{\eta(v_i)} - I$ to $X_{\eta(v_{i+1})} - I$ for all $i \in \{0, 1, \ldots, k-1\}$ depending on whether v_{i+1} is a left (or right) neighbor of its parent. Let P be the left specified $\gamma(t_0)$-linkage and Q be the right specified $\gamma(t_0)$-linkage. Then $U \cup P$ is a left t_0-linkage and $U \cup Q$ is a right t_0-linkage. We designate $U \cup P$ to be the left specified t_0-linkage and $U \cup Q$ to be the right specified t_0-linkage. It is easy to see that this choice satisfies the conclusion of the lemma. □

Let (T, X) be a tree-decomposition of a graph G, and let η be an ordered cascade with specified linkages in (T, X) of height h and size $|I| + s$, where I is the common intersection set. We say that η is *regular* if there exist sets $A, B \subseteq \{1, 2, \ldots, s\}$, and sets C and D such that the confinement sets A_{t_0}, B_{t_0}, C_{t_0} and D_{t_0} satisfy $A_{t_0} = A$, $B_{t_0} = B$, $C_{t_0} = C$ and $D_{t_0} = D$ for every major vertex $t_0 \in V(T_h)$.

Lemma 5.4. *For every two positive integers a and s there exists a positive integer $h = h(a, s)$ such that the following holds. Let (T, X) be a linked tree-decomposition of a graph G. If there exists an injective cascade η of height h in (T, X), then there exists a regular cascade $\eta' : T_a \hookrightarrow T$ of height a in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_a)$ such that η' has the same size and common intersection set as η.*

Proof. Let η be an injective cascade of size $|I| + s$ and height h in (T, X), where we will specify h in a moment. By Lemma 5.1 η can be turned into an ordered cascade with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. For every major vertex $t_0 \in V(T_h)$, the number of possible quadruples $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ is a finite number $k = k(s)$ that depends only on s.

Consider each choice of $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ as a color; then by Lemma 5.2, there exists a positive integer $h = h(a, k)$ such that there exists a monotone homeomorphic embedding $\gamma : T_a \hookrightarrow T_h$ such that the quadruple $(A_{\gamma(t)}, B_{\gamma(t)}, C_{\gamma(t)}, D_{\gamma(t)})$ for η is the same for every $t \in V(T_a)$. Now, let $\eta' = \eta \circ \gamma : T_a \to T$. Then η' is as desired by Lemma 5.3. □

The following is the main result of this section.

Theorem 5.5. *For any two positive integers a and w, there exists a positive integer $p = p(a, w)$ such that the following holds. Let G be a 2-connected graph of tree-width less than w and path-width at least p. Then G has a tree-decomposition (T, X) such that:*

- *(T, X) has width less than w,*
- *(T, X) satisfies (W1)–(W7), and*
- *for some s, where $2 \leq s \leq w$, there exists a regular cascade $\eta : T_a \hookrightarrow T$ of height a and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_a)$.*

Proof. Given positive integers a and w let h be as in Lemma 5.4, and let $p = p(h, w)$ be as in Lemma 4.6. We claim that p satisfies the conclusion of the theorem. To see that let G be a graph of tree-width less than w and path-width at least p. By Theorem 3.4, G admits a tree-decomposition (T, X) of width less than w satisfying (W1)–(W7). By Lemma 4.6 there is an injective cascade of height h in (T, X). Let s be the size of this cascade, then $s \leq w$. If G is 2-connected, then $s \geq 2$. The last conclusion of the theorem follows from Lemma 5.4. □

6 Taming Linkages

Lemma 6.6, the main result of this section, states that there are essentially only two types of linkage.

Let $s > 0$ be an integer. Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) of size $|I| + s$ and with orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1, t_2, t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. We say that t_0 *has property* AB_{ij} *in* η if there exist disjoint paths L_i, L_j and disjoint paths R_i, R_j in G' such that the two ends of L_m are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for each $m \in \{i, j\}$ and the two ends of R_m are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for each $m \in \{i, j\}$.

If P is a path and $u, v \in V(P)$, then by uPv we denote the subpath of P with ends u and v.

Lemma 6.1. *Let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_1 \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height one and size $s + |I|$, where I is the common intersection set. Let t_0 be the major vertex in T_1, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. If t_0 has property AB_{ij} in η, then t_0 has either property A_{ij} or property B_{ij} in η.*

Proof. Let (t_1, t_2, t_3) be the trinity at t_0. Let G' be the η-torso at t_0. Since t_0 has property AB_{ij} in η, there exist disjoint paths L_i, L_j and disjoint paths R_i, R_j in G' such that two endpoints of L_m are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for all $m \in \{i, j\}$, and two endpoints of R_m are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for all $m \in \{i, j\}$.

We may choose L_i, L_j, R_i, R_j such that $|E(L_i) \cup E(L_j) \cup E(R_i) \cup E(R_j)|$ is as small as possible.

Let $x_k = \xi_{t_1}(k)$ and $z_k = \xi_{t_3}(k)$ for $k \in \{i, j\}$. Starting from z_i, let a be the first vertex where R_i meets $L_i \cup L_j$, and starting from z_j, let b be the first vertex where R_j meets $L_i \cup L_j$. If a and b are not on the same path (one on L_i and the other on L_j), then by considering L_i, L_j and the parts of R_i and R_j from z_i to a and from z_j to b we see that t_0 has property A_{ij} in η.

If a and b are on the same path, then we may assume they are on L_i. We may also assume that $a \in L_i[y_i, b]$. Then following R_i from a away from z_i, the paths R_i and L_i eventually split; let c be the vertex where the split occurs. In other words, c is such that $aL_ic \cap aR_ic$ is a path and its length is maximum. Let d be the first vertex on $cR_ix_i \cap (L_i \cup L_j) - \{c\}$ when traveling on R_i from c to x_i. If $d \in V(L_i)$, then by replacing

cL_id by cR_id we obtain a contradiction to the choice of L_i, L_j, R_i, R_j. Thus $d \in V(L_j)$. Now L_i, L_j and the paths z_iR_id and z_jR_jb show that t_0 has property B_{ij} in η. □

Let (T, X) be a tree-decomposition of a graph G and let $\eta : T_h \hookrightarrow T$ be an injective cascade in (T, X) of height h and size $|I| + s$, where I is the common intersection set. Let v be a vertex of T_h and let Y consist of $\eta(v)$ and the vertex-sets of all components of $T\backslash\eta(v)$ that do not contain the image under η of the minor root of T_h. Let H be the subgraph of G induced by $\bigcup_{t\in Y} X_t - I$. We will call H the *outer graph at* v.

Lemma 6.2. *Let (T, X) be a tree-decomposition satisfying (W6) of a graph G and let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) of height h and size $|I| + s$, where I is the common intersection set. Let v be a minor vertex of T_h at height at most $h - 1$, let H be the outer graph at v, and let $x, y \in X_{\eta(v)}$. Then there exists a path of length at least two with ends x and y and every internal vertex in $V(H) - X_{\eta(v)}$.*

Proof. Let v_0 be the child of v, let v_1 be a child of v_0, and let B be the component of $T - \eta(v)$ that contains $\eta(v_1)$. We show that x is B-tied. This is obvious if $x \in I$, and so we may assume that $x \notin I$. Since η is ordered, there exist s disjoint paths from $X_{\eta(v)} - I$ to $X_{\eta(v_1)} - I$ in $G\backslash I$. It follows that each of the paths uses exactly one vertex of $X_{\eta(v)} - I$, and that vertex is its end. Let P be the one of those paths that ends in x, and let x' be the neighbor of x in P. The vertex x' exists, because $X_{\eta(v)} \cap X_{\eta(v_1)} = I$. By (W1) there exists a vertex $t \in V(T)$ such that $x, x' \in X_t$. Since $P - x$ is disjoint from $X_{\eta(v)}$, it follows from Lemma 3.1 applied to the path $P - x$ and vertices t and $\eta(v_1)$ of T that $t \in V(B)$. Thus x is B-tied and the same argument shows that so is y. Hence the lemma follows from (W6). □

We will refer to a path as in Lemma 6.2 as a W6-*path.*

Let h, h' be integers. We say that a homeomorphic embedding $\gamma : T_{h'} \hookrightarrow T_h$ is *weakly monotone* if for every two vertices $t, t' \in V(T_{h'})$

- if t' is a descendant of t in $T_{h'}$, then the vertex $\gamma(t')$ is a descendant of $\gamma(t)$ in T_h
- if t is a minor vertex of $T_{h'}$, then the vertex $\gamma(t)$ is minor in T_h.

Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be a cascade in (T, X) and let $\gamma : T_{h'} \hookrightarrow T_h$ be a weakly monotone homeomorphic embedding. Then the composite mapping $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ is a cascade in (T, X) of height h', and we will call it a *weak subcascade of* η.

Lemma 6.3. *Let $s \geq 2$ be an integer, let (T, X) be a tree-decomposition of a graph G satisfying (W6), and let $\eta : T_5 \hookrightarrow T$ be a regular cascade in (T, X) of height five and size $|I| + s$ with specified linkages that are minimal, where I is the common intersection set of η. Then either there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A_{ij} or B_{ij} in η' for some distinct integers $i, j \in \{1, 2, \ldots, s\}$, or the major root of T_5 has property C in η.*

Proof. We will either construct a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_5$ such that in $\eta' = \eta \circ \gamma$ the major root of T_1 will have property AB_{ij} for some distinct $i, j \in \{1, 2, ..., s\}$, or establish that the major root of T_5 has property C in η. By Lemma 6.1 this will suffice.

Since η is regular, there exist sets A, B, C, D as in the definition of regular cascade. Let t_0 be the unique major vertex of T_1 and let (t_1, t_2, t_3) be its trinity. Let u_0 be the major root of T_5 and let (v_1, v_2, v_3) be its trinity. Let u_1 be the major vertex of T_5 of height one that is adjacent to v_3 and let (v_3, v_4, v_5) be its trinity. Let us recall that for a major vertex u of T_5 we denote the paths in the specified left u-linkage by $P_i(u)$ and the paths in the specified right u-linkage by $Q_i(u)$. If there exist two distinct integers $i, j \in A \cap B$, then the paths $P_i(u_0), P_j(u_0), Q_i(u_0), Q_j(u_0)$ show that u_0 has property AB_{ij} in η. Let $\gamma : T_1 \hookrightarrow T_5$ be the homeomorphic embedding that maps t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_3, respectively. Then $\eta' = \eta \circ \gamma$ is as desired. We may therefore assume that $|A \cap B| \leq 1$.

For $i \in \{1, 2, \ldots, s\} - A$ the path $P_i(u_0)$ exits and re-enters the η-torso at u_0, and it does so through two distinct vertices of $X_{\eta(v_3)}$. But $|X_{\eta(v_3)} - I| = s$, and hence $|A| \geq s/2$. Similarly $|B| \geq s/2$. By symmetry we may assume that $|B| \geq |A|$. It follows that $|A| = \lceil s/2 \rceil$, and hence for $i \in \{1, 2, \ldots, s\} - A$ and every major vertex w of T_5 the path $P_i(w)$ exits and re-enters the η-torso at w exactly once. The set C includes an element of the form (i, l, m), which means that the vertices $\xi_{w_1}(i), \xi_{w_3}(l), \xi_{w_3}(m), \xi_{w_2}(i)$ appear on the path $P_i(w)$ in the order listed. Let $l_i := l, m_i := m$, $x_i(w) := \xi_{w_3}(l)$, $y_i(w) := \xi_{w_3}(m)$, $X_i(w) := \xi_{w_1}(i)P_i(w)x_i(w)$ and $Y_i(w) := y_i(w)P_i(w)\xi_{w_2}(i)$. Thus $X_i(w)$ and $Y_i(w)$ are subpaths of the η-torso at w. We distinguish two main cases.

Main case 1: $|A \cap B| = 1$. Let j be the unique element of $A \cap B$. We claim that $B - A \neq \emptyset$. To prove the claim suppose for a contradiction that $B \subseteq A$. Thus $|B| = 1$, and since $|B| \geq |A|$ we have $|A| = 1$, and hence $s = 2$. We may assume, for the duration of this paragraph, that $A = B = \{1\}$. The paths $P_1(u_0), X_2(u_0), Y_2(u_0)$ are pairwise disjoint, because they are subgraphs of the specified left u_0-linkage. The path $Q_2(u_0)$ is unconfined, and hence it has a subpath R joining $\xi_{v_2}(1)$ and $\xi_{v_2}(2)$ in the outer graph at v_2. It follows that $P_1(u_0) \cup R \cup Y_2(u_0)$ and $X_2(u_0)$ are disjoint paths from $X_{\eta(v_1)}$ to $X_{\eta(v_3)}$, and it follows from the minimality of the specified u_0-linkage that they form the specified right u_0-linkage, contrary to $1 \in A$. This proves the claim that $B - A \neq \emptyset$, and so we may select an element $i \in B - A$.

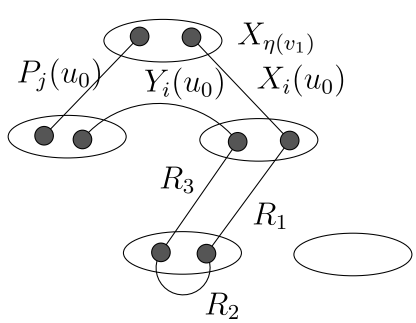

Figure 1: First case of the construction of the path R.

Let us assume as a case that either $l_i \in A$ or $l_i \notin B$. In this case we let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_5, respectively, and we will prove that t_0 has property AB_{ij} in η'. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cup Q_i(u_1)$ and $Q_j(u_0) \cup Q_j(u_1)$. The second pair will consist of $P_j(u_0)$ and another path from $\xi_{v_1}(i)$ to $\xi_{v_2}(i)$ which is a subgraph of a walk that we are about to construct. It will consist of $X_i(u_0) \cup Y_i(u_0)$ and a walk R in the outer graph of v_3 with ends $x_i(u_0)$ and $y_i(u_0)$. To construct the walk R we will construct paths R_1, R_2 and a walk R_3, whose union will contain the desired walk R. If $l_i \in A$, then we let $R_1 := P_{l_i}(u_1)$. If $l_i \notin B$, then the path $Q_{l_i}(u_1)$ is unconfined, and hence includes a subpath R_1 from $x_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the η-torso at u_1. We need to distinguish two subcases depending on whether $m_i \in B$. Assume first that $m_i \notin B$ and refer to Figure 1. Then similarly as above the path $Q_{m_i}(u_1)$ is unconfined, and hence includes a subpath R_3 from $y_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the η-torso at u_1, and we let R_2 be a W6-path in the outer graph at v_4 joining the ends of R_1 and R_3 in $X_{\eta(v_4)}$. This completes the subcase $m_i \notin B$, and so we may assume that $m_i \in B$. In this subcase we define $R_3 := Y_i(u_1) \cup Q_{m_i}(u_1)$ and we define R_2 as above. See Figure 2. This completes the case that either $l_i \in A$ or $l_i \notin B$.

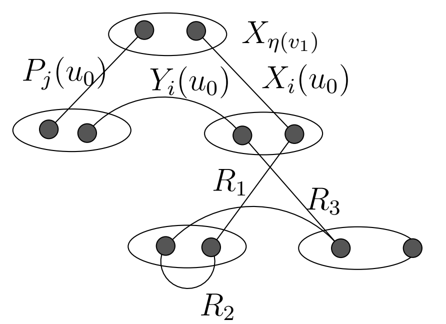

Figure 2: Second case of the construction of the path R.

Next we consider the case $l_i \in B$ and $m_i \notin A - B$. We proceed similarly as in the previous paragraph, but with these exceptions: the homeomorphic embedding γ will map t_3 to v_4, rather than v_5, the first pair of disjoint paths will now be $Q_i(u_0) \cup P_i(u_1)$ and $Q_j(u_0) \cup P_j(u_1)$, and for the second pair we define $R_1 = Q_{l_i}(u_1)$, $R_3 = X_{m_i}(u_1)$ if $m_i \notin A$ and $R_3 = Q_{m_i}(u_1)$ if $m_i \in B$, and R_2 will be a W6-path in the outer graph of v_5 joining the ends of R_1 and R_3.

Therefore assume that $l_i \in B - A$ and $m_i \in A - B$ for every $i \in B - A$. Let u_2 be the major vertex of T_5 at height two whose trinity includes v_5 and assume its trinity is (v_5, v_6, v_7). Let u_3 be the major vertex of T_5 at height three whose trinity includes v_7 and assume its trinity is (v_7, v_8, v_9). Let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_8, respectively. Then t_0 also has property AB_{ij} in η'. To see that the first pair of disjoint paths is $Q_i(u_0) \cup Q_i(u_1) \cup Q_i(u_2) \cup P_i(u_3)$ and $Q_j(u_0) \cup Q_j(u_1) \cup Q_j(u_2) \cup P_j(u_3)$. The first path of the second pair is $P_j(u_0)$. Let $R_1 = Y_i(u_0) \cup Q_{m_i}(u_1) \cup P_{m_i}(u_2)$, $R_2 = P_j(u_2) \cup Q_j(u_2) \cup Q_j(u_3)$, and $R_3 = X_i(u_0) \cup Q_{l_i}(u_1) \cup X_{l_i}(u_2) \cup X_{l_{l_i}}(u_3)$. Then the second path of the second pair is a path from $\xi_{v_1}(i)$ to $\xi_{v_2}(i)$ that is a subgraph of $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, where R_4 is a W6-path in the outer graph of v_6 joining the ends of R_1 and R_2, and R_5 is a W6-path in the outer graph of v_9 joining the ends of R_2 and R_3. See Figure 3. This completes main case 1.

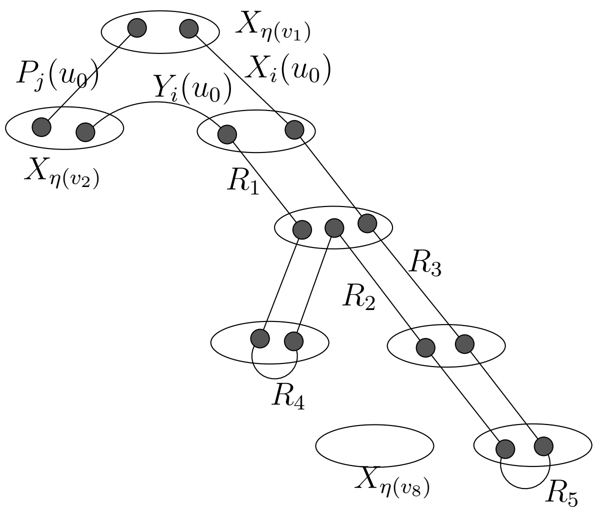

Figure 3: Second pair when $l_i \in B - A$ and and $m_i \in A - B$.

Main case 2: $A \cap B = \emptyset$. It follows that s is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$. But the integers l_i, m_i are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. We may therefore assume that $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$. Let us recall that u_2 is the child of v_5 and (v_5, v_6, v_7) is its trinity. We let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_6, respectively, and we will prove that t_0 has property AB_{ij} in η'. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cap Q_i(u_1) \cap P_i(u_2)$ and $Q_j(u_0) \cap Q_j(u_1) \cap P_j(u_2)$. The first path of the second pair will consist of the union of $X_i(u_0)$ with a subpath of $Q_{l_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and $Y_i(u_0)$ with a subpath of $Q_{m_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and a suitable W6-path in the outer graph of v_4 joining their ends, and the second path will consist of the union of $X_j(u_0) \cup Q_{l_j}(u_1) \cup Q_{l_j}(u_2)$ and $Y_j(u_0) \cup Q_{m_j}(u_1) \cup Q_{m_j}(u_2)$ and a suitable W6-path in the outer a graph of v_7 joining their ends. See Figure 4. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$.

We may therefore assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B. Let us recall that for every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(l_i)$ to $\xi_{v_3}(m_i)$ in the outer graph at v_3 and is disjoint from the η-torso at u_0, except for its ends. Let J be the union of these subpaths; then J is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the η-torso at u_0. For $i \in A$ the intersection of the path $Q_i(u_0)$ with the η-torso at u_0 consists of two paths, one from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and the other from $X_{\eta(v_2)}$ to $X_{\eta(v_3)}$. Let L denote the union of these subpaths over all $i \in A$. It follows that $J \cup L \cup \bigcup_{i \in B} Q_i(u_0)$ is a linkage from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and so by the minimality of the specified u_0-linkages, it is equal to the specified left u_0-linkage. It follows that u_0 has property C in η. $\square$

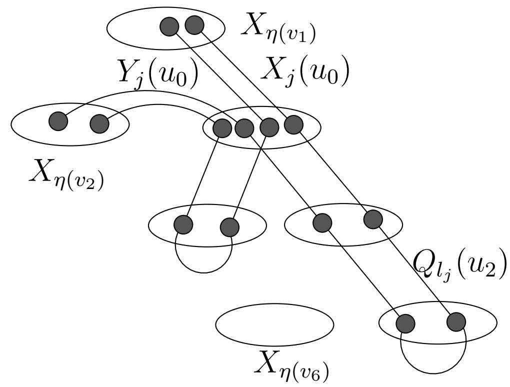

Figure 4: Second pair when $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$.

Lemma 6.4. *Let (T, X) be a tree-decomposition of a graph G satisfying (W6) and (W7). If there exists a regular cascade $\eta : T_3 \hookrightarrow T$ with orderings ξ_t in which every major vertex has property C, then there is a weak subcascade η' of η of height one such that the major vertex in η' has property C_{ij} for some i, j.*

Proof. Let the common confinement sets for η be A, B, C, D. For a major vertex $w \in V(T_3)$ with trinity (v_1, v_2, v_3) there are disjoint paths in the η-torso at w as in the definition of property C. For $a \in A$ and $b \in B$ let $R_a(w)$ denote the path with ends $\xi_{v_1}(a)$ and $\xi_{v_2}(a)$, let $R_b(w)$ denote the path with ends $\xi_{v_1}(b)$ and $\xi_{v_3}(b)$, and let $R_{ab}(w)$ denote the path with ends $\xi_{v_2}(b)$ and $\xi_{v_3}(a)$.

Assume the major root of T_3 is u_0 and its trinity is (v_1, v_2, v_3), and let I be the common intersection set of η. Then $\eta(v_1), \eta(v_2), \eta(v_3)$ is a triad in T with center $\eta(u_0)$ and for all $i \in \{1, 2, 3\}$ we have $X_{\eta(v_i)} \cap X_{\eta(u_0)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}$, and hence the triad is not X-separable by (W7). Thus by Lemma 3.1 there is a path $R(u_0)$ connecting two of the three sets of disjoint paths in the η-torso at u_0. Assume without loss of generality that one end of $R(u_0)$ is in a path $R_i(u_0)$, where $i \in A$. Then the other end of $R(u_0)$ is either in a path $R_j(u_0)$, where $j \in B$; or in a path $R_{aj}(u_0)$, where $j \in B$ and $a \in A$. In the former case we define $a \in A$ to be such that $R_{aj}(u_0)$ is a path in the family.

Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let $\gamma(t_0) = u_0$, $\gamma(t_1) = v_1$, $\gamma(t_2) = v_2$. Let the major vertex that is the child of v_3 be u_1, and the trinity of u_1 be (v_3, v_4, v_5). Let $\gamma(t_3) = v_5$. We will prove that t_0 has property C_{ij} in $\eta' = \eta \circ \gamma$. Let $b \in B$ be such that $R_{ib}(u_1)$ is a member of the family of the disjoint paths in the η-torso at u_1 as in the definition of property C. By Lemma 6.2, there exists a $W6$-path P in the outer graph at v_4 joining $\xi_{v_4}(a)$ and $\xi_{v_4}(b)$. By considering the paths $R_a(u_0)$, $R_j(u_0) \cup R_j(u_1)$, $R_{aj}(u_0) \cup R_a(u_1) \cup P \cup R_{ib}(u_1)$ and $R(u_0)$ we find that t_0 has property C_{ij} in η', as desired. □

Lemma 6.5. *Let $s \geq 2$ be an integer and let (T, X) be a tree-decomposition of a graph G satisfying (W6). Let $\eta : T_3 \hookrightarrow T$ be an ordered cascade in (T, X) of height three and size $|I| + s$ with orderings ξ_t and common intersection set I such that every major vertex of T_3*

has property C_{ij} for some distinct $i, j \in \{1, 2, \ldots, s\}$. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property B_{ij} in η'.

Proof. Assume that the three major vertices at height zero and one of T_3 are u_0, u_1, u_2. Let the trinity at u_0 be (v_1, v_2, v_3), the trinity at u_1 be (v_2, v_4, v_5), and the trinity at u_2 be (v_3, v_6, v_7). Assume the major vertex of T_1 is t_0, and its trinity is (t_1, t_2, t_3). For a major vertex $w \in V(T_3)$ let $R_i(w), R_j(w), R_{ij}(w)$ and $R(w)$ be as in the definition of property C_{ij}.

We need to find a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_3$ such that $\eta' = \eta \circ \gamma$ satisfies the requirement. Set $\gamma(t_0) = u_0$ and $\gamma(t_1) = v_1$. Our choice for $\gamma(t_2)$ will be v_4 or v_5, depending on which two of the three paths $R_i(u_1), R_j(u_1), R_{ij}(u_1)$ in the torso at u_1 the path $R(u_1)$ is connecting. If $R(u_1)$ is between $R_i(u_1)$ and $R_j(u_1)$, then choose either v_4 or v_5 for $\gamma(t_2)$. If $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_4$, and if it is between $R_j(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_5$. Do this similarly for $\gamma(t_3)$. Then $\eta' = \eta \circ \gamma$ will satisfy the requirement. In fact, we will prove this for the case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$. See Figure 5. The other five cases are similar.

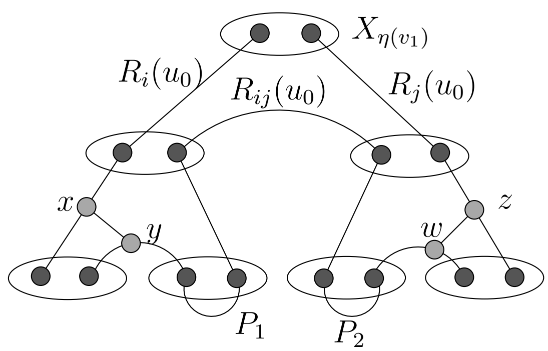

Figure 5: The case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$.

In this case, our choice is $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_4, \gamma(t_3) = v_7$. Assume the two endpoints of $R(u_1)$ are x and y and the two endpoints of $R(u_2)$ are w and z. By Lemma 6.2, there exists a W6-path P_1 between $\xi_{v_5}(i)$ and $\xi_{v_5}(j)$ in the outer graph at v_5 and a W6-path P_2 between $\xi_{v_6}(i)$ and $\xi_{v_6}(j)$ in the outer graph at v_6. Now let

$$P = yR_{ij}(u_1)\xi_{v_5}(i) \cup P_1 \cup R_j(u_1) \cup R_{ij}(u_0) \cup R_i(u_2) \cup P_2 \cup \xi_{v_6}(j)R_{ij}(u_2)w,$$

$$L_i = R_i(u_0) \cup R_i(u_1) \cup R(u_1) \cup P \cup wR_{ij}(u_2)\xi_{v_7}(i)$$

and

$$L_j = R_j(u_0) \cup R_j(u_2) \cup R(u_2) \cup P \cup yR_{ij}(u_1)\xi_{v_4}(j).$$

The tripods L_i and L_j show that the major vertex of $\eta' = \eta \circ \gamma : T_1 \hookrightarrow T$ has property B_{ij}. □

Lemma 6.6. *For every positive integers h' and $w \geq 2$ there exists a positive integer $h = h(h', w)$ such that the following holds. Let s be a positive integer such that $2 \leq s \leq w$. Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W6) and (W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified linkages that are minimal, where I is its common intersection set. Then there exist distinct integers $i, j \in \{1, 2, \dots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that*

- *every major vertex of $T_{h'}$ has property A_{ij} in η', or*
- *every major vertex of $T_{h'}$ has property B_{ij} in η'*

Proof. Let $h(a, k)$ be the function of Lemma 5.2, let $a_3 = 3h'$, $a_2 = h(a_3, 2\binom{w}{2})$, $a_1 = 5a_2$ and $h = h(a_1, 2)$. Consider having property C or not having property C as colors, then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property C in η for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property C in η for every major vertex $t \in V(T_{a_1})$. By Lemma 5.3 $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either t has property C in η_1 for every major vertex $t \in V(T_{a_1})$ or t does not have property C in η_1 for every major vertex $t \in V(T_{a_1})$.

If t has property C in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.4 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property C_{ij} in η_2 for some distinct $i, j \in \{1, 2, ..., s\}$. Consider each choice of pair i, j as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, ..., s\}$, $\gamma_1(t)$ has property C_{ij} in η_2 for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 5.3 this implies t has property C_{ij} in η_3 for every major vertex $t \in V(T_{a_3})$. Then by Lemma 6.5 there exists a weak subcascade $\eta_4 : h' \hookrightarrow a_3$ of η_3 such that every major vertex of $T_{h'}$ has property B_{ij} in η_4. Hence η_4 is as desired.

If t does not have property C in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.3 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property A_{ij} or B_{ij} for some distinct $i, j \in \{1, 2, ..., s\}$. Consider each property A_{ij} or B_{ij} as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{h'} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, ..., s\}$, either $\gamma_1(t)$ has property A_{ij} in η_2 for every major vertex $t \in V(T_{h'})$ or $\gamma_1(t)$ has property B_{ij} in η_2 for every major vertex $t \in V(T_{h'})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then t has property A_{ij} in η_3 for every major vertex $t \in V(T_{h'})$ or t has property B_{ij} in η_3 for every major vertex $t \in V(T_{h'})$ by Lemma 5.3. Hence η_3 is as desired. □

7 Proof of Theorem 1.3

By Lemmas 2.2 and 2.4 Theorem 1.3 is equivalent to the following theorem.

Theorem 7.1. *For any positive integer k, there exists a positive integer $p = p(k)$ such that for every 2-connected graph G, if G has path-width at least p, then G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$.*

We need the following lemma.

Lemma 7.2. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height h and size $s + I$, where I is the common intersection set, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct and such that every major vertex of T_h has property B_{ij} in η. Let t be the minor root of T_h, and let w_1w_2 be the base edge of $\mathcal{Q}_h$. Then G has a minor isomorphic to $\mathcal{Q}_h - w_1w_2$ in such a way that $\xi_t(i)$ belongs to the node of w_1 and $\xi_t(j)$ belongs to the node of w_2 .*

Proof. We proceed by induction on h. Let t_0 be the major root of T_h, let (t_1, t_2, t_3) be its trinity, and let L_i and L_j be the tripods in the η-torso at t_0 as in the definition of property B_{ij}. The graph $L_i \cup L_j$ contains a path P joining $\xi_{t_1}(i)$ to $\xi_{t_1}(j)$, which shows that the lemma holds for $h = 1$.

We may therefore assume that $h > 1$ and that the lemma holds for $h-1$. For $k \in \{2, 3\}$ let R_k be the subtree of T_h rooted at t_k, let η_k be the restriction of η to R_k, and let G_k be the subgraph of G induced by $\bigcup\{X_r : r \in sp(\eta_k)\}$. By the induction hypothesis applied to η_k and G_k, the graph G_k has a minor isomorphic to $\mathcal{Q}_{h-1} - u_1u_2$ in such a way $\xi_{t_k}(i)$ belongs to the node of u_1 and $\xi_{t_k}(j)$ belongs to the node of u_2, where u_1u_2 is the base edge of $\mathcal{Q}_{h-1}$. By using these two minors, the path P and the rest of the tripods L_i and L_j we find that G has the desired minor. □

We deduce Theorem 7.1 from the following lemma.

Lemma 7.3. *Let k and w be positive integers. There exists a number $p = p(k, w)$ such that for every 2-connected graph G, if G has tree-width less than w and path-width at least p, then G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$.*

Proof. Let $h' = 2k + 1$, let $h = h(h', w)$ be the number as in Lemma 6.6, and let p be as in Theorem 5.5 applied to $a = h$ and w. We claim that p satisfies the conclusion of the lemma. By Theorem 5.5, there exists a tree-decomposition (T, X) of G such that:

- (T, X) has width less than w,
- (T, X) satisfies (W1)–(W7), and
- for some s, where $2 \le s \le w$, there exists a regular cascade $\eta : T_h \hookrightarrow T$ of height h and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.

Let I be the common intersection set of η, let ξ_t be the orderings, and let $s_1 = s - |I|$. Then $s_1 \ge 1$ by the definition of injective cascade.

Assume first that $s_1 = 1$. Since $s \ge 2$, it follows that $I \ne \emptyset$. Let $x \in I$. Let R be the union of the left and right specified t-linkage with respect to η, over all major vertices $t \in V(T_h)$ at height at most $h - 2$. The minimality of the specified linkages implies that R has a subtree isomorphic to a subdivision of $CT_{\lfloor (h-1)/2 \rfloor}$. Let t be a minor vertex of T_h at height $h - 1$. By Lemma 6.2 there exists a W6-path with ends $\xi_t(1)$ and x and every internal vertex in the outer graph at t. The union of R and these W6-paths shows that G has a $\mathcal{P}_k$ minor, as desired.

We may therefore assume that $s_1 \geq 2$. By Lemma 6.6 there exist distinct integers $i, j \in \{1, 2, \ldots, s\}$ and a subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that

- every major vertex of $T_{h'}$ has property A_{ij} in η', or
- every major vertex of $T_{h'}$ has property B_{ij} in η'

Assume next that every major vertex of $T_{h'}$ has property A_{ij} in η', and let R be the union of the corresponding tripods, over all major vertices $t \in V(T_{h'})$ at height at most $h' - 2$. It follows that R is the union of two disjoint trees, each containing a subtree isomorphic to $CT_{(h'-1)/2}$. Let t be a minor vertex of $T_{h'}$ at height $h' - 1$. By Lemma 6.2 there exists a W6-path with ends $\xi_t(i)$ and $\xi_t(j)$ in the outer graph at t. By contracting one of the trees comprising R and by considering these W6-paths we deduce that G has a $\mathcal{P}_k$ minor, as desired.

We may therefore assume that every major vertex of $T_{h'}$ has property B_{ij} in η'. It follows from Lemma 7.2 that G has a minor isomorphic to $\mathcal{Q}_{h'-1}$, as desired. □

Proof of Theorem 7.1. Let a positive integer k be given. By Theorem 1.1 there exists an integer w such that every graph of tree-width at least w has a minor isomorphic to $\mathcal{P}_k$. Let $p = p(k, w)$ be as in Lemma 7.3. We claim that p satisfies the conclusion of the theorem. Indeed, let G be a 2-connected graph of path-width at least p. By Theorem 1.1, if G has tree-width at least w, then G has a minor isomorphic to $\mathcal{P}_k$, as desired. We may therefore assume that the tree-width of G is less than w. By Lemma 7.3 G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$, as desired. □

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