Graphs without $K_4$ and Well-Quasi-Ordering

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It is proved that given an infinite sequence $G_1, G_2, G_3, \ldots$ of series-parallel graphs there are indices $i < j$ such that $G_i$ contains an induced subgraph contractable onto $G_j$. An example is given showing that for planar graphs the preceding theorem fails.

0. INTRODUCTION

By a Kuratowski type theorem for a property $P$ of graphs we mean any assertion of the form:

$G$ does not have $P$ iff $H \leq G$ for some $H$ in $L$, where $L$ is a finite list of graphs and $\leq$ is a quasi-ordering (i.e., a reflexive and transitive relation). Such a theorem exists, e.g., for the properties “to be planar” (the classical Kuratowski’s theorem) or “to be embeddable in the projective plane” (see [1, 2]), where the lists are explicitly known. We are interested not in constructing such lists, but in their existence. The connection with the well-quasi-ordering theory may be stated as follows: If the property $P$ is $\leq$-closed (i.e., $G$ has $P$ and $H \leq G$ implies $H$ has $P$) and the class of all graphs is wqo by $\leq$, then there is a Kuratowski type theorem for $P$. In this context, the following conjecture due to Wagner is important:

Conjecture. The class of all graphs is wqo by the relation $\leq (G \leq H$ if $H$ contains a subgraph contractable onto $G$).

Partial results are due to Kruskal [4], Mader [5], and Robertson and Seymour [7]. Very recently the existence of a Kuratowski type theorem for higher surfaces has been proved by Robertson and Seymour.

There are properties, which are not $\leq$-closed, for example, the property $P =$ “to be a string graph” (see [8]). On the other hand this property $P$ is $\preceq$-closed, where $\preceq$ is a strengthening of $\leq$ (see Sect. 2), and that leads to a natural question whether the class of all graphs is wqo by $\preceq$. A negative
answer to this is given in Section 3, and an affirmative result concerning \( \preceq \) is found in Section 2. Section 1 is devoted to definitions only.

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1. DEFINITIONS.

A graph is a pair \((V, E)\), where \(V\) is a finite set and \(E\) is a subset of the collection of all 2-element subsets of \(V\). \(\mathcal{X}_4\) denotes the class of connected graphs which contain no subdivision of \(K_4\).

Let \(G, H\) be arbitrary graphs. A contraction of \(H\) onto \(G\) is a mapping \(f: V(H) \rightarrow V(G)\) such that

1. For every \(v \in V(G)\) the graph induced by \(f^{-1}(v)\) in \(H\) is connected.
2. For every \(u \neq v \in V(G)\) the following equivalence holds: \(v \in E(G)\) if and only if there are \(y, z \in V(H)\), \(y \in f^{-1}(u), z \in f^{-1}(v)\) such that \(\{y, z\} \in E(H)\).

Let \(G\) be a graph and \(v_1, \ldots, v_n\) distinct vertices of \(G\). The pair \((G, v_1, \ldots, v_n)\) is called an \(n\)-rooted graph and will be denoted simply \((1, \ldots, n)\). The graph \(G\) itself is considered to be a 0-rooted graph. If \((1, \ldots, n)\) is an \(n\)-rooted graph and \(Q\) an arbitrary set we define a \(Q\)-labelled \(n\)-rooted graph to be an \((n+2)\)-tuple \((g, G, v_1, \ldots, v_n)\), where \(g(V(G)) \rightarrow Q\). For \(|Q| = 1\), the mapping \(g\) brings no further structure on \(G\), the \(n\)-rooted graph \(G(v_1, \ldots, v_n)\) itself will be considered to be \(Q\)-labelled, e.g., \(Q = \{0\}\).

If \(\mathcal{G}\) is a class of \(n\)-rooted graphs and \(Q\) an arbitrary set, then \(\mathcal{G}(Q)\) denotes the collection of all \(Q\)-labelled graphs \((g, G, v_1, \ldots, v_n)\) such that \(1, \ldots, n) \in \mathcal{G}\).

But also

\[
\begin{align*}
&= \{G(u, v) : G \text{ is a block from } \mathcal{X}_4 \text{ and } \{u, v\} \in E(G)\} \\
&= \{G(u, v) : \{u, v\} \notin E(G), H \text{ is 2-connected and } H \in \mathcal{X}_4, \text{ where } H \text{ is obtained from } G \text{ by joining the edge } \{u, v\}\} \\
&= \mathcal{B}^+ \cup \mathcal{B}^-.
\end{align*}
\]

2. AFFIRMATIVE RESULT

We shall briefly recall some facts concerning wqo theory. For a nice presentation of the method used the reader is referred to [6]. Let \(Q\) be a set which a quasi-ordering (i.e., reflexive and transitive relation) \(\preceq\) is defined. Such sets are said to be quasi-ordered (qo). By a \(Q\)-sequence we mean any mapping \(f: X \rightarrow Q\), where \(X\) is an infinite subset of \(Q\). The letters \(X, Y\) (with or without dashes or suffixes) will always denote infinite subsets of \(Q\). A \(Q\)-sequence \(f: X \rightarrow Q\) is called good, if there are \(i \leq j \in X\) such that \(f(i) \preceq f(j)\) and is called bad otherwise. A \(Q\)-set \(Q\) is called well-quasi-ordered (wqo) if every \(Q\)-sequence is good. If \(Q, Q'\) are quasi-ordered via quasi-orderings \(\preceq, \preceq'\), then we define a quasi-ordering \(\preceq \times \preceq'\) on \(Q \times Q'\) as follows:

\[
[q_1, q'_1] \preceq \times [q_2, q'_2] \iff q_1 \preceq q_2 \text{ and } q'_1 \preceq' q'_2.
\]

PROPOSITION 2.1 (Higman [3]). If \(Q, Q'\) are wqo, then \(Q \times Q'\) is wqo.

By \(Q^{<\infty}\) we mean the set of finite sequences of elements of \(Q\). \(Q^{<\infty}\) will be quasi-ordered by the rule that \((a_1, \ldots, a_n) \preceq (b_1, \ldots, b_m)\) if there is a strictly increasing mapping \(f: \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}\) such that \(a_i \leq b_{f(i)}\) for any \(i = 1, \ldots, n\).

PROPOSITION 2.2 (Higman [3]). If \(Q\) is wqo, then \(Q^{<\infty}\) is wqo.

Suppose that a quasi-ordering \(\preceq\) on a set \(Q\) is given and suppose that \(\gamma = (g, G, v_1, \ldots, v_n), \eta = (h, H, w_1, \ldots, w_n)\) are \(Q\)-labelled \(n\)-rooted graphs. Define \(\gamma \prec \eta\) if \(H\) contains an induced subgraph \(H'\) and there is a contraction \(f\) of \(H'\) onto \(G\) such that

- For any \(1 \leq i \leq n, w_i \in V(H')\) and \(f(v_i) = w_i\).
- For any \(v \in V(G),\) there is \(w \in E^{-1}(v)\) such that \(g(v) \leq h(w)\).

We will say that \(\gamma\) is smaller than \(\eta\) (and write \(\gamma \prec \eta\)) if \(\gamma \preceq \eta\) and \(|E(G)| < |E(H)|\). Note that according to our agreements \(\preceq\) and \(\prec\) are defined on the class of \(n\)-rooted graphs as well as on the class of graphs itself. In what follows we shall be concerned with the wqo property of the quasi-ordering \(\prec\).

Let \(\mathcal{G}\) be an arbitrary class of graphs and \(Q\) a qo set, and let \(f: X \rightarrow \mathcal{G}(Q), f': X \rightarrow \mathcal{G}(Q)\) be two \(\mathcal{G}(Q)\)-sequences. We define \(f \preceq f'\) if \(X \preceq X'\) and \(f(i) \leq f'(i)\) for every \(i \in X\) and similarly \(f \prec f'\) if \(X \prec X'\) and \(f(i) \prec f'(i)\) for every \(i \in X\). A \(\mathcal{G}(Q)\)-sequence \(f: X \rightarrow \mathcal{G}(Q)\) is called minimal bad if it is bad and there is no bad \(\mathcal{G}(Q)\)-sequence \(f' \prec f\).

LEMMA 2.3. If \(f: X \rightarrow \mathcal{G}(Q)\) is a bad \(\mathcal{G}(Q)\)-sequence, then there is a minimal bad \(\mathcal{G}(Q)\)-sequence \(f_0 \preceq f\).

Proof. Let \(X = \{i_1 < i_2 < \cdots\}\). Choose \(f_0(i_1)\) such that it is a first term of a bad \(\mathcal{G}(Q)\)-sequence which is \(\preceq f\) and there is no smaller element of \(\mathcal{G}(Q)\) with this property. Then choose \(f_0(i_2)\) such that \(f_0(i_1), f_0(i_2)\) (in that order) are first two terms of a bad \(\mathcal{G}(Q)\)-sequence which is \(\preceq f\) and there is no smaller element of \(\mathcal{G}(Q)\) with this property. Continuing this process
get a bad $f_0: X \to \mathcal{B}(Q)$. We claim it is the desired one. For if there is a $1 f': X' \to \mathcal{B}(Q), f' \not\leq f_0$, we may define $e: Y \to \mathcal{B}(Q)$ by

\[
Y = \{i \in X: i < \min X' \} \cup X'
\]

\[
e(i) = \begin{cases} f_0(i) & i < \min X', i \in X, \\ f'(i) & i \in X'. \end{cases}
\]

Now $e$ contradicts the choice of $f_0$.

Lemma 2.4. If $\mathcal{B}(Q)$ is not wqo, then there is a minimal bad $\mathcal{B}(Q)$-sequence $f_0: X_0 \to \mathcal{B}(Q)$ such that $\text{Im } f_0 \subseteq \mathcal{B}^-(Q)$.

Proof. By Lemma 2.3 there is a minimal bad $f: X \to \mathcal{B}(Q)$, we denote

\[
(f_i, G_i, u_i, v_i).
\]

We may assume that either $\text{Im } f \subseteq \mathcal{B}^-(Q)$ or $\text{Im } f \subseteq \mathcal{B}^+(Q)$. In the second case we define $f^{-1}: X \to \mathcal{B}^-(Q)$ by $f^{-1}(i) = (G_{i-}, u_i, v_i)$, where $G_{i-}$ is obtained from $G_i$ by removing the edge $\{u_i, v_i\}$. $f^{-1}$ is bad but it may fail to be minimal bad. By Lemma 2.3 there is minimal bad $f_1: X_1 \to \mathcal{B}(Q)$ such that $f_1 \leq f^{-1}$. We may again assume that either $\text{Im } f_1 \subseteq \mathcal{B}^-(Q)$ or $\text{Im } f_1 \subseteq \mathcal{B}^+(Q)$. In the first case we are done and in the second one $f_1 \not\leq f$, which contradicts the minimality of $f$.

Lemma 2.5. If $G(u, v) \in \mathcal{B}^-$ then either

(i) there exist $G_1(u_1, v_1), G_2(u_2, v_2) \in \mathcal{B}$, vertex-disjoint and smaller in $G(u, v)$, so that $G(u, v)$ is obtained by identifying $u_1 = u_2 = u, v_1 = v_2 = v$, or

(ii) there exist $G_1(u, w_1), G_2(w_2, v) \in \mathcal{B}$, vertex-disjoint and smaller than $G(u, v)$, so that $G(u, v)$ is obtained by identifying $w_1 = w_2$.

Proof. Is well-known and we shall just sketch it. Let $G(u, v) \in \mathcal{B}^-$ be a bad graph. Then either there are two disjoint paths joining $u$ and $v$ or (by Hunger's theorem) there is a cutpoint between $u$ and $v$. The first case given while the second one leads to (ii).

Lemma 2.6. If $Q$ is wqo, then $\mathcal{B}(Q)$ is wqo by $\leq$.

Proof. Suppose that the lemma fails for some $Q$ which is wqo. Then by Lemma 2.4 there is a minimal bad $\mathcal{B}(Q)$-sequence $f: X \to \mathcal{B}(Q)$ such that $f \subseteq \mathcal{B}^-(Q)$. Denote $f(i) = (g_i, G_i, u_i, v_i)$. We may assume that either

(i) holds for $G_i(u_i, v_i)$ for any $i \in X$

(ii) holds for $G_i(u_i, v_i)$ for any $i \in X$.

Define $g_1, g_2$ to be the restrictions of $g_i$ to $V(G_1)$, $V(G_2)$, respectively. Then $f_1, f_2: X \to \mathcal{B}(Q)$ defined by

\[
f_1(i) = (g_1, G_1, u_1, v_1), \quad f_2(i) = (g_2, G_2, u_2, v_2)
\]

are smaller than $f$. The sets $\text{Im } f_1, \text{Im } f_2$ are wqo by minimality of $f$, and by Proposition 2.1 there are $i < j \in X$ such that $f_1(i) \not\leq f_1(j), f_2(i) \not\leq f_2(j)$. This yields

\[
f(i) \not\leq f(j)
\]

which contradicts the badness of $f$ and proves the lemma.

Theorem 2.7. Let $Q$ be wqo and let $\mathcal{F}$ be a class of 1-rooted blocks satisfying

\[
\text{if } R \text{ is wqo then } \mathcal{F}(R) \text{ is wqo by } \leq.
\]

(*)

Denote

\[
\mathcal{G} = \{G(v): G \text{ is connected and } B(b) \in \mathcal{F} \text{ for every block } B \text{ of } G \text{ and every } b \in V(B)\}.
\]

Then $\mathcal{B}(Q)$ is wqo by $\leq$.

Proof. Suppose that $Q$ and $\mathcal{F}$ satisfy the assumptions, but $\mathcal{B}(Q)$ is not wqo. Let $f: \omega \to \mathcal{B}(Q)$ be a minimal bad $\mathcal{B}(Q)$-sequence, we denote $f(i) = (g_i, G_i, u_i, v_i)$. Clearly each $G_i$ contains at least two vertices and thus we may choose a block $B_i$ in each $G_i$ such that $v_i \in V(B_i)$. Denote $W_i = \{w: \{w\} = V(B_i) \cap V(B) \text{ for some block } B \text{ in } G \text{ distinct from } B_i\}$ and let $w_{1, i}, ..., w_{n, i}$ be elements of $W_i$. Consider the graph obtained from $G_i$ by deleting edges from $B_i$ and denote by $B'_i$ that component of this graph which contains $w_{1, i}$. Define $h_i'$ to be the restriction of $g_i$ to $V(B'_i)$. Put $\mathcal{H} = \{(h_i', H_i', w_{1, i}^i): i = 1, ..., s\}, \mathcal{H} = \bigcup_{i \in \omega} \mathcal{H}_i \cup \{Q\}$. We define a quasi-ordering on $\mathcal{H}$, the least one containing the restriction of $\leq$ to $\bigcup_{i \in \omega} \mathcal{H}_i$. It follows easily from minimality of $f$ that $\mathcal{H}$ is wqo. Define $b_i: V(B_i) \to \mathcal{H} \times Q$ by the rule

\[
\begin{align*}
&x \mapsto ((h_i', H_i', w_{1, i}^i), g_i(x)) \quad \text{for } x = w_{1, i}^i \\
&x \mapsto ((Q, g_i(x))) \quad \text{otherwise}.
\end{align*}
\]

By Proposition 2.1, $\mathcal{H} \times Q$ is wqo and hence $\mathcal{F}(\mathcal{H} \times Q)$ is wqo. Thus $e: \omega \to \mathcal{F}(\mathcal{H} \times Q)$ defined by

\[
e(i) = (b_i, B_i, v_i)
\]
Suppose not, thus there are $4 < k < n$ such that $G_k \cong G_n$. Then there is an induced subgraph $G'_k$ of $G_n$ and a contraction $f$ of $G'_k$ onto $G_k$. Denote by $H_n$ the graph induced by the set $f^{-1}(V(C_k))$ in $G'_k$. Every connected subgraph of $G'_k$ containing neither 0 nor $2n+1$ has at most four neighbours, but 0 and $2k+1$ (vertices of $G_k$) are of degree at least five, and hence $0, 2n+1 \in G'_k$ and $f(0), f(2n+1) \in \{0, 2k+1\}$. This yields $H_n \subseteq C_n$, and moreover, since $C_k$ is a cycle, $H_n$ must contain a cycle and it follows that $H_n = C_n$. This shows that $G'_n = G_n$. Thus there are $i, j$ adjacent in $C_n$ such that $f(i) = f(j)$. Now $f(i)$ is of degree four but that is impossible since any vertex from $C_k$ is of degree three.

4. Concluding Remarks

1. Theorem 2.8 can be extended to infinite graphs. This is a more technical statement which will appear elsewhere.

2. We remark that the above results concerning the quasi-ordering $\leq$ do not cover all possible cases. For instance, any of the graphs from Example 3.1 can be contracted onto $K_5^-$ (the complete graph $K_5$ minus one edge). It is natural to ask which classes of graphs are wqo by $\leq$. Particularly we propose the following problem:

   Is the class of all graphs that cannot be contracted onto $K_5^-$ wqo by $\leq$? (Compare [3].)

   The referee points out that concerning this conjecture the following (unpublished) theorem of Seymour may be relevant:

   The only 3-connected graphs not contractable onto $K_5^-$ are

   (i) wheels

   (ii) $K_{3,3}$ and the prism

   (iii) graphs with $\leq 4$ vertices.

3. Negative Result

The following example shows that the class of planar graphs is not wqo by $\leq$.

Example 3.1. Define $G_n = (V_n, E_n)$, where

$V_n = \{0, \ldots, 2n, 2n+1\}$

$E_n = \{\{1, 2\}, \{2, 3\}, \ldots, \{2n-1, 2n\}, \{2n, 1\}\} \cup \{\{0, 2k\} | k = 1, \ldots, n\}$

$\cup \{\{2n+1, 2k+1\} | k = 0, \ldots, n-1\}.$

Denote by $C_n$ the graph obtained from $G_n$ by removing vertices 0 and $2n+1$. We claim that the sequence

$G_5, G_6, G_7, \ldots$

is bad (with respect to $\leq$).

References


7. N. ROBERTSON AND P. D. SEYMOUR, Graph width and well-quasi-ordering, preprint.