\textit{K}_t \textit{ MINORS IN LARGE }\textit{t-CONNECTED GRAPHS}

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joint work with Sergey Norin
$K_t$ MINORS IN LARGE $t$-CONNECTED GRAPHS

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joint work with Serguei Norine
• A minor of $G$ is obtained by taking subgraphs and contracting edges.
• Preserves planarity and other properties.
• $G$ has an $H$ minor ($H \leq_m G$) if $G$ has a minor isomorphic to $H$.
• A $K_5$ minor:
Excluding $K_t$ minors

- $G \not\geq_m K_3 \iff G$ is a forest (tree-width $\leq 1$)
- $G \not\geq_m K_4 \iff G$ is series-parallel (tree-width $\leq 2$)
- $G \not\geq_m K_5 \iff$ tree-decomposition into planar graphs and $V_8$ (Wagner 1937)
- $G \not\geq_m K_6 \iff$ ???
Graphs with no $K_6$

• apex ($G \setminus v$ planar for some $v$)
Graphs with no $K_6$

- apex ($G \setminus v$ planar for some $v$)
- planar + triangle
Graphs with no $K_6$

- apex ($G\backslash v$ planar for some $v$)
- planar + triangle
- double-cross
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GRAPHS WITH NO $K_t$ MINOR

REMARK
$G \not \cong_m K_t \Rightarrow (G + \text{universal vertex}) \not \cong_m K_{t+1}$

REMARK
$G \setminus X$ planar for $X \subseteq V(G)$ of size $\leq t-5 \Rightarrow G \not \cong_m K_t$
THEOREM (Robertson & Seymour)
\[ G \not\cong_m K_t \implies G \text{ has "structure"} \]

Roughly structure means tree-decomposition of pieces that \( k \)-almost embed in a surface that does not embed \( K_t \), where \( k = k(t) \).

Converse not true, but:
\[ G \text{ has "structure"} \implies G \not\cong_m K_t \text{ for some } t' >> t \]

Our objective is to find a simple iff statement
Extremal results for $K_t$

• $G \not\supseteq K_3 \implies |E(G)| \leq n-1$
Extremal results for $K_t$

- $G \not\supseteq K_3 \Rightarrow |E(G)| \leq n - 1$
- $G \not\supseteq K_4 \Rightarrow |E(G)| \leq 2n - 3$
Extremal results for $K_t$

- $G \nsubseteq K_3 \Rightarrow |E(G)| \leq n-1$
- $G \nsubseteq K_4 \Rightarrow |E(G)| \leq 2n-3$
- $G \nsubseteq K_5 \Rightarrow |E(G)| \leq 3n-6$ (Wagner)
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- $G \not\supseteq K_7 \Rightarrow |E(G)| \leq 5n-15$ (Mader)
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So

- $G \not\ni K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$ for $t \leq 7$
Extremal results for $K_t$

- $G \nsubseteq K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$ for $t \leq 7$
- $G \nsubseteq K_8 \Rightarrow |E(G)| \leq 6n-21$, because of $K_{2,2,2,2,2}$
- $G \nsubseteq K_t \Rightarrow |E(G)| \leq ct(\log t)^{1/2}n$ (Kostochka, Thomason)

CONJ (Seymour, RT)
$G \nsubseteq K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$
Extremal results for $K_t$

- $G \not
\cong K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$ for $t \leq 7$
- $G \not
\cong K_8 \not\Rightarrow |E(G)| \leq 6n-21$, because of $K_{2,2,2,2,2}$
- $G \not
\cong K_t \Rightarrow |E(G)| \leq ct(\log t)^{1/2}n$ (Kostochka, Thomason)

**CONJ** (Seymour, RT) $G$ is $(t-2)$-connected, big $G \not
\cong K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$

- $G \not
\cong K_8 \Rightarrow |E(G)| \leq 6n-21$, unless $G$ is a $(K_{2,2,2,2,2,5})$-cockade (Jorgensen)
- $G \not
\cong K_9 \Rightarrow |E(G)| \leq 7n-28$, unless…. (Song, RT)
$K_t$ minors naturally appear in:

Structure theorems:
- series-parallel graphs (Dirac)
- characterization of planarity (Kuratowski)
- linkless embeddings (Robertson, Seymour, RT)
- knotless embeddings (unproven)

Hadwiger’s conjecture: $K_t \not\subseteq_m G \Rightarrow \chi(G) \leq t-1$
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- For $t=6$ implied by the 4CT by THM (Robertson, Seymour, RT) Every minimal counterexample to Hadwiger for $t=6$ is apex ($G \setminus v$ is planar for some $v$)
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**THM (Robertson, Seymour, RT)** Every minimal counterexample to Hadwiger for \( t=6 \) is apex (\( G\setminus v \) is planar for some \( v \))

Hadwiger’s conjecture is open for \( t>6 \)

Open even for \( G \) with no 3 pairwise non-adjacent vertices; **HC** implies any such \( G \geq_m K_{[n/2]} \)
Hadwiger’s conjecture: \( K_t \not\leq_m G \Rightarrow \chi(G) \leq t-1 \)

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Theorem implied by

Jorgensen’s conjecture: If $G$ is 6-connected and $K_{6 \not\subseteq m} G$, then $G$ is apex.
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Jorgensen’s conjecture: If $G$ is 6-connected and $K_6 \not\subseteq_m G$, then $G$ is apex.

**THM** (DeVos, Hegde, Kawarabayashi, Norin, RT, Wollan) True for big graphs: There exists $N$ such that every 6-connected graph $G \not\subseteq_m K_6$ on $\geq N$ vertices is apex.

**MAIN THM** (with Norin) $\forall t \exists N_t$ $\forall t$-connected graph $G \not\subseteq_m K_t$ on $\geq N_t$ vertices $\exists X \subseteq V(G)$ with $|X| \leq t-5$ such that $G \setminus X$ is planar.
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NOTES

• Gives iff characterization
• \( t \)-connected and \( |X| \leq t-5 \) best possible
• \( N_t \) needed for \( t>7 \)
• Proved for \( 31t/2 \)-connected graphs by Kawarabayashi, Maharry, Mohar
MAIN THM (with Norin) \( \forall \ t \ \exists \ N_t \forall \ t \)-connected graph \( \exists_m K_t \) on \( \geq N_t \) vertices \( \exists \ X \subseteq V(G) \) with \( |X| \leq t-5 \) such that \( G \setminus X \) is planar.

INGREDIENTS IN THE PROOF

- “Brambles” (“tangles”)
- Thm of DeVos-Seymour on graphs in a disk
- No big bramble \( \Rightarrow \) bounded tree-width method
- Excluded \( K_t \) theorem of Robertson & Seymour to examine the structure of a big bramble
THM (DeVos, Seymour) If $G$ is drawn in a disk with at most $k$ vertices on the boundary and every interior vertex has degree $\geq 6$, then $G$ has $\leq f(k)$ vertices.
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DEF A bramble $\mathcal{B}$ in $G$ is a set of connected subgraphs that pairwise touch (intersect or are joined by an edge). The order of $\mathcal{B}$ is $\min\{|X| : X \cap B \neq \emptyset \text{ for every } B \in \mathcal{B}\}$.

EXAMPLE $G=kk$ grid, $\mathcal{B}=\{\text{all crosses}\}$, order is $k$
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EXAMPLE $G = k \times k$ grid, $\mathcal{B} = \{\text{all crosses}\}$, order is $k$
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THEOREM (Seymour, RT) tree-width$(G) = \max$ order of a bramble $+ 1$

THEOREM (Robertson, Seymour) All brambles in $G$ form a tree-decomposition.
CASE 1 $G$ has bounded tree-width

PROOF Let $(T, W)$ be a tree-decomposition of bounded width. $T$ has a vertex of big degree or a long path.
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PROOF Let \((T,W)\) be a tree-decomposition of bounded width. \(T\) has a vertex of big degree or a long path. This suffices to get a \(K_7\) minor. For bigger cliques we need a more sophisticated argument.
CASE 2 There is a bramble $B$ of large order

By the excluded $K_t$ theorem of Robertson and Seymour we reduce to the same problem as above.
SUMMARY

MAIN THM (with Norin) $\forall t \exists N_t \forall t$-connected graph $G \not\ni_m K_t$ on $\geq N_t$ vertices $\exists X \subseteq V(G)$ with $|X| \leq t-5$ such that $G \setminus X$ is planar.

COR $G$ is $t$-connected, $\geq N_t$ vertices, $G \not\ni_m K_t \Rightarrow |E(G)| \leq (t-2)n-(t-1)(t-2)/2$

CONJ Corollary holds for $(t-2)$-connected graphs