3 - Induction and Recursion

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Using Recurrence Equations (1)

Basic Problem  How many regions are determined by \( n \) lines that intersect in general position?

Answer
\[ d_1 = 2 \]
\[ d_{n+1} = d_n + n+1 \text{ when } n \geq 0. \]

So
\[ d_2 = 2 + (1+1) = 4 \]
\[ d_3 = 4 + (2+1) = 7 \]
\[ d_4 = 7 + (3+1) = 11 \]

What are \( d_5 \) and \( d_6 \)?
**Basic Problem**  How many regions are determined by $n$ circles that intersect in general position?

**Answer**

$d_1 = 2$

$d_{n+1} = d_n + 2n$  when  $n \geq 0$.

So  

$d_2 = 2 + 2*1 = 4$

$d_3 = 4 + 2*2 = 8$

$d_4 = 8 + 2*3 = 14$

What are $d_5$ and $d_6$?
Basic Problem  How many ways to tile a \( 2 \times n \) grid with dominoes of size \( 1 \times 2 \) and \( 2 \times 1 \)?

Answer
\[
d_1 = 1 \\
d_2 = 2 \\
d_{n+2} = d_{n+1} + d_n 	ext{ when } n \geq 0.
\]

So \( d_3 = 2 + 1 = 3 \)
\( d_4 = 3 + 2 = 5 \)

What are \( d_5 \) and \( d_6 \)?
Basic Problem  How many ways to tile a $3 \times n$ grid with tiles of the four shapes illustrated here?

Partial Answer  

\[ d_1 = 1 \]
\[ d_2 = 2 \]
\[ d_3 = 4 \]

What are $d_5$ and $d_6$?

Cash Prize  One dollar to first person who can correctly evaluate $d_{20}$. 
Basic Problem  How ternary sequences do not contain 01 in consecutive positions?

Answer
\[ t_1 = 3 \]
\[ t_2 = 8 \]
\[ t_n = 3t_{n-1} - t_{n-2} \text{ when } n \geq 2. \]

So  \[ t_3 = 3 \times 8 - 3 = 21 \]
\[ t_4 = 3 \times 21 - 8 = 55 \]

What is  \[ t_5? \]
Question: If you know that:

\[ a_1 = 14 \]
\[ a_2 = 23 \]
\[ a_3 = -96 \]
\[ a_4 = 52 \]

and

\[ a_{n+4} = 9a_{n+3} - 7a_{n+2} + 8a_{n+1} + 13a_n \] when \( n \geq 1 \), then you can calculate \( a_n \) for any positive integer \( n \). Is this good enough, or would you like to know even more about \( a_n \)?
The Principle of Math Induction

Postulate  If \( S \) is a set of positive integers, 1 is in \( S \), and \( k + 1 \) is in \( S \) whenever \( k \) is in \( S \), then \( S \) is the set of all positive integers.

Consequence  To prove that a statement \( S_n \) is true for all \( n \), it suffices to do the following two tasks. First show that \( S_n \) holds when \( n = 1 \). Second, assume that \( S_n \) is true when \( n = k \) and show that it then holds when \( n = k + 1 \).
int my_function (int a) {
    if (a == 1) {
        return 42; /* The Secret */
    } else return 3*my_function (a -1) - 80;
}

What is the value of:

my_function (3)

Answer  58
A More Challenging Example

```c
int update_value(int a) {
    if (a % 2 == 0) {  /* a % 2 = a mod 2 */
        return a/2;
    } else return 3*a + 1;
}

int collatz_sequence(int a) {
    printf("%d \n", a);
    do while (a != 1) {a = update (a);}
    printf("Success!\n");
}
```
Applying Math Induction (1)

**Theorem**  The sum of the first $n$ odd integers is $n^2$, i.e.,
$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2.$$

**Proof**  $2 \times 1 - 1 = 1^2 = 1$, so true when $n = 1$.  
Assume true when $n = k$, i.e., assume 
$$1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2.$$  

Then 
$$1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$  
$$= k^2 + 2k + 1$$  
$$= (k +1)^2$$  

QED
Theorem  The sum of the first $n$ odd integers is $n^2$, i.e.,

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2.$$  

But  ... can we really be certain about what is meant with the expression of the left hand side? Let’s take out the ambiguity. In the English language, we might say “the sum of the first $n$ odd integers is $n^2$.”

Here’s an even more precise way. First, for a sequence $\{a_n: \ n \geq 1\}$, we define:

$$\sum_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \sum_{i=1}^{k+1} a_i = a_{k+1} + \sum_{i=1}^{k} a_i$$
Avoiding Ambiguity (2)

Theorem \[ \sum_{i=1}^{n} 2i - 1 = n^2 \]

Proof \[ \sum_{i=1}^{1} 2i - 1 = 2(1) - 1 = 1 = 1^2 \]

Now assume \[ \sum_{i=1}^{k} 2i - 1 = k^2 \]

Then \[ \sum_{i=1}^{k+1} 2i - 1 = k^2 + [2(k + 1) - 1] \]
\[ = k^2 + 2k + 1 \]
\[ = (k + 1)^2 \]

QED
Theory vs. Practice

Remark  In practice most mathematicians, computer scientists and engineers prefer the informal notation as they feel it is more intuitive. However, whenever truly pressed, they could if absolutely forced, go the more formal and absolutely unambiguous route.

Also  A combinatorial proof is usually preferable to a formal inductive proof ... as this helps us to understand what is really going on behind the scenes.

Remember  Usually means usually and not always.
Applying Math Induction (2)

**Exercise**  Show that the following formula is valid:
\[1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.\]

**Proof**  
1\(^2 = 1 = 1(1+1)(2*1+1)/6, so true when \ n = 1.\)  
Assume true when \ n = k, i.e., assume  
\[1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.\]

Then  
\[1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2\]
\[= \left[(2k^3 + 3k^2 +k)+(6k^2+12k+6)\right]/6\]
\[= (2k^3 + 9k^2 + 13k + 6)/6\]
\[= \frac{(k +1)(k + 2)(2k + 3)}{6}\]

QED
Applying Math Induction (3)

**Theorem**  For all \( n \geq 1 \), \( n^3 + (n + 1)^3 + (n + 2)^3 \) is divisible by 9.

**Proof**  When \( n = 1 \), \( 1^3 + 2^3 + 3^3 = 1 + 8 + 25 = 36 \).

Assume true when \( n = k \). Then, if \( n = k + 1 \),

\[
(k+1)^3 + (k+2)^3 + (k+3)^3 \\
= (k+3)^3 + (k+1)^3 + (k+2)^3 \\
= (k^3 + 9k^2 + 27k + 27) + (k+1)^3 + (k+2)^3 \\
= [(k^3 + (k+1)^3 + (k+2)^3)] + [9k^2 + 27k + 27]
\]

QED
An Exercise in Math Induction (1)

Exercise  Show that for all \( n \geq 2 \),

\[
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}
\]

Solution  (Which turned out to be more substantive than our other examples presented thus far.)

The base case is \( n = 2 \). Here the left hand is \( 1 + \frac{1}{\sqrt{2}} \) while the right hand side is \( \sqrt{2} \), so we want to show that \( 1 + \frac{1}{\sqrt{2}} > \sqrt{2} \).
Exercise (continued) Squaring both sides, this is equivalent to showing that

\[
1 + 2/\sqrt{2} + 1/2 > 2 \quad \text{and this is equivalent to} \quad \sqrt{2} > 1/2 \quad \text{which is true since } \sqrt{2} > 1.
\]

So we have established that the inequality is valid when \( n = 2 \). Now assume that it is valid for some integer \( k \), i.e.,

\[
1/\sqrt{1} + 1/\sqrt{2} + 1/\sqrt{3} + ... + 1/\sqrt{k} > \sqrt{k}
\]
Exercise (continued) It follows that

\[ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}. \]

Now what we want to prove is that

\[ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}, \]

so it suffices to prove that

\[ \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \]
Exercise (continued) Squaring both sides, the last inequality is equivalent to

\[ k + 2 \sqrt{\frac{k}{k+1}} + \frac{1}{k+1} > k + 1, \text{ which is equivalent to } \]

\[ 2 \sqrt{\frac{k}{k+1}} + \frac{1}{k+1} > 1. \text{ But this inequality holds if } \]

\[ 2 \sqrt{\frac{k}{k+1}} > 1, \text{ which is not equivalent to } \]

\[ 4k > k+1, \text{ which is true. } \]

QED (Whew!)
Exercise

Show that $n^2 > 5n + 13$ when $n \geq 7$.

Attempt at Solution

Base Case: $7^2 = 49 > 5 \cdot 7 + 13 = 48$. This works!

Inductive Step

Assume $k^2 > 5k + 13$ for some $k \geq 7$.

Then $(k + 1)^2 = k^2 + 2k + 1$

$> (5k + 13) + (2k + 1)$

$= (5k + 5) + (2k + 9)$

But I need to show that

$(k + 1)^2 > 5(k + 1) + 13 = (5k + 5) + 13$

So I need $2k + 9 \geq 13$. Is this true?
Exercise  If \( n \geq 2 \), then \( 2n + 9 \geq 13 \)

Proof  If \( n \geq 2 \), then \( 2n \geq 4 \), so that \( 2n + 9 \geq 4 + 9 = 13 \).

Exercise  Show that \( n^2 > 5n + 13 \) when \( n \geq 7 \).

Base Case  \( 7^2 = 49 > 5 \cdot 7 + 13 = 48 \). Check!

Inductive Step  Assume \( k^2 > 5k + 13 \) for some \( k \geq 7 \). Then

\[
(k + 1)^2 = k^2 + 2k + 1 \\
> (5k + 13) + (2k + 1) \\
= (5k + 5) + (2k + 9) \\
\geq 5(k + 1) + 13 \quad \text{QED}
\]
Alternative Forms of Induction

**Strategy 1** To argue by contradiction, if a statement $S_n$ is not true for all $n \geq 1$, there is a least positive integer for which it fails.

**Strategy 2** To prove that a statement $S_n$ holds for all $n \geq 1$, it is enough to do the following two steps:

**Base Step** Verify that the statement $S_1$ is valid.

**Strong Inductive Step** Assume that for some $k \geq 1$, the statement $S_m$ is valid for all $m$ with $1 \leq m \leq k$. Then show that statement $S_{k+1}$ is valid.
Basis for Long Division

**Theorem**  If \( m \) and \( n \) are positive integers, there are unique integers \( q \) and \( r \) with \( q \geq 0 \) and \( 0 \leq r < m \) so that

\[
n = q \cdot m + r
\]

**Question**  Is this obvious or does it require an explanation/proof?

**Yes!!**  It does require an argument.
Strategy  Make the following statement $S_n$: For all positive integers $m$, there exist $q$ and $r$ with $q \geq 0$ and $0 \leq r < m$ so that $n = q \cdot m + r$.

Proof  When $n = 1$, if $m = 1$, then $1 = 1 \cdot 1 + 0$, and if $m > 1$, then $1 = 0 \cdot m + 1$. So $S_1$ is true. Now assume $S_k$ is true, and let $m$ be a positive integer. Choose $q$ and $r$ so that $k = q \cdot m + r$. Then $k + 1 = q \cdot m + (r + 1)$ works unless $r + 1 = m$. In this case, $k + 1 = (q + 1) \cdot m + 0$.

The uniqueness part is just high school algebra.
Finding Greatest Common Divisors

**Problem** If \( n \) and \( m \) are positive integers with \( n \geq m \), find their greatest common divisor.

**Solution** The following loop always works.

```c
int gcd (int n, int m) {
    int gotit = 0;
    int answer = m;
    while (gotit == 0) do {
        if (n % answer == 0) return answer;
        gotit = 1;
        answer = answer - 1;
    }
}
```
Remark  There is no computer on the planet that will solve the following problem using the algorithm on the preceding slide:

\[ \gcd(275887499882303013399012285973582, 3747754982288837599088247) \]

Comment  Maple reported that they are relatively prime in less than one second.
The Euclidean Algorithm

Setup  Suppose $n$ and $m$ are positive integers with $n \geq m$. Choose $q$ and $r$ with $q \geq 0$ and $0 \leq r < m$ so that $n = qm + r$.

Fact  If $r = 0$, then $\gcd(n, m) = m$.

Fact  If $r > 0$, then $\gcd(n, m) = \gcd(m, r)$.

Explanation  $n/d = (qm + r)/d = q (m/d) + r/d$. 
An Improved Algorithm

```c
int gcd (int n, int m) {
    int gotit = 0;
    while (gotit == 0) do {
        r = n % m; /* r = n mod m */
        if (r == 0) return m;
        gotit = 1;
        else n = m;
            m = r;
    }
}
```
Concrete Example

Problem  Find $\gcd(10262736, 85470)$.

\[
\begin{align*}
10262736 \mod 85470 &= 6336 \\
85470 \mod 6336 &= 3102 \\
6336 \mod 3102 &= 132 \\
3102 \mod 132 &= 66 \\
132 \mod 66 &= 0
\end{align*}
\]

Answer  $66 = \gcd(10262736, 85470)$
Quotients and Remainders

Problem Find \( \text{gcd} (n, m) \) when \( n = 10262736 \) and \( m = 85470 \).

\[
\begin{align*}
10262736 &= 120 \times 85470 + 6336 \\
85470 &= 13 \times 6336 + 3102 \\
6336 &= 2 \times 3102 + 132 \\
3102 &= 23 \times 132 + 66 \\
132 &= 2 \times 66 + 0
\end{align*}
\]

\[
\begin{align*}
6336 &= 10262736 - 120 \times 85470 \\
3102 &= 85470 - 13 \times 6336 \\
132 &= 6336 - 2 \times 3102 \\
66 &= 3102 - 23 \times 132
\end{align*}
\]

Problem Use back-tracking to find integers \( a \) and \( b \) so that \( an + bm = \text{gcd} (n, m) \).
**Fact** When $n$ and $m$ are positive integers, there are integers $a$ and $b$ so that

$$\gcd(n, m) = an + bm$$

**Fact** We can find $a$ and $b$ by back-tracking with the information gained in carrying out the Euclidean algorithm.
Back Tracking Details

Problem Find $a$ and $b$ so that $\gcd(n, m) = an + bm$ when $n = 10262736$ and $m = 85470$

\[
66 = 3102 - 23 \times 132 \quad \text{and} \quad 132 = 6336 - 2 \times 3102
\]
\[
= -23 \times 6336 + 47 \times 3102 \quad \text{and} \quad 3102 = 85470 - 13 \times 6336
\]
\[
= 47 \times 85470 - 634 \times 6336 \quad \text{and} \quad 6336 = 10262736 - 120 \times 85470
\]
\[
= -634 \times 10262736 + 76127 \times 85470
\]

Solution $a = -634$ and $b = 76127$
Preferring Loops

Recommendation

Check out the program `gcd_lcm.c` on the course website and see how to compute gcd’s and solve the Diophantine equation \( an + bm = \gcd(n, m) \) using a loop with no backtracking and very little memory.