KEY

Student Name and ID Number

MATH 3012 Quiz 1, February 9, 2006, WTT

1. Consider strings whose symbols are drawn from the set of capital letters.

a. How many strings of length 14 can be formed if repetition of letters is *not* permitted?

$P(26, 14)$

b. How many strings of length 14 can be formed if repetition of letters is permitted?

26^{14}

c. How many strings of length 14 can be formed using exactly 4 A's, 3 B's, 5 C's, and 2 D's?

$\binom{14}{4,3,5,2}$

d. Let $S = \{W, I, L, A, M, T, R, O, E\}$. These 9 letters appear in the Professor's name!. How many strings of length 14 are possible if repetition is allowed and *exactly* 6 of the 14 symbols in the string come from S.

$\binom{14}{6} 9^6\, 17^8$

e. How many strings of length 14 are possible if repetition is *not* allowed and *exactly* 6 of the 14 symbols in the string come from the set S defined in part d?

$\binom{14}{6} P(9,6)\; P(17,8)$

2. How many lattice paths from $(0,0)$ to $(15,19)$ pass through $(4,6)$ and $(8,11)$?

$\binom{10}{4}\binom{9}{4}\binom{15}{8}$

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 + x_4 = 47$, all $x_i > 0$.

$\binom{46}{3}$

b. $x_1 + x_2 + x_3 + x_4 = 47$, all $x_i \geq 0$.

$\binom{50}{3}$

c. $x_1 + x_2 + x_3 + x_4 \leq 47$, all $x_i \geq 0$.

$\binom{51}{4}$

d. $x_1 + x_2 + x_3 + x_4 = 47$, all $x_i \geq 0$, $x_4 \geq 8$.

$\binom{42}{3}$

e. $x_1 + x_2 + x_3 + x_4 = 47$, all $x_i \geq 0$, $x_4 \leq 7$.

$\binom{50}{3} - \binom{42}{3}$

4. Define a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ by $f(1) = 3$ and $f(k+1) = 4k - 2 + f(k)$. Use math induction to prove that $f(n) = 2n^2 - 4n + 5$, for all $n \geq 1$.

We first verify for $n=1$: $f(1) = 3$ and $2(1)^2 - 4(1) + 5 = 2 - 4 + 5 = 3$, so the formula is valid for $n = 1$. Now suppose it holds for some $k \geq 1$. Then

$$f(k+1) = 4k - 2 + f(k) = 4k - 2 + 2k^2 - 4k + 5 = 2k^2 + 3 = 2(k+1)^2 - 4(k+1) + 5.$$

Therefore, $f(n) = 2n^2 - 4n + 5$ for all $n \geq 1$.

5. Use the Euclidean algorithm to find $d = \gcd(42, 858)$.

$$858 = 20 \cdot 42 + 18$$
$$42 = 2 \cdot 18 + 6$$
$$18 = 3 \cdot 6 + 0$$

Thus, $\gcd(42, 858) = 6$.

6. Use your work in the preceding problem to find integers x and y so that $d = 42x + 858y$.

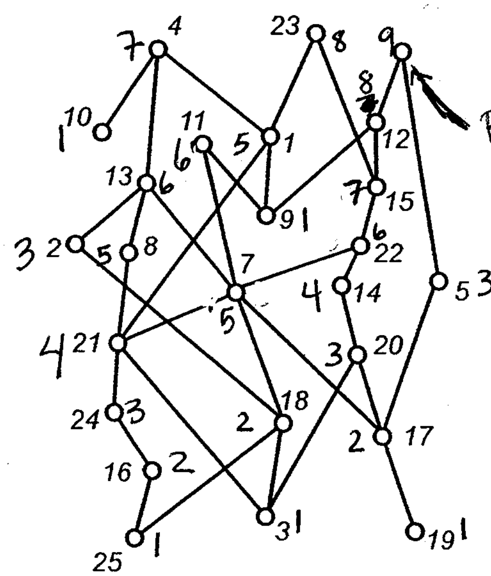
$$6 = 42 - 2 \cdot 18$$
$$= 42 - 2(858 - 20 \cdot 42)$$
$$= 41 \cdot 42 - 2 \cdot 858$$

Therefore $x = 41$, $y = -2$ satisfy the requirement.

7. Find the coefficient of $x^7y^{14}z^{11}$ in $(3x - 2y + 5z)^{32}$.

$$\binom{32}{7,14,11} 3^7 (-2)^{14} 5^{11}$$

8. Consider the partially ordered set (poset) P shown below:



Point number 6. (Accidentally got hit with white out.)

a. Find the set of minimal elements of P.

$\{25, 3, 19, 9, 10\}$

b. How many elements of P are comparable with the point labeled 14?

10 (They are 23, 6, 12, 15, 22, 14, 20, 17, 19, and 3.)

c. How many elements of P are incomparable with the point labeled 10?

23 (All of them <u>except</u> 10 and 4.)

d. Find a 4-element maximal chain containing 3 and 7 but not 18.

$\{3, 21, 7, 11\}$

e. Explain why $\{16, 17, 18\}$ is not a maximal antichain.

Because 10 is incomparable to 16, 17, and 18, so $\{10, 16, 17, 18\}$ is a larger antichain containing $\{16, 17, 18\}$.

f. Let $h = \text{height}(P)$, i.e., h is the size of a maximum chain in P. Then for each $x \in P$, let height(x) denote the maximum size of a chain in P having x as its greatest element. Note that $1 \leq \text{height}(x) \leq h$, for all $x \in P$. Writing directly on the diagram, label each point with the integer representing its height. Note that this will determine a partition of P into antichains, and among all such partitions, this one will use the minimum number of antichains.

g. Find the height h of the poset P.

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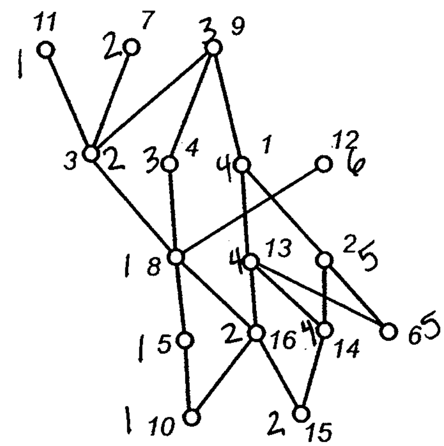
h. Find a chain of h points in P.

$\{25, 16, 24, 21, 7, 22, 15, 12, 6\}$

9. Consider the partially ordered set (poset) Q shown below:



a. Find an antichain of 6 points in this poset.

$\{11, 7, 4, 13, 2, 12\}$

b. Determine a partition of P into 6 chains by labeling directly on the figure each point with an integer from $\{1, 2, 3, 4, 5, 6\}$ so that for each i, all points labeled i form a chain.