Solutions

Student Name and ID Number

## MATH 3012 Quiz 1, February 5, 2018, WTT

9

3×3

**1.** Consider the 52-element set consisting of all letters of the English alphabet, including both lower case and upper case.

**a.** How many strings of length 42 can be formed if repetition of symbols is *not* permitted?

$P(52, 42)$

**b.** How many strings of length 42 can be formed if repetition of symbols is permitted?

$52^{42}$

**c.** How many strings of length 42 can be formed using exactly seven $A$'s, twenty $a$'s, ten $B$'s and five $b$'s?

$\binom{42}{7, 20, 10, 5}$ OR $\frac{42!}{7!\,20!\,10!\,5!}$

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**2.** How many lattice paths from $(0, 0)$ to $(14, 14)$ do not pass through any point above the diagonal?

The Catalan number $\frac{\binom{28}{14}}{15}$

12

4×3

**3.** How many integer valued solutions to the following equations and inequalities:

**a.** $x_1 + x_2 + x_3 + x_4 + x_5 = 81$, all $x_i > 0$.

$\binom{80}{4}$ 80 gap, choose 4

**b.** $x_1 + x_2 + x_3 + x_4 + x_5 = 81$, all $x_i \geq 0$.

$\binom{85}{4}$ Add 5 artificial element 85 gaps, choose

**c.** $x_1 + x_2 + x_3 + x_4 + x_5 < 81$, all $x_i > 0$.

$\binom{80}{5}$ Add positive slack variable 80 gaps, choose 5

**d.** $x_1 + x_2 + x_3 + x_4 + x_5 = 81$, $x_1, x_3, x_4, x_5 > 0$, $7 < x_2 < 13$.

Part 1. # of solutions with $x_2 > 7$ Set aside 7. 74 objects, 73 gaps choose 4. So part 1 = $\binom{73}{4}$. Part 2. # of solutions with $x_2 \geq 13$. Set aside 12. 69 objects 68 gaps, choose 4. = $\binom{68}{4}$

Answer is difference $\boxed{\binom{73}{4} - \binom{68}{4}}$

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**4.** Find the coefficient of $x^4y^{26}z^8$ in $(7xy - 9y^2 + 5z)^{23}$

$i_1 = 4$ from power of $x$; $i_3 = 8$ from power of $z$ : so $i_2 = 11$

Note power of $y = 4 + 2 \cdot 11 = 26$

$$\boxed{\binom{23}{4, 11, 8} 7^4 (-9)^{11} 5^8}$$

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**5.** Use the Euclidean algorithm to find $d = \gcd(2366, 520)$.

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$520 \overline{)2366}$ quotient 4; 2080; 286

$286 \overline{)520}$ quotient 1; 286; 234

$234 \overline{)286}$ quotient 1; 234; 52

$52 \overline{)234}$ quotient 4; 208; 26

$26 \overline{)52}$ quotient 2; 52; 0

$$\boxed{26 = \gcd(2366, 520)}$$

**6.** Use your work in the preceding problem to find integers $a$ and $b$ so that $d = 2366a + 520b$.

8

$26 = 1 \cdot 234 - 4 \cdot 52$

$52 = 1 \cdot 286 - 1 \cdot 234$

$234 = 1 \cdot 520 - 1 \cdot 286$

$286 = 1 \cdot 2366 - 4 \cdot 520$

$= 1 \cdot 234 - 4[1 \cdot 286 - 1 \cdot 234]$

$= 5 \cdot 234 - 4 \cdot 286$

$= 5[1 \cdot 520 - 1 \cdot 286] - 4 \cdot 286$

$= 5 \cdot 520 - 9 \cdot 286$

$= 5 \cdot 520 - 9[1 \cdot 2366 - 4 \cdot 520]$

$= -9 \cdot 2366 + 41 \cdot 520$

So $\boxed{a = -9 \text{ and } b = 41}$

**7.** For a positive integer $n$, let $t_n$ count the number of ways to tile a $2 \times n$ grid with tiles of three shapes: (1) a $2 \times 2$ square, (2) a $1 \times 2$ rectangular tile and (3) a $2 \times 1$ rectangular tile. Develop initial conditions and a recurrence relation satisfied by $t_n$ and use it to compute $t_1$, $t_2$, $t_3$, $t_4$ and $t_5$.

12

3 ways to cover the top right square.

①

$t_{n-2}$

②

$t_{n-1}$

③

must be another 1×2

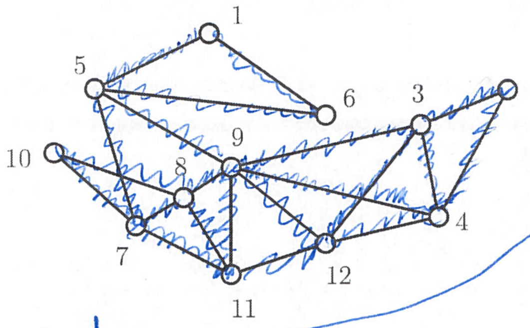
$t_{n-2}$

$$\boxed{\begin{aligned} t_n &= 2t_{n-2} + t_{n-1} \\ t_1 &= 1 \\ t_2 &= 3 \\ t_3 &= 3 + 2 \cdot 1 = 5 \\ t_4 &= 5 + 2 \cdot 3 = 11 \\ t_5 &= 11 + 2 \cdot 5 = 21 \end{aligned}}$$

/28

**8.** Use the greedy algorithm developed in class (always proceed to the lowest legal vertex) to find an Euler circuit in the graph $G$ shown below (use node 1 as root):

12



(1, 5, 6, 1)

(5, 7, 8, 9, 3, 2, 4, 3, 12, 4, 9, 5)

4

1, 5, 7, 8, 9, 3, 2, 4, 3, 12, 4, 9, 5, 6, 1)

7, 10, 8, 11, 7

1, 5, 7, 10, 8, 11, 7, 8, 9, 3, 2, 4, 3, 12, 4, 9, 5, 6, 1

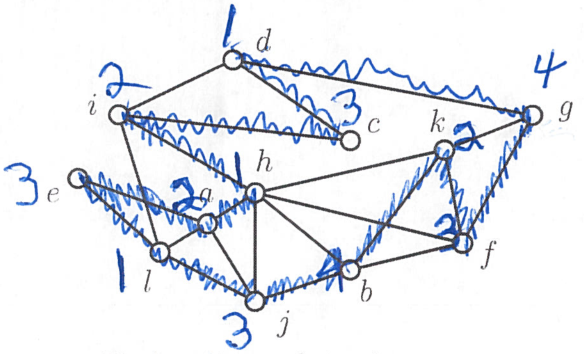
11, 9, 12, 11

1, 5, 7, 10, 8, 11, 9, 12, 11, 7, 8, 9, 3, 2, 4, 3, 12, 4, 9, 5, 6, 1

15

5×3

**9.** For the graph shown below:



Many correct answers for 4-coloring

**a.** Find a clique of size 4. {b, f, k, h}

**b.** Find an induced cycle of size 5. {d, g, k, h, i}

**c.** Show that $\chi(G) \leq 4$ by producing a proper coloring using the elements of $\{1, 2, 3, 4\}$ as colors. Write directly on the figure to give your answer.

**d.** Explain why this graph does not have an Eulerian circuit. Vertices d and g have odd degree.

**e.** Show that the graph is Hamiltonian by listing an appropriate sequence of vertices below.

(e, a, h, i, c, d, g, f, k, b, j, l)

other correct answers

(OK to list e on both ends)

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⑨

9×1

**10.** True–False. Mark in the left margin.

F 1. $P(10,4) = 720$. $\boxed{10 \cdot 9 \cdot 8} \cdot 7 > 720$

T 2. $C(10,4) = 210$. $\frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7}{1 \cdot \cancel{2} \cdot 3 \cdot \cancel{4}} = 210$ ✓

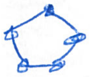
F 3. Any connected graph with an even number of edges has an Euler circuit.

T 4. There is a connected graph with 500 vertices and 5000 edges which does not have a Hamiltonian cycle.

F 5. The number of lattice paths from $(0,0)$ to $(12,12)$ which pass through $(6,8)$ is $C(12,6)C(12,8)$.

$C(14,6)\ C(10,4)$

F 6. If $G$ is a graph and $\chi(G) = 3$, then $\omega(G) = 3$.



T 7. The running time of an optimal implementation of the Euler Circuit algorithm is linear in the input size.

T 8. The decision problems: Is $\chi(G) \leq 3$? and Is $G$ Hamiltonian? belong to the class $NP$.

T 9. The decision problems: Is $\chi(G) \leq 2$? and Is $G$ Eulerian? belong to the class $P$.

10. *Just for fun and not graded.* Connected tilings of recursive multinomial graphs admit Hamiltonian circuits with vertices having clique size larger than their chromatic number, except when they have useless certificates whose correctness can be estimated in polynomial time by a biased referee.

Completely true!!

$C_{499}$

6, 4